Due Date: 29 December 2014, Monday – Class time NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 503 Complex Analysis – Take-Home Midterm Exam 2 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Let $\zeta(z)$ be the Riemann zeta function, which is meromorphic on $\mathbb{C}$ with a simple pole at $z = 1$ and holomorphic elsewhere, and set $\eta(z) = \dfrac{\zeta'(z)}{\zeta(z)}$ for $\operatorname{Re} z > 1$.

Show that for any z_0 with $\operatorname{Re} z_0 \geq 1$, we have

$$\lim_{z \to z_0} (z - z_0)\eta(z) = N,$$

where N is an integer. How do we determine the sign of N?

Solution:

For any meromorphic function $f(z)$, the Laurent expansion of f'/f around any point z_0 is given as

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + F(z),$$

where F is analytic around z_0 and m is an integer denoting the order of f at z_0: if f vanishes to order n at z_0, then $m = n$, and if f has a pole of order n at z_0, then $m = -n$. If on the other hand $f(z_0) \neq 0$, then $m = 0$.

It follows that $\lim\limits_{z \to z_0} \dfrac{f'(z)}{f(z)} = m$ is an integer. Now putting $f(z) = \zeta(z)$ solves the problem.

NAME: STUDENT NO:

Q-2) Assume that $\dfrac{\zeta'(z)}{\zeta(z)} = -\sum_{n=1}^{\infty} \dfrac{\Lambda(n)}{n^z}$ for $\operatorname{Re} z > 1$, where the Λ function is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and positive integer } m, \\ 0 & \text{otherwise.} \end{cases}$$

Continuing from Question 1, show that for every $\epsilon > 0$ and any $t \in \mathbb{R}$, we must have

$$\operatorname{Re} \eta(1+\epsilon+it) = -\sum_{n=1}^{\infty} \Lambda(n)\, n^{-(1+\epsilon)} \cos(t \log n).$$

Solution:

We need to find the real part of $n^{-(1+\epsilon+it)}$.

$$\begin{aligned} n^{-(1+\epsilon+it)} &= n^{-(1+\epsilon)}\, n^{-it} = n^{-(1+\epsilon)}\, e^{-it\log n} \\ &= n^{-(1+\epsilon)}[\cos(t\log n) - i\sin(t\log n)]. \end{aligned}$$

Therefore $\operatorname{Re} \Lambda(n) n^{-z} = \Lambda(n)\, n^{-(1+\epsilon)} \cos(t\log n)$. Now summing these up we get the result.

NAME: STUDENT NO:

Q-3) Continuing from the previous questions, show that for all $\epsilon > 0$, we have

$$3\,\mathrm{Re}\,\eta(1+\epsilon) + 4\,\mathrm{Re}\,\eta(1+\epsilon+it) + \mathrm{Re}\,\eta(1+\epsilon+2it) \leq 0.$$

Solution:

Using the previous result, what we want to prove here is equivalent to showing that

$$3 + 4\cos\alpha + \cos 2\alpha \geq 0,$$

where $\alpha = t\log n$. This however follows from an obvious trigonometric identity.

$$\begin{aligned} 3 + 4\cos\alpha + \cos 2\alpha &= 3 + 4\cos\alpha + 2\cos^2\alpha - 1 \\ &= 2(\cos\alpha + 1)^2 \\ &\geq 0. \end{aligned}$$

NAME: STUDENT NO:

Q-4) Continuing from the previous questions, show that neither $\zeta(it)$ nor $\zeta(1+it)$ vanishes for any $t \in \mathbb{R}$.

Solution:

Suppose that ζ vanishes at $(1+it)$ to order N. Letting $z = 1+\epsilon+it$ and $z_0 = 1+it$, we see that $z \to z_0$ is equivalent to $\epsilon \to 0$. Using the result of Question 1, we have

$$\lim_{\epsilon \to 0}[3\epsilon\eta(1+\epsilon) + 4\epsilon\eta(1+\epsilon+it) + \epsilon\eta(1+\epsilon+2it)] = -3+4N > 0,$$

but this contradicts the result of Question 2. This shows that $\zeta(1+it) \neq 0$ for ant $t \in \mathbb{R}$.

It follows from the Riemann functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\,\zeta(1-z)$$

that if $\zeta(it) = 0$, then $\zeta(1-it) = 0$. But this contradicts our finding above. So ζ function has no zeros on the $\operatorname{Re} z = 0$ and $\operatorname{Re} z = 1$ lines.