Due Date: 29 December 2014, Monday – Class time NAME:....................................................

Ali Sinan Sertöz STUDENT NO:....................................................

**Math 503 Complex Analysis – Take-Home Midterm Exam 2 –**

| 1 | 2 | 3 | 4 | TOTAL |
|---|---|---|---|---|
| | | | | |
| 25 | 25 | 25 | 25 | 100 |

*Please do not write anything inside the above boxes!*

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

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NAME: STUDENT NO:

**Q-1)** Let $\zeta(z)$ be the Riemann zeta function, which is meromorphic on $\mathbb{C}$ with a simple pole at $z = 1$ and holomorphic elsewhere, and set $\eta(z) = \dfrac{\zeta'(z)}{\zeta(z)}$ for $\operatorname{Re} z > 1$.

Show that for any $z_0$ with $\operatorname{Re} z_0 \geq 1$, we have

$$\lim_{z \to z_0} (z - z_0)\eta(z) = N,$$

where $N$ is an integer. How do we determine the sign of $N$?

**Solution:**

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**Q-2)** Assume that $\dfrac{\zeta'(z)}{\zeta(z)} = -\sum_{n=1}^{\infty} \dfrac{\Lambda(n)}{n^z}$ for $\operatorname{Re} z > 1$, where the $\Lambda$ function is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and positive integer } m, \\ 0 & \text{otherwise.} \end{cases}$$

Continuing from Question 1, show that for every $\epsilon > 0$ and any $t \in \mathbb{R}$, we must have

$$\operatorname{Re} \eta(1 + \epsilon + it) = -\sum_{n=1}^{\infty} \Lambda(n)\, n^{-(1+\epsilon)} \cos(t \log n).$$

**Solution:**

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**Q-3)** Continuing from the previous questions, show that for all $\epsilon > 0$, we have

$$3\,\mathrm{Re}\,\eta(1+\epsilon) + 4\,\mathrm{Re}\,\eta(1+\epsilon+it) + \mathrm{Re}\,\eta(1+\epsilon+2it) \leq 0.$$

**Solution:**

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**Q-4)** Continuing from the previous questions, show that neither $\zeta(it)$ nor $\zeta(1+it)$ vanishes for any $t \in \mathbb{R}$.

**Solution:**