Due Date: 22 December 2016, Thursday
Class Time
Instructor: Ali Sinan Sertöz

NAME:..
STUDENT NO:.......................................

Math 503 Complex Analysis - Final Exam – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Take-Home Assignments

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you **write your answers alone**.

(2) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(3) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you **exhibit your total understanding of the ideas involved**.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

Please sign here:

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Q-1) Show that there is an analytic function f defined on the punctured unit disc

$$D^* = \{z \mid 0 < |z| < 1\},$$

such that f' never vanishes and $f(D^*) = D$, where

$$D = \{z \mid |z| < 1\}.$$

Solution:

The map $z \mapsto \dfrac{z+1}{z-1}$ maps D^* onto the half plane $\operatorname{Re} z < 0$ with -1 missing. The exponential map maps that region onto $D \setminus \{0, 1/e\}$. Now use $z \mapsto \dfrac{1+z}{1-z}$ to map $D \setminus \{0, 1/e\}$ onto the right half plane, $\operatorname{Re} z > 0$, with 1 and $a = \dfrac{e+1}{e-1}$ missing.

Define a branch of the logarithm function with $-\pi < \theta < \pi$. Send the above half plane with this branch of the logarithm. The image is the horizontal strip $-\dfrac{\pi}{2} < \operatorname{Im} z < \dfrac{\pi}{2}$, with 1 and $\ln a$ missing.

Rotate everything by $z \mapsto iz$. The new region is the vertical strip $-\dfrac{\pi}{2} < \operatorname{Re} z < \dfrac{\pi}{2}$, with 0 and $i \ln a$ missing.

Dilate everything by $z \mapsto \dfrac{\pi z}{\ln a}$, to obtain the vertical strip $-\dfrac{\pi^2}{2 \ln a} < \operatorname{Re} z < \dfrac{\pi^2}{2 \ln a}$ with 0 and $i\pi$ missing.

Let $R = \dfrac{\pi^2}{2 \ln a}$. Note that $R \approx 6.39$.

Define the annulus $A_R = \{z \in \mathbb{C} \mid \dfrac{1}{R} < |z| < R\}$.

Use the map $z \mapsto e^z$ to map the last vertical strip to A_R with ± 1 missing.

Let $\alpha = e^{-i\pi/3} = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$.

Use the map $z \mapsto \alpha z$ to send the region $A_R \setminus \{\pm 1\}$ onto $A_R \setminus \{\pm \alpha\}$.

For any $r > 1$ define $E_r \subset \mathbb{C}$ to be the interior of the ellipse given by

$$\frac{4x^2}{\left(r + \frac{1}{r}\right)^2} + \frac{4y^2}{\left(r - \frac{1}{r}\right)^2} = 1,$$

where as usual $z = x + iy$.

Consider the map $\phi(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$. Note that $\phi'(z) = 0$ only at ± 1 which are missing in $A_R \setminus \{\pm 1\}$. Also note that $\phi(A_R \setminus \{\pm 1\}) = E_R \setminus \{\pm 1\}$.

Now consider the polynomial map $P(z) = z^3 - 3z$. This polynomial is chosen to have its derivatives vanish only at ± 1.

Consider the map $\phi(z) = z + \frac{1}{z}$, and let $E'_R = \phi(A_R)$ be the corresponding ellipse.

We now assume that $E_R \setminus \{\pm 1\}$ is conformally equivalent to $E'_R \setminus \{\pm 1\}$.

By direct computation check that $P(E'_R) = \phi(A_{R^3}) = E'_{R^3}$ and hence is a simply connected bounded region in $\mathbb{C}$.

Notice that since $R > 2$, the points ± 2 belong to $E'_R \setminus \{\pm 1\}$.

Also note that $P(1) = P(-2) = -2$ and $P(-1) = P(2) = 2$. Hence $P(E'_R \setminus \{\pm 1\}) = G$ is an open, simply connected and bounded region in $\mathbb{C}$.

By the Riemann mapping theorem there is a one-to-one analytic function h from G onto D.

Composing all the above described maps gives us the reqiured map f.

Without assuming that $E_R \setminus \{\pm 1\}$ is conformally equivalent to $E'_R \setminus \{\pm 1\}$ we proceed as follows.

We continue to use the notation of the previous paragraph.

Any point in $\mathbb{C}$ is of the form $z = \phi(re^{i\theta})$ for some unique $r \geq 1$ and some θ. In particular if z is inside E_R, then $P(z) = \phi(z^3)$.

Suppose there is a loop γ in $P(E_R)$ which is not null homotopic. Then there is a point q inside this loop which does not belong to $P(E_R)$. The point q is of the form $\phi(z^3)$ for some $z = re^{i\theta}$ with $r \geq 1$. Let q' be $\phi(z)$.

Let F_0 be the ellipse which is the image under ϕ of the circle with radius r and F_1 the ellipse which is the image under ϕ of the circle with radius r^3. We consider F_0 in the same plane as E_R and F_1 as in the same plane as $P(E_R)$.

We have $q \in F_1$ and $q' \in F_0$.

The images of the ray through z is both orthogonal to F_0 and F_1. This image intersects γ on both sides of q, so there must be points on the image of this ray on both sides of q'. But this is impossible as the ellipses of the form $\phi(re^{i\theta})$ form an increasing sequence of nested sets and once q' is outside E_R, the points on the orthogonal ray on one side of that ellipse are never in E_R.

Hence no loop in $P(E_R)$ can be null homotopic.

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Q-2 Let G be a simply connected and bounded region in $\mathbb{C}$. Fix a point $a \in G$. Assume that for every real valued harmonic function $u(z)$ on ∂G, there exists a real valued harmonic function $U(z)$ on G such that $U(z) = u(z)$ for all $z \in \partial G$. Construct an analytic function $f \colon G \to D$ which vanishes only at a. Here D is the unit disc $|z| < 1$.
This is how Riemann started to prove his famous mapping theorem. After this step he uses some intricate analysis to show that the above constructed f is a conformal equivalence.

Solution:

Using the existence assumption of the problem (the Dirichlet Principle), let $U(z)$ be a real valued harmonic function defined on G such that

$$U(z) = -\log|z-a| \quad \text{for} \quad z \in \partial G.$$

Since G is simply connected there exists a harmonic conjugate $V(z)$ for $U(z)$ on G. Let g be the analytic function on G defines by $g(z) = U(z) + iV(z)$. Now check that

$$f(z) = (z-a)e^{g(z)}$$

maps G into D and vanishes only at $z = a$.

Q-3 Show that there is an analytic function f on $D = \{z \mid |z| < 1\}$ which is not analytic on any open set G which properly contains D.

Solution:

We can use two different approaches. First we can use Theorem 5.15 on page 170 of Conway.

Theorem: *Let G be a region and let $\{a_n\}$ be a sequence of distinct points in G with no limit point in G; and let $\{m_i\}$ be a sequence of positive integers. Then there is an analytic function f defined on G whose only zeros are at the points a_n; moreover, a_n is a zero of f of multiplicity m_n.*

In our case we take $G = D$ and $a_n = (1 - \frac{1}{n})e^{in}$. Also take each $m_n = 1$. Then there exists an analytic function whose only zeros are simple zeros at the points a_n. The sequence a_n has the boundary of D as its accumulation set so f cannot extend beyond D.

The second approach uses a theorem from the book of Bak and Newman; Theorem 18.5 on page 231.

Theorem: *Suppose*

$$f(z) = \sum_{k=0}^{\infty} c_k z^{n_k} \quad \textit{with} \quad \liminf_{k \to \infty} \frac{n_{k+1}}{n_k} > 1.$$

Then the circle of convergence of the power series is a natural boundary for f.

This means that such an f is an example whose existence we are asked to show in the problem. Now check that

$$f(z) = \sum_{k=0}^{\infty} z^{k!}$$

is analytic in D where $|z| = 1$ is a natural boundary.

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Q-4) Let $\zeta(z)$ be the Riemann zeta function. Prove that for $\mathrm{Re}\, z > 2$,

$$\frac{\zeta(z-1)}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^z},$$

where $\phi(n)$ is the Euler totient function which counts the number of positive integers less than n that are relatively prime to n.

Solution:

We first consider the product $\displaystyle\sum_{n=1}^{\infty} \frac{\phi(n)}{n^z} \sum_{n=1}^{\infty} \frac{1}{n^z}$. Suppose $ab = n$ and $a \leq b$. Then from

$$\left(\cdots + \frac{\phi(a)}{a^z} + \cdots + \frac{\phi(b)}{b^z} + \cdots \right) \left(\cdots + \frac{1}{a^z} + \cdots + \frac{1}{b^z} + \cdots \right)$$

we see that the term $\dfrac{\phi(a) + \phi(b)}{(ab)^z}$ is contributed. Hence we have

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^z} \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{\sum_{d|n} \phi(d)}{n^z}.$$

Let C_n be a cyclic group of order n, and let ω be a generator. For any d dividing n, $\omega^{n/d}$ generates a subgroup C_d of order d. There are $\phi(d)$ generators of the group C_d. These generators are the only elements of C_n with order d. Since every element of C_n has an order d which divides n, we have

$$\sum_{d|n} \phi(d) = n.$$

Thus we proved that

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^z} \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{\sum_{d|n} \phi(d)}{n^z} = \sum_{n=1}^{\infty} \frac{n}{n^z} = \zeta(z-1),$$

and this proves the claim of the problem.