Due Date: 13 October 2016, Thursday
Class Time

NAME:..

STUDENT NO:.......................................

Math 503 Complex Analysis - Homework 1

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework Assignments

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you **write your answers alone**.

(2) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(3) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you **exhibit your total understanding of the ideas involved**.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

Please sign here:

NAME: STUDENT NO: DEPARTMENT:

Q-1) Let Λ be a circle lying on the unit sphere $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$. Show that the stereographic projection of Λ to $\mathbb{C}$ is a straight line if Λ passes through the North pole, and that it is a circle otherwise.

First Solution: Let us first immediately dispose the case when Λ passes through the North pole. In that case the plane π which cuts Λ is also the plane which contains all the lines from the North pole to the points on Λ. These rays intersect the $\mathbb{C}$ along the line of intersection of $\mathbb{C}$ with π.

Now assume that Λ does not pass through the North pole.

Let the plane π which cuts Λ be given by

$$b_1x_1 + b_2x_2 + b_3x_3 = d,$$

where $b_3 \neq d$ since the circle Λ, and hence the plane π does not pass through $(0, 0, 1)$. Moreover we normalize the coefficients by forcing

$$b_1^2 + b_2^2 + b_3^2 = 1.$$

In this case, by Cauchy-Schwartz inequality we have $|d| < 1$. Equality would hold when Λ is a degenerate circle, i.e. a point.

Under the stereographic projection we have

$$(x_1, x_2, x_3) \mapsto (X, Y) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}\right).$$

At this point, by trial and error, we try to find some constants X_0 and Y_0 such that

$$(X - X_0)^2 + (Y - Y_0)^2 > 0,$$

which would mean that all such (X, Y) lie on a circle in $\mathbb{C}$. After some playing around we find that

$$(X - X_0)^2 + (Y - Y_0)^2 = \left(\frac{x_1}{1 - x_3} - \frac{b_1}{d - b_3}\right)^2 + \left(\frac{x_2}{1 - x_3} - \frac{b_2}{d - b_3}\right)^2 = \frac{1 - d^2}{(d - b_3)^2} > 0,$$

as expected.

Another approach: Finding the constants X_0 and Y_0 may be easy on a clear day but on other days we can try the following approach. Suppose that (X, Y) is a point in the image of Λ under the stereographic projection as above. The inverse stereographic projection gives

$$(X, Y) \mapsto \left(\frac{2X}{X^2 + Y^2 + 1}, \frac{2Y}{X^2 + Y^2 + 1}, \frac{X^2 + Y^2 - 1}{X^2 + Y^2 + 1}\right).$$

These points must satisfy the equation of the plane π. Substituting these into that equation and simplifying we get

$$(b_3 - d)(X^2 + Y^2) + 2b_1X + 2b_2Y - (d + b_3) = 0.$$

Since we assumed that $b_3 \neq d$, this shows that the points (X, Y) lie on a circle as expected. Note that this is exactly the same circle equation we obtained above by guessing the center.

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Q-2) Show that $\{\operatorname{cis} k \mid k = 0, 1, 2, \dots\}$ is dense in $T = \{z \in \mathbb{C} \mid |z| = 1\}$.

Solution:

This is a classical result which you should work out yourself at least once in your life.

It suffices to show that $\{n \mod 2\pi \mid n \in \mathbb{Z}\}$ is dense in $[0, 2\pi)$. Then for any point $\operatorname{cis}\theta \in T$ and for any $\epsilon > 0$ we can find an integer k such that $|\theta - (k \mod 2\pi)|$ is small enough to make $\operatorname{cis}\theta - \operatorname{cis} k||\epsilon$, proving the claim of the problem.

For this however it suffices to show that for any irrational number α, the set $\{n\alpha \mod 1 \mid n \in \mathbb{Z}\}$ is dense in $[0, 1)$; using this it is immediate to see that $\{n \mod (1/\alpha) \mid n \in \mathbb{Z}\}$ is dense in $[0, 1/\alpha)$. Taking $\alpha = 1/(2\pi)$ will take us to where we want.

So we want to prove that for any irrational $\alpha > 0$ the set of all $n\alpha$ where n is an integer is dense in $[0, 1)$.

For this I will follow the logic and the notation of a classical article:

> John H. Staib and Miltiades S. Demos, *On the Limit Points of the Sequence* $\sin n$,
> Mathematics Magazine, Vol. 40, No. 4 (Sep., 1967), pp. 210-213
> Published by: Mathematical Association of America.

First note that for any $x \in \mathbb{R}$,

$$\lfloor x \rfloor = \text{ the greatest integer } n \text{ with } n \leq x.$$

Using the above article we define for any $x \in \mathbb{R}$

$$(x) = x - \lfloor x \rfloor.$$

Clearly $(x) = x \mod 1$ and is in the interval $[0, 1)$. We first prove a theorem.

Theorem 1: For any $x, y \in \mathbb{R}$,

$$(x + y) = \begin{cases} (x) + (y) & \text{if } \ (x) + (y) < 1, \\ (x) + (y) - 1 & \text{if } \ (x) + (y) \geq 1. \end{cases}$$

Proof: If $0 \leq (x) + (y) < 1$, writing the corresponding definitions we have

$$0 \leq x + y - (\lfloor x \rfloor + \lfloor y \rfloor) < 1.$$

Subtracting $x + y$ from all sides and multiplying by -1 we get

$$x + y - 1 < \lfloor x \rfloor + \lfloor y \rfloor \leq x + y.$$

This shows that $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$, and it follows that

$$(x + y) = x + y - \lfloor x + y \rfloor = x - \lfloor x \rfloor + y - \lfloor y \rfloor = (x) + (y).$$

The other case is proved in exactly the same manner except that now we start with

$$0 \leq (x) + (y) - 1 < 2.$$

$\square$

Corollary 1: For any real x which is not an integer, we have $(-x) = 1 - (x)$.
Proof: In the theorem put $-x$ for y. Then

$$0 = (x - x) = (x) + (-x) + \delta,$$

where $\delta \in \{0, -1\}$. But since both (x) and $(-x)$ are strictly positive, we need to have $\delta = -1$. $\square$

Corollary 2: If $(z) > (x)$, then $(z - x) = (z) - (x)$.
Proof: In the theorem write $z - x$ for y to get

$$(z) = (x + z - x) = (z) + (z - x) + \delta,$$

where $\delta \in \{0, -1\}$. Since $(z) - (x) = (z - x) + \delta > 0$, we must have $\delta = 0$. $\square$

Corollary 3: For any positive integer n and any real x, if we have $n(x) < 1$, then $n(x) = (nx)$.
Proof: Clearly holds for $n = 1$. Assume that $[n-1](x) < 1$ implies $[n-1](x) = ([n-1]x)$. Now assume $n(x) < 1$. Then clearly $[n-1](x) < 1$ holds, and we have by the theorem and the induction hypothesis that

$$(nx) = ([n-1]x + x) = ([n-1]x) + (x) = [n-1](x) + (x) = n(x),$$

as claimed. $\square$

Lemma: Let $\alpha > 0$ be an irrational number. For any $\epsilon > 0$, there exists a positive integer n such that $(n\alpha) < \epsilon$.
Proof: Without loss of generality we may assume that $\epsilon < 1$. Choose an integer N such that $N > 1/\epsilon$. Consider the set

$$R = \{0, (\alpha), (2\alpha), \ldots, (N\alpha)\}.$$

Let $b = \max R$. Then the points in R partitions the interval $[0, b] \subset [0, 1)$ into N subintervals. The length of the smallest of these subintervals must not exceed b/N. This means that there are distinct integers j and k, $0 \leq j, k \leq N$, such that

$$0 < (k\alpha) - (j\alpha) \leq b/N < 1/N < \epsilon.$$

It follows from Corollary 2 that

$$0 < ([k - j]\alpha) < \epsilon.$$

If $k - j > 0$, then we are done by setting $n = k - j$. So suppose $k - j < 0$, and let $m = j - k$. We now have

$$(-m\alpha) = (-m\alpha) - \lfloor -m\alpha \rfloor.$$

Letting $(-m\alpha = \epsilon^*$, we have $\epsilon^* < \epsilon < 1$, and

$$-m\alpha = \quad \text{negative integer} \quad + \epsilon^*.$$

Let p be the largest integer such that $p\epsilon^* < 1$, and multiply the above equation by p to get

$$-pm\alpha = \quad \text{negative integer} \quad + p\epsilon^*.$$

This shows that $(-pm\alpha) = p\epsilon^*$. Moreover the choice of p assures us that $0 < 1 - p\epsilon^* < \epsilon^*$. Therefore we have

$$0 < 1 - (-pm\alpha) < \epsilon^* < \epsilon,$$

and by Corollary 1,

$$0 < (pm\alpha) < \epsilon,$$

finishing the proof by setting $n = pm$. □

After all this preparation we finally prove what we set out to prove.

Theorem 2: For any irrational α, the set of points $(n\alpha)$ where n is an integer is dense in $[0, 1]$.
Proof: We lose no generality if we assume $\alpha > 0$ and show denseness in $(0, 1)$. Take any number $u \in (0, 1)$, and any $\epsilon > 0$ with $0 < \epsilon < u$. By the above lemma there exists a positive integer k such that $(k\alpha) < \epsilon$. Take j as the largest integer such that

$$j(k\alpha) \leq u < j(k\alpha) + (k\alpha).$$

It follows from this that

$$0 \leq u - j(k\alpha) < (k\alpha) < \epsilon.$$

Since $u < 1$, we must have $j(k\alpha) < 1$, and by Corollary 3 we have $j(k\alpha) = (jk\alpha)$. Hence taking $n = jk$ we finally have

$$0 \leq u - (n\alpha) < \epsilon,$$

proving the denseness property. □

NAME: STUDENT NO: DEPARTMENT:

Q-3 Let $\{f_n\}$ be a sequence of uniformly continuous functions from a metric space (X, d) into another metric space (Y, p) and suppose that $f = u - \lim f_n$ exists. Prove that f is uniformly continuous.

Solution:

Let $\epsilon > 0$ be given. The uniform convergence of f_n says that there exists an index N such that for all indices $n \geq N$ and for all $z \in X$ we have

$$p(f(z), f_n(z)) < \epsilon.$$

Moreover since each f_n is uniformly continuous on X, there exists a $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$, we have

$$p(f_n(x), f_n(y)) < \epsilon.$$

Now for every $x, y \in X$ with $d(x, y) < \delta$, and any fixed index n with $n \geq N$, we have

$$p(f(x), f(y)) \leq p(f(x), f_n(x)) + p(f_n(x), f_n(y)) + p(f_n(f_n(y), f(y)) < 3\epsilon,$$

showing that f is uniformly continuous on X.

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Q-4) Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be C^2-functions with $f(0) = 1$ and $h(0) = 0$. Moreover assume that the complex function

$$\phi(x + iy) = \sin x \cdot f(y) + ig(x)h(y)$$

is analytic on $\mathbb{C}$. (Here x and y are real variables.) Find f, g, h explicitly.

Solution:

This basically follows from Cauchy-Riemann equations. Let

$$\phi(z) = u(x, y) + iv(x, y),$$

where

$$u(x, y) = \sin x \; f(y), \;\; v(x, y) = g(x) \, h(y).$$

First, from $u_x = v_y$, we have $\cos x \; f(y) = g(x) \; h'(y)$. Puting $y = 0$ we get

$$\boxed{g(x) = \frac{1}{\alpha} \; \cos x}, \quad \text{where} \quad \alpha = h'(0) :$$

Second, from $u_y = -v_x$ we get $\sin x \; f'(y) = (1/\alpha) \sin x \; h(y)$, which gives

$$f'(y) = \frac{1}{\alpha} \, h(y), \quad \text{with} \quad f'(0) = 0, \quad \text{since} \quad h(0) = 0.$$

Third, from $v_{yy} = -v_{xx}$ we get $(1/\alpha) \cos x \; h''(y) = (1/\alpha) \cos x \; h(y)$, which gives

$$h''(y) = h(y).$$

This last ODE together with $h(0) = 0$ and $h'(0) = \alpha$ gives us

$$\boxed{h(y) = \alpha \sinh y}.$$

Finally, since we know that $f'(y) = (1/\alpha)h(y)$, we immediately know that $f'(y) = \sinh y$. Together with $f(0) = 1$, this gives

$$\boxed{f(y) = \cosh y}.$$

Now we can put these together to conclude that

$$\phi(z) = \sin x \cosh y + i \cos x \sinh y.$$

This is an analytic function which restricts to $\sin x$ on the real line. Later we will see that it is the unique such function.