Due Date: 10 November 2016,
Thursday, Class Time

NAME:...

STUDENT NO:.......................................

Math 503 Complex Analysis - Homework 2

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework Assignments

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you **write your answers alone**.

(2) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(3) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you **exhibit your total understanding of the ideas involved**.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

Please sign here:

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Q-1) We know (by Theorem 7.2 on page 97) that for any entire function f and any $R > 0$,

$$\frac{1}{2\pi i}\int_{|z|=R}\frac{f'(z)}{f(z)}\,dz = \text{The number of zeros of } f\text{, counting multiplicities, that lie in } |z|<R.$$

Using this theorem give a proof of the Fundamental Theorem of Algebra.

Solution:

Let f be a polynomial of degree $n > 0$, and let m_R be the number of zeros of f inside the disc $|z| < R$. We will show that $\lim_{R\to\infty} m_R = n$.

We have

$$m_R = \frac{1}{2\pi i}\int_{|z|=R}\frac{f'(z)}{f(z)}\,dz, \quad \text{and} \quad 1 = \frac{1}{2\pi i}\int_{|z|=R}\frac{1}{z}\,dz.$$

We now compute the difference $|m_R - n|$ as $R \to \infty$.

$$\begin{aligned}
\lim_{R\to\infty}|m_R - n| &= \frac{1}{2\pi}\lim_{R\to\infty}\left|\int_{|z|=R}\left(\frac{f'(z)}{f(z)} - \frac{n}{z}\right)dz\right| \\
&\le \frac{1}{2\pi}\lim_{R\to\infty}\int_{|z|=R}\left|\frac{f'(z)}{f(z)} - \frac{n}{z}\right|\,|dz| \\
&= 0,
\end{aligned}$$

since $\deg(zf'(z) - nf(z)) + 1 < \deg zf(z)$. We now conclude that $m_R = n$ for all large R since m_R is always an integer.

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Q-2) Let γ be the polygon $[0, 2, 2+2i, 2i, 0]$. Find $\displaystyle\int_\gamma f$ for

(a) $f(z) = \dfrac{1}{(z-\frac{1}{2}-i)(z-1-\frac{3}{2}i)(z-1-\frac{i}{2})(z-\frac{3}{2}-i)}$.

(b) $f(z) = \dfrac{1}{(z-\frac{1}{4}[1+i])(z-\frac{1}{2}[1+i])(z-\frac{3}{4}[1+i])}$.

Solution:

(a)

$$f(z) = \frac{1}{(z-\frac{1}{2}-i)(z-1-\frac{3}{2}i)(z-1-\frac{i}{2})(z-\frac{3}{2}-i)}$$

$$= \frac{-2}{z-\frac{1}{2}-i} + \frac{2\,i}{z-1-\frac{3}{2}i} + \frac{-2\,i}{z-1-\frac{i}{2}} + \frac{2}{z-\frac{3}{2}-i}$$

Integrating each of these around γ will give

$$2\pi i(-2+2i-2i+2) = 0,$$

which is the answer.

(b)

$$f(z) = \frac{1}{(z-\frac{1}{4}[1+i])(z-\frac{1}{2}[1+i])(z-\frac{3}{4}[1+i])}$$

$$= \frac{-4\,i}{z-\frac{1}{4}[1+i]} + \frac{8\,i}{z-\frac{1}{2}[1+i]} + \frac{-4\,i}{z-\frac{3}{4}[1+i]}$$

Integrating each of these around γ will give

$$2\pi i(-4i+8i-4i) = 0,$$

which is the answer.

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Q-3 Give an example of a closed rectifiable curve γ such that for any integer k there is a point $a \notin \gamma$ with $n(\gamma; a) = k$.

Solution:

For any integer $n \geq 0$, define the path

$$\gamma_n(t) = \begin{cases} \frac{1}{2^n}\left[\sin(2t) + i(1-\cos(2t))\right] & 0 \leq t \leq \pi, \\ \frac{1}{2^n}\left[-\sin(2t) - i(1-\cos(2t))\right] & \pi \leq t \leq 2\pi. \end{cases}$$

Now let

$$\gamma = \gamma_0 + \gamma_1 + \cdots + \gamma_n + \cdots .$$

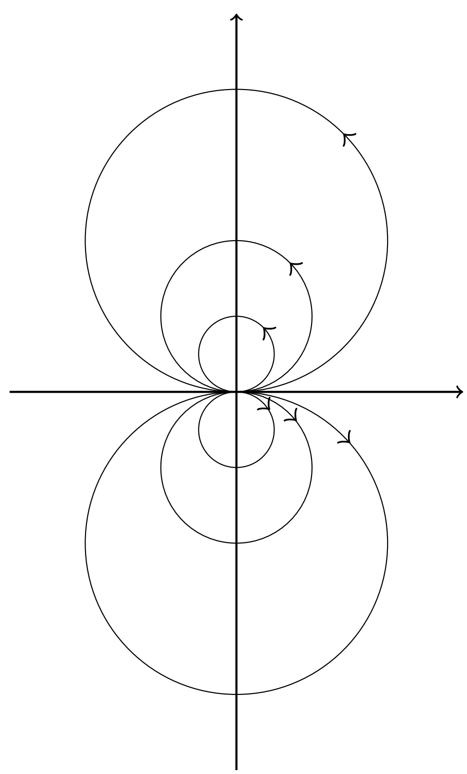
Length of γ is 4π. Let

$$D_n = \{z = x + iy \in \mathbb{C} \mid x^2 + (y \pm \frac{1}{2^n})^2 < \frac{1}{2^{2n}}\}.$$

These form a nested sequence of open sets

$$D_0 \supset D_1 \supset \cdots$$

For any point p in the complement of the closure of D_0, the index of γ around p is zero. For any positive integer n, let p be a point in $D_n - \bar{D}_{n+1}$, where bar denotes the closure. Then the index of γ around p is n if p is in the upper half plane, and is $-n$ otherwise.



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Q-4) Evaluate the following integral for $n = 1$ and $n = 2$.

$$\int_{|z-\frac{3}{2}|=\frac{3}{2}} \left(\frac{z}{z^2-3z+2}\right)^n \, dz.$$

Solution:

$$\begin{aligned}
\int_{|z-\frac{3}{2}|=\frac{3}{2}} \frac{z}{z^2-3z+2} \, dz &= \int_{|z-1|=\frac{1}{2}} \frac{\frac{z}{z-2}}{z-1}; dz + \int_{|z-2|=\frac{1}{2}} \frac{\frac{z}{z-1}}{z-2} \, dz \\
&= 2\pi i \left(\left. \frac{z}{z-2} \right|_{z=1} \right) + 2\pi i \left(\left. \frac{z}{z-1} \right|_{z=2} \right) \\
&= 2\pi i.
\end{aligned}$$

$$\begin{aligned}
\int_{|z-\frac{3}{2}|=\frac{3}{2}} \left(\frac{z}{z^2-3z+2}\right)^2 \, dz &= \int_{|z-1|=\frac{1}{2}} \frac{\left(\frac{z}{z-2}\right)^2}{(z-1)^2} \, dz + \int_{|z-2|=\frac{1}{2}} \frac{\left(\frac{z}{z-1}\right)^2}{(z-2)^2} \, dz \, dz \\
&= 2\pi i \left(\left(\left. \frac{d}{dz} \right|_{z=1} \right) \left(\frac{z}{z-2} \right)^2 + \left(\left. \frac{d}{dz} \right|_{z=2} \right) \left(\frac{z}{z-1} \right)^2 \right) \\
&= 0.
\end{aligned}$$