Due Date: 8 December 2016,
Thursday, Class Time

NAME:..

STUDENT NO:.......................................

Math 503 Complex Analysis - Homework 3

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework Assignments

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you **write your answers alone**.

(2) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(3) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you **exhibit your total understanding of the ideas involved**.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

Please sign here:

NAME: STUDENT NO: DEPARTMENT:

Q-1) Construct a sequence of functions $f_n(x)\colon \mathbb{R} \to \mathbb{R}$ such that each $f_n(x)$ is real analytic, i.e. at every $x_0 \in \mathbb{R}$ each $f_n(x)$ has a Taylor expansion converging to the function itself, and moreover $f_n(x)$ converges uniformly to $f(x)$ on $\mathbb{R}$ where $f(x) = |x|$.
Note that the uniform limit of these real analytic functions is not analytic. This never happens with complex analytic functions.

Solution:

Define $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, $n = 1, 2, \ldots$. It is clear that each f_n is real analytic.

We claim that $f_n(x) - f(x) \leq \frac{1}{\sqrt{n}}$ for all $x \in \mathbb{R}$ and all $n = 1, 2, \ldots$. Assuming that this is true, for any $\epsilon > 0$ let N be any integer such that $\frac{1}{\sqrt{N}} < \epsilon$. Then for any $n \geq N$ and any $x \in \mathbb{R}$, we will have $|f_n(x) - f(x)| < \epsilon$, proving the uniform convergence.

Since each f_n is even, and also f is even, it suffices to prove the above claim only for $x \geq 0$. For this first consider the function for each positive integer n.

$$\phi_n(x) = \sqrt{x^2 + \frac{1}{n}} + x - \frac{1}{\sqrt{n}}$$

for $x \geq 0$. We have $\phi_n(0) = 0$ and $\phi_n'(x) > 0$. This shows that $\phi_n(x) \geq 0$ for all $x \geq 0$. Then for $x \geq 0$ and every $n = 1, 2, \ldots$, we have the following inequalities.

$$\begin{aligned}
\frac{1}{\sqrt{n}} &\leq \sqrt{x^2 + \frac{1}{n}} + x \\
\frac{\frac{1}{\sqrt{n}}}{\sqrt{x^2 + \frac{1}{n}} + x} &\leq 1 \\
\frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + x} &\leq \frac{1}{\sqrt{n}} \\
\sqrt{x^2 + \frac{1}{n}} - x &\leq \frac{1}{\sqrt{n}} \\
f_n(x) - f(x) &\leq \frac{1}{\sqrt{n}}
\end{aligned}$$

which completes the proof.

NAME: STUDENT NO: DEPARTMENT:

Q-2) Let D be the unit disc. Find all analytic functions $f\colon D \to D$ with at least two fixed points.

Solution:

Let $f(p) = p$ and $f(q) = q$ with $p, q \in D$ and $p \neq q$. Assume without loss of generality that $q \neq 0$. Let ϕ be that automorphism of D sending p to 0. Consider the function $g = \phi \circ f \circ \phi^{-1}$. Let $\phi(q) = r$. Note that $r \neq 0$.

We now have $g(0) = 0$ and $g(r) = r$. By Schwarz's lemma $g(z) = cz$ for some $|c| = 1$, since $|g(z)| = |z|$ holds for $r \neq 0$. But since $g(r) = r$, c must be 1.

Now we have $\phi \circ f \circ \phi^{-1}(z) = z$ or $f \circ \phi^{-1}(z) = \phi^{-1}(z)$ for every $z \in D$. This shows that f is the identity map.

Hence only analytic function from D to D with at least two fixed points is the identity function.

NAME: STUDENT NO: DEPARTMENT:

Q-3

(a) Does there exist an analytic surjective map $f\colon D^* \to D$? Here D is the unit disc around the origin and D^* is D with the origin removed.

(b) Does there exist an analytic surjective map $f\colon D \to D^*$?

Solution:

(a) Recall that for any $a \in D$, the function

$$\phi_a(z) = \frac{z-a}{1-\bar{a}\,z}$$

is an automorphism of D. For any $a \in D^*$, the function

$$f(z) = (\phi_a(z))^2 = \left(\frac{z-a}{1-\bar{a}\,z}\right)^2$$

does the trick. Clearly $\phi_a(D^*) = D \setminus \{\phi_a(0)\} = D \setminus \{-a\}$. So we have to find another point of D^* which maps to $f(0) = a^2$. Check that

$$f\left(\frac{2a}{1+|a|^2}\right) = a^2,$$

hence f is surjective.

(b) Check that

$$f(z) = \exp\frac{z+1}{z-1}$$

does the job! The Möbius transformation sends D onto $\operatorname{Re} z < 0$, and the exponential map sends $\operatorname{Re} z < 0$ onto D^*.

NAME: STUDENT NO: DEPARTMENT:

Q-4) For any positive integer n calculate the integral

$$I_n = \int_0^\infty \frac{dx}{(x^2+1)^n}.$$

Solution:

Let $\phi(z) = (z+i)^{-n}$. Since ϕ is analytic at $z = i$, it has a Taylor expansion around i.

$$\phi(z) = a_0 + a_1(z-i) + \cdots + a_k(z-i)^k + \cdots .$$

Consider the function

$$f_n(z) = \frac{1}{(z^2+1)^n} = \frac{\phi(z)}{(z-i)^n}.$$

Then

$$\mathrm{Res}(f(z), i) = a_{n-1} = \frac{\phi^{(n-1)}(i)}{(n-1)!}.$$

By induction we find that

$$\phi^{(k)}(z) = (-1)^k \frac{(n+k-1)!}{(n-1)!} (z+i)^{-n-k}.$$

Hence

$$\phi^{(n-1)}(z) = (-1)^{n-1} \frac{(2n-2)!}{(n-1)!} (z+i)^{-2n+1}, \quad \text{and} \quad \phi^{(n-1)}(i) = \frac{(2n-2)!}{(n-1)!} \frac{1}{i\, 2^{2n-1}}.$$

Evaluate $f_n(z)$ around the closed path going from $-R$ to R along the real line and then following the semicircle with center the origin and radius R back to $z = -R$. The integral along the circular path goes to zero as R goes to infinity by standard arguments. The only singularity within the path is $z = i$. The integrand is even. So we have

$$\int_0^\infty \frac{dx}{(x^2+1)^n} = \frac{(2n-2)!}{[(n-1)!]^2} \frac{\pi}{2^{2n-1}}$$

$$= \frac{\pi}{2} \frac{2n-3}{2n-2} \frac{2n-5}{2n-4} \cdots \frac{1}{2}.$$