Due Date: 24 November 2016, Thursday
Class Time

NAME:..

STUDENT NO:..

Math 503 Complex Analysis - Midterm Exam 2 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Take-Home Assignments

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you **write your answers alone**.

(2) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(3) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you **exhibit your total understanding of the ideas involved**.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

Please sign here:

NAME: STUDENT NO: DEPARTMENT:

Q-1) Let f be analytic on $\bar{B}(0;R)$ with $|f(z)| \leq M$ for $|z| \leq R$ and $|f(0)| = a > 0$. Let $\alpha > 2$ be any real number. Show that the number of zeros of f in $B(0;\frac{R}{\alpha})$ is less than or equal to $\dfrac{1}{\log(\alpha-1)} \log\left(\dfrac{M}{a}\right)$.

Solution:

Let $z_1, \ldots, z_n$ be the zeros of f in $B(0;R/\alpha)$, repeated according to multiplicity if necessary. Consider the function

$$g(z) = \frac{f(z)\, z_1 \cdots z_n}{(z-z_1)\cdots(z-z_n)}.$$

Clearly g is analytic in $\bar{B}(0;R)$ and $|g(0)| = |f(0)| = a > 0$. By the Maximum Modulus Principle there is some z_0 with $|z_0| = R$ such that $|g(z_0)| \geq |g(z)|$ for all $z \in B(0;R)$. In particular $|g(z_0)| \geq |g(0)| = a > 0$. Thus we have

$$0 < a \leq \frac{|f(z_0)|\,|z_1 \cdots z_n|}{|z_0 - z_1| \cdots |z_0 - z_n|} \leq \frac{M\,|z_1 \cdots z_n|}{|z_0 - z_1| \cdots |z_0 - z_n|},$$

or

$$|z - z_1| \cdots |z - z_n| \leq \frac{M}{a}\,|z_1 \cdots z_n|.$$

But for each $1 \leq k \leq n$ we have

$$|z_0 - z_k| \geq |z_0| - |z_k| \geq R - \frac{R}{\alpha} = (\alpha - 1)\,\frac{R}{\alpha},$$

and

$$|z_k| < \frac{R}{\alpha}.$$

Putting these together we get

$$(\alpha-1)^n\,\frac{R^n}{\alpha^n} \leq \frac{M}{a}\frac{R^n}{\alpha^n},$$

or, after cancellation

$$(\alpha-1)^n \leq \frac{M}{a}.$$

Taking the logarithm of both sides we finally get

$$n\,\log(\alpha-1) \leq \log\left(\frac{M}{a}\right).$$

Since $\log(\alpha-1) > 0$, we can divide both sides by $\log(\alpha-1)$ and get

$$n \leq \frac{1}{\log(\alpha-1)}\log\left(\frac{M}{a}\right).$$

NAME: STUDENT NO: DEPARTMENT:

Q-2 Let f be a non-constant analytic function in $B(0;R)$ and for $0 \leq r < R$ define $A(r) = \max\{\operatorname{Re} f(z) \mid |z| = r\}$. Show that $A(r)$ is a strictly increasing function.

Solution:

Assume not. Then choose $0 \leq r_0 < R$ to be the smallest radius for which there is some $\epsilon > 0$ such that $A(r_0) \geq A(r)$ for all $r_0 < r < r_0 + \epsilon$.

Choose z_0 with $|z_0| = r_0$ and $|f(z_0)| = A(r_0)$.

Let $U_0 = B(z_0; \epsilon)$ and $W_0 = f(U_0)$. Since f is analytic W_0 is open.

Since $f(z_0)$ is in W_0, and since W_0 is open, there is some $\delta > 0$ such that $B(f(z_0); \delta) \subset W_0$. In particular $f(z_0) + \delta/2 \in B(f(z_0); \delta) \subset W_0$. But $\operatorname{Re}(f(z_0) + \delta/2) > \operatorname{Re} f(z_0)$, so there is no point in U_0 mapping to $f(z_0) + \delta/2$. But then $f(z_0) + \delta/2 \notin W_0$. This is a contradiction.

Hence $A(r)$ must be strictly increasing.

NAME: STUDENT NO: DEPARTMENT:

Q-3 Evaluate the improper integral $\displaystyle\int_0^\infty \left(\frac{1}{x^2} - \frac{1}{x\sinh x}\right) dx$.

Solution:

Let $f(z) = \dfrac{\sinh z - z}{z^2 \sinh z}$. This function has a removable singularity at $z = 0$. In fact

$$f(z) = \frac{1}{6} - \frac{7}{360}\, z^2 + \frac{31}{1529}\, z^3 + \cdots .$$

The only singularities of f are simple poles at $z = k\pi i$ where k is a non-zero integer, and hence

$$\mathrm{Res}(f(z), k\pi i) = -\frac{i}{\pi}\,\frac{(-1)^{k+1}}{k}.$$

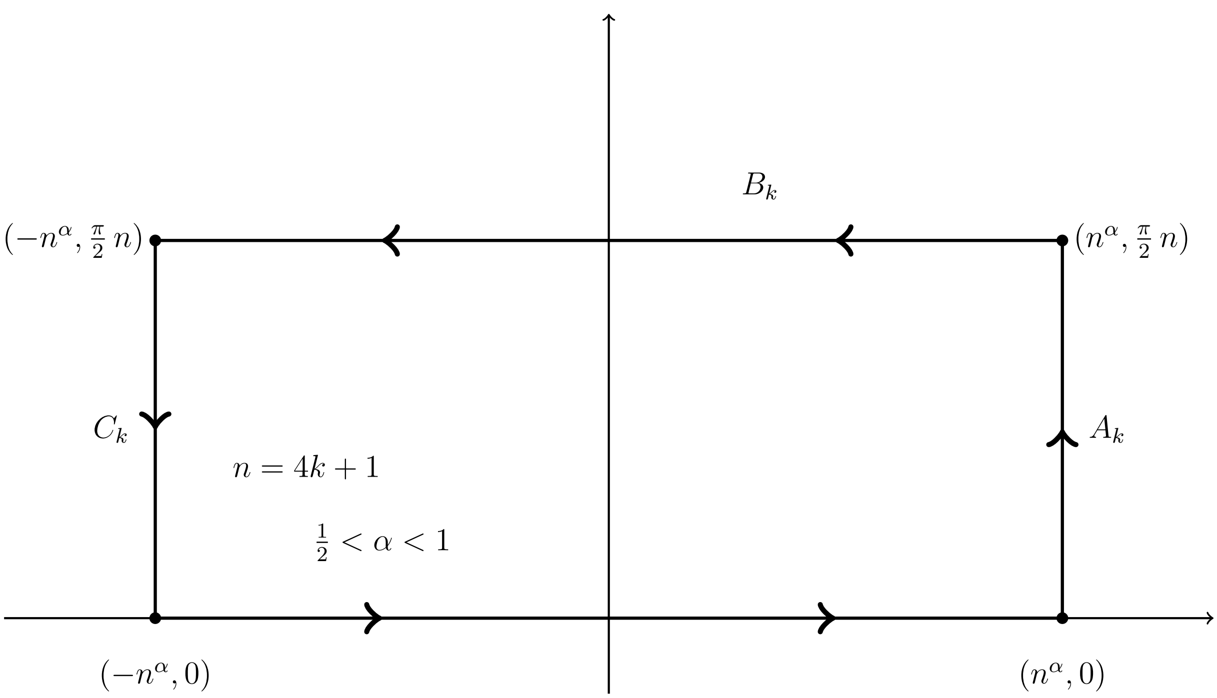
Note that

$$\sum_{k=1}^{\infty} \mathrm{Res}(f(z), k\pi i) = -\frac{i}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = -\frac{i}{\pi}\,\log 2.$$

We will write

$$f(z) = \frac{1}{z^2} - \frac{1}{z\sinh z},$$

and integrate it over the closed path γ_k as shown below.



$$\gamma_k = A_k + B_k + C_k + [-n^\alpha, n^\alpha]$$

We note the following facts about $\sinh z$.

1. $\sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$
2. $|\sinh(x+iy)|^2 = \sinh^2 x + \sin^2 y \geq \sinh^2 x$

3. $|\sinh x| \geq |x|$ for all $x \in \mathbb{R}$
4. In particular $|\sinh(x+iy)| \geq |x|$ for all $x, y \in \mathbb{R}$.

Integrating $f(z)$ around γ_k we get

$$\int_{\gamma_k} f(z)\,dz = 2\pi i \sum_{j=1}^{2k} \operatorname{Res}(f(z), i\pi j) = 2\sum_{j=1}^{2k}(-1)^{j+1}\frac{1}{j}.$$

When we take the limit of all sides as k goes to infinity, the rightmost side will converge to $2\log 2$. Now we examine the integral on left hand side.

on the path A_k:
Here $z = n^\alpha + iy$ where $0 \leq y \leq \frac{\pi}{2}n$. Therefore $|z| \geq n^\alpha$ and $|\sinh z| \geq n^\alpha$. Then

$$|f(z)| \leq \frac{1}{|z|^2} + \frac{1}{|z|\,|\sinh z|} \leq \frac{2}{n^{2\alpha}}.$$

Letting $\ell(A_k)$ denote the length of the path A_k, we have

$$\ell(A_k) = \frac{\pi}{2}n.$$

It follows that

$$\left|\int_{A_k} f(z)\,dz\right| \leq \left(\frac{\pi}{2}n\right)\left(\frac{2}{n^{2\alpha}}\right) \to 0 \quad \text{as } k \to \infty.$$

on the path C_k:
Here $z = -n^\alpha + iy$ where $0 \leq y \leq \frac{\pi}{2}n$. Hence the inequalities of the analysis of path A_k hold here. Also we have $\ell(C_k) = \ell(A_k)$. So we again have

$$\left|\int_{C_k} f(z)\,dz\right| \leq \left(\frac{\pi}{2}n\right)\left(\frac{2}{n^{2\alpha}}\right) \to 0 \quad \text{as } k \to \infty.$$

on the path B_k:
Here $z = x + i\frac{\pi}{2}n$, with $-n^\alpha \leq x \leq n^\alpha$. We have $\ell(B_k) = 2n^\alpha$ and $|z| \geq \frac{\pi}{2}n$. Recalling that $n = 4k+1$, we have

$$\sinh(x + \frac{\pi}{2}n) = i\cosh x,$$

and hence $|\sinh z| \geq 1$ for $z \in B_k$. Then we have

$$\left|\int_{B_k} f(z)\,dz\right| \leq (2n^\alpha)\left(\frac{4}{\pi^2 n^2} + \frac{2}{\pi n}\right) = \frac{8n^\alpha}{\pi^2 n^2} + \frac{4n^\alpha}{\pi n} \to 0 \quad \text{as } k \to \infty \quad .$$

on the path $[-n^\alpha, n^\alpha]$:
Since f is an even function here we have

$$\int_{[-n^\alpha, n^\alpha]} f(z)\,dz = 2\int_0^{n^\alpha}\left(\frac{1}{x^2} - \frac{1}{x\sinh x}\right)dx \to 2\int_0^{\infty}\left(\frac{1}{x^2} - \frac{1}{x\sinh x}\right)dx \quad \text{as } k \to \infty \quad .$$

Putting these together we finally have

$$\int_0^\infty \left(\frac{1}{x^2} - \frac{1}{x\sinh x}\right)dx = \log 2.$$

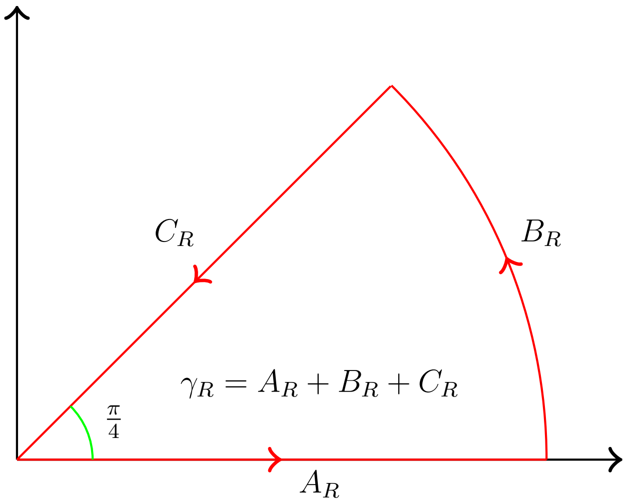
NAME: STUDENT NO: DEPARTMENT:

Q-4) Evaluate the following improper integrals.

(a) $\displaystyle\int_0^\infty \sin x^2 \, dx$

(b) $\displaystyle\int_0^\infty \cos x^2 \, dx$

Solution:



Let $f(z) = e^{iz^2}$, and consider the integral $\int_{\gamma_R} f(z)\,dz$, where γ_R is shown in the above figure.

On A_R: $z = x$, $0 \le x \le R$ and $dz = dx$.

$$\int_{A_R} f(z)\,dz = \int_0^R e^{ix^2}\,dx = \int_0^R \cos x^2\,dx + i\int_0^R \sin x^2\,dx.$$

On C_R: $z = \alpha x$, $0 \le x \le R$ and $dz = \alpha dx$ where $\alpha = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$. Note that $\alpha^2 = i$. Then

$$\int_{C_R} f(z)\,dz = \alpha\int_R^0 e^{i\alpha^2x^2}\,dx = -\alpha\int_0^R e^{i\alpha^2x^2}\,dx = -\alpha\int_0^R e^{-x^2}\,dx.$$

On B_R: $z = Re^{i\theta} = R\cos\theta + iR\sin\theta$, $0 \le \theta \le \frac{\pi}{4}$, $dz = iRe^{i\theta}\,d\theta$. Note $z^2 = R^2\cos 2\theta + iR^2\sin 2\theta$. Hence

$$\int_{B_R} f(z)\,dz = iR\int_0^{\pi/4} e^{-R^2\sin 2\theta + iR^2\cos 2\theta}\,e^{i\theta}\,d\theta.$$

Recall that $\sin 2\theta \ge 2\theta\dfrac{2}{\pi}$ for $0 \le 2\theta \le \pi/2$. Therefore

$$\left|\int_{B_R} f(z)\,dz\right| \le R\int_0^{\pi/4} e^{-R^2\sin 2\theta}\,d\theta \le R\int_0^{\pi/4} e^{-\frac{4R^2\theta}{\pi}}\,d\theta = R\left(-\frac{\pi}{4R^2}e^{-\frac{4R^2\theta}{\pi}}\Big|_{\theta=0}^{\theta=\pi/4}\right).$$

Hence

$$\left|\int_{B_R} f(z)\,dz\right| \le \frac{\pi}{4R}\left(1 - e^{-R^2}\right).$$

Since $f(z)$ is analytic inside the path γ_R, by Cauchy theorem we have

$$\int_{\gamma_R} f(z)\, dz = 0,$$

or equivalently

$$\int_{A_R} f(z)\, dz + \int_{B_R} f(z)\, dz + \int_{C_R} f(z)\, dz = 0.$$

Taking the limits of both sides as $R \to \infty$, we first observe that

$$\lim_{R\to\infty} \int_{B_R} f(z)\, dz = 0,$$

and

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = -\alpha \int_0^\infty e^{-x^2}\, dx = -\alpha \frac{\sqrt{\pi}}{2} = -\left(\frac{\sqrt{\pi}}{2\sqrt{2}} + i\frac{\sqrt{\pi}}{2\sqrt{2}}\right).$$

Moreover

$$\lim_{R\to\infty} \int_{A_R} f(z)\, dz = \int_0^\infty \cos x^2\, dx + i \int_0^\infty \sin x^2\, dx.$$

Finally, putting these together we find

$$\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \frac{\sqrt{\pi}}{2\sqrt{2}} \approx 0.626....$$

Here is a graph of $y = \cos x^2$ and $y = \sin x^2$.