Due Date: 4 January 2018, Thursday

NAME:..

Due Time: 17:00

STUDENT NO:..

Math 503 Complex Analysis - Final Exam – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

General Rules for Take-Home Assignments

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you **write your answers alone**.

(2) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(3) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you **exhibit your total understanding of the ideas involved**.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

Please sign here:

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Q-1) Let $\eta(z) = \dfrac{\zeta'(z)}{\zeta(z)}$ for $\operatorname{Re} z > 1$, where ζ is the Riemann zeta function. Show that $\lim\limits_{z\to z_0}(z - z_0)\eta(z)$ is always an integer for $\operatorname{Re} z_0 \geq 1$. What is this integer? Make sure you cover all the cases of $\operatorname{Re} z_0 \geq 1$.

Solution:

Note that $\eta(z)$ is defined ana analytic for all z with $\operatorname{Re} z \geq 1$ with the exception of $z = 1$ where it has a pole of order 1 with residue 1. Also note that $\eta(z)$ has no zero in this domain. Thus with the exception of $z_0 = 1$, $\lim_{z\to z_0} \eta(z)$ exists for all z_0 wirh $\operatorname{Re} z_0 \geq 1$ Thus

$$\lim_{z\to z_0}(z - z_0)\eta(z) = 0,\ \operatorname{Re} z_0 \geq 1, z_0 \neq 1.$$

At $z = 1$, $\zeta(z)$ has the Laurent expansion

$$\zeta(z) = \frac{1}{z-1} + h(z),$$

where $h(z)$ is analytic in some open ball around $z = 1$. Then

$$\eta(z) = \frac{-\frac{1}{(z-1)^2} + h'(z)}{\frac{1}{z-1} + h(z)} = \frac{-1}{z-1}\,\frac{1-(z-1)^2 h2(z)}{1+(z-1)h(z)}.$$

Therefore

$$\lim_{z\to 1}(z-1)\eta(z) = -1.$$

Conclusion:

$$\text{When } \operatorname{Re} z_0 \geq 1, \quad \lim_{z\to z_0}(z - z_0)\eta(z) = \begin{cases} 0 & z_0 \neq 1, \\ -1 & z_0 = 1. \end{cases}$$

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Q-2) Let ϕ be an analytic function on $\operatorname{Re} z > 0$, and satisfy the conditions

(a) $\phi(1) = 2017$,

(b) $\phi(z+1) = z\phi(z)$,

(c) $\displaystyle\lim_{n\to\infty} \frac{\phi(z+n)}{n^z\phi(n)} = 2018$.

Find $\displaystyle\lim_{z\to\pi} \frac{\phi(z)}{\Gamma(z)}$.

Solution:

First note that (b) implies that $\phi(z+n) = z(z+1)(z+2)\cdots(z+n-1)\phi(z)$. Together with (a) this gives $\phi(n) = 2017(n-1)!$.

Next we decipher the left hand side of (c).

$$\begin{aligned}
\frac{\phi(z+n)}{n^z\phi(n)} &= \frac{z(z+1)\cdots(z+n-1)\phi(z)}{n^z 2017(n-1)!} \cdot \frac{z+n}{z+n} \cdot \frac{n}{n} \\
&= \frac{z(z+1)\cdots(z+n)}{n^z n!} \cdot \frac{n}{z+n} \cdot \frac{\phi(z)}{2017}. \\
&= \left(\Gamma(z)\,\frac{z(z+1)\cdots(z+n)}{n^z n!}\right) \cdot \frac{n}{z+n} \cdot \left(\frac{\phi(z)}{\Gamma(z)}\right) \frac{1}{2017}.
\end{aligned}$$

Taking the limit of both sides as n goes to infinity and using Gauss's formula for the Γ function we get

$$2018 = \left(\lim_{n\to\infty} \frac{\phi(z)}{\Gamma(z)}\right) \frac{1}{2017} = \frac{\phi(z)}{\Gamma(z)}\,\frac{1}{2017}.$$

Thus we find that

$$\phi(z) = 2017 \times 2018\,\Gamma(z).$$

Since $\Gamma(z)$ is continuous at $z = \pi$, we get

$$\lim_{z\to\pi} \frac{\phi(z)}{\Gamma(z)} = 2017 \times 2018 = 4070306.$$

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Q-3 Show that

$$\sinh \pi z = \pi z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2}\right).$$

Solution:

This follows from a simple calculation using the definition and the factorization of the sine function.

$$\begin{aligned}
\sinh z &= \frac{e^z - e^{-z}}{2} = \frac{e^{-i(iz)} - e^{i(iz)}}{2} \\
&= -i\,\frac{e^{i(iz)} - e^{-i(iz)}}{2i} = -i\,\sin iz \\
&= (-i)(iz) \prod_{n=1}^{\infty} \left(1 - \frac{(iz)^2}{\pi^2 n^2}\right) \\
&= z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{\pi^2 n^2}\right).
\end{aligned}$$

Now it follows that

$$\sinh \pi z = \pi z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2}\right).$$

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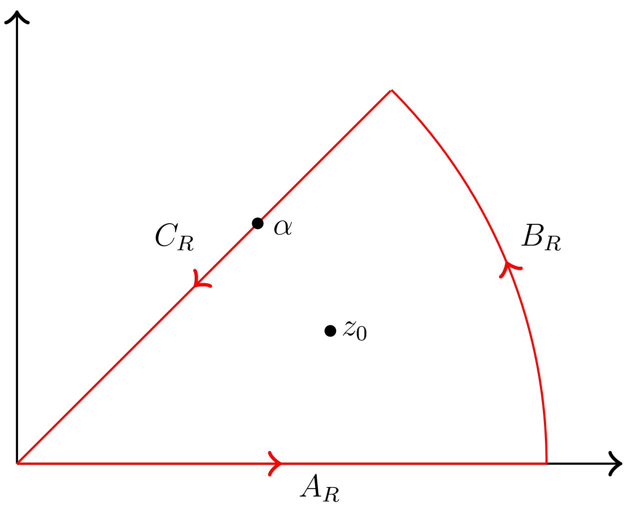
Q-4) For any positive integer n, calculate

$$I_n = \int_0^\infty \frac{dx}{1+x^{2n+1}}.$$

Solution:



$$\alpha = \exp(\tfrac{2\pi i}{2n+1})$$

$$z_0 = \exp(\tfrac{\pi i}{2n+1})$$

$$\gamma_R = A_R + B_R + C_R$$

Let $f(z) = \dfrac{1}{1+z^{2n+1}}$. Then

$$\operatorname{Res}(f(z), z_0) = \frac{1}{(2n+1)z_0^{2n}} = -\frac{z_0}{2n+1},$$

since the pole is simple. Now check that

$$\lim_{R\to\infty}\int_{A_R} f(z)\,dz = I_n, \quad \lim_{R\to\infty}\int_{B_R} f(z)\,dz = 0, \quad \lim_{R\to\infty}\int_{C_R} f(z)\,dz = -\alpha\, I_n.$$

The residue theorem now gives us

$$(1-\alpha)\, I_n = 2\pi\, i \left(\frac{-z_0}{2n+1}\right).$$

Finally simplifying this expression to your heart's content, you find

$$I_n = \frac{\pi}{2n+1}\operatorname{cosec}\frac{\pi}{2n+1}, \quad n = 1, 2, \ldots.$$