Due Date: 2 November 2017, Thursday
Instructor: Ali Sinan Sertöz

NAME:...

STUDENT NO:.......................................

Math 503 Complex Analysis - Homework 2 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

General Rules for Take-Home Assignments

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you **write your answers alone**.

(2) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(3) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you **exhibit your total understanding of the ideas involved**.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

Please sign here:

NAME: STUDENT NO: DEPARTMENT:

Q-1) Let $f(z) = u(x, y) + iv(x, y)$ be a complex differentiable function with continuous first partials in a simply connected open set G. Use Green's theorem to show that

$$\int_\gamma f(z)\, dz = 0$$

for every smooth loop γ in G.

Solution:

Let γ be a smooth loop and let D be the inside of γ. When $M(x, y)$ and $N(x, y)$ have continuous first partials on an open region containing γ and D, then Green's theorem states that

$$\int_\gamma M\, dx + N\, dy = \int_D (N_y - M_y)\, dx\, dy.$$

Note here that if $\omega = M\, dx + N\, dy$, then $d\omega = (N_y - M_y)\, dx$. Moreover observe that $\partial D = \gamma$, where ∂ denotes the boundary. Thus this is a special case of the Stokes' theorem which says that

$$\int_{\partial D} \omega = \int_D d\omega,$$

when certain smoothness conditions hold.

In our case we have

$$\omega = f(z)\, dz = (u + iv)(dx + idy) = (u\, dx - v\, dy) + i(v\, dx + u\, dy),$$

and

$$d\omega = -(v_x + u_y)\, dxdy + i(u_x - v_y)\, dxdy.$$

But $d\omega = 0$ because of the Cauchy-Riemann equations which follow from the assumption that f is analytic. Now the green's theorem gives the required vanishing.

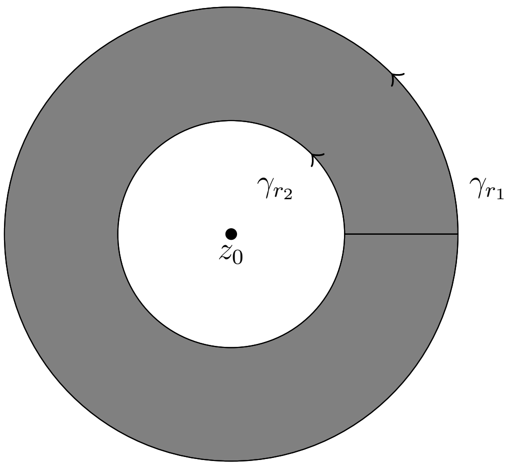
NAME: STUDENT NO: DEPARTMENT:

Q-2) Let $f(z) = u(x, y) + iv(x, y)$ be a complex differentiable function with continuous first partials in a simply connected open set G. Let $z_0 \in G$ be a fixed point. Show that the integral

$$\int_{|z-z_0|=r} \frac{f(z)}{z - z_0}\, dz$$

is independent of $r > 0$ provided that the closed ball of radius r centered at z_0 totally lies in G.

Solution:



In the above figure, the function $g(x) = \dfrac{f(z)}{z - z_0}$ is analytic on the shaded region which lies between the curves γ_{r_1} and γ_{r_2}. By Cauchy theorem the integral of g along the boundary of this region is zero. But the boundary is $\gamma = \gamma_{r_1} - \gamma_{r_2}$. This gives

$$\int_{\gamma_{r_1}} g = \int_{\gamma_{r_2}} g,$$

showing that the value of the integral does not depend on the radius.

NAME: STUDENT NO: DEPARTMENT:

Q-3 In the previous question you showed that the value of the given integral is independent of r. Find that value.

Solution:

Let $r > 0$ be such that the closed disk D_r with radius r and center z_0 totally lies in G. Define a new function on G as

$$h(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & z \neq z_0, \\ f'(z_0) & z = z_0. \end{cases}$$

Then g is continuous on G since f is analytic on G. Let $M > 0$ be such that $|g(z)| \leq M$ for all $z \in D_r$. We then have

$$\lim_{r\to 0} \left| \int_{|z-z_0|=r} g(z)\, dz \right| \leq \lim_{r\to 0} \int_{|z-z_0|=r} |g(z)|\, dz \leq \lim_{r\to 0} 2\pi r M = 0.$$

Since the integral $\displaystyle\int_{|z-z_0|=r} \frac{f(z)}{z-z_0}\, dz$ is independent of r, its value does not change if we take the limit as r goes to zero. We then have

$$\begin{aligned}
\int_{|z-z_0|=r} \frac{f(z)}{z-z_0}\, dz &= \lim_{r\to 0} \int_{|z-z_0|=r} \frac{f(z)}{z-z_0}\, dz \\
&= \lim_{r\to 0} \int_{|z-z_0|=r} \frac{f(z)}{z-z_0}\, dz - f(z_0) + f(z_0) \\
&= \lim_{r\to 0} \int_{|z-z_0|=r} \frac{f(z)-f(z_0)}{z-z_0}\, dz + \lim_{r\to 0} \int_{|z-z_0|=r} \frac{f(z_0)}{z-z_0}\, dz \\
&= 0 + \lim_{r\to 0} 2\pi i f(z_0) \\
&= 2\pi i f(z_0).
\end{aligned}$$

NAME: STUDENT NO: DEPARTMENT:

Q-4) Redo question 1, but this time do not assume the continuity of the first partials. This is Goursat's theorem. For the purposes of this homework, instead of giving all the details of the proof, it suffices to summarize the main ideas used in the proof.

Solution:

For this question I will refer you to two articles. The first one is Goursat's origina article:
Goursat, E. Sur la définition générale des fonctions analytiques, d'après Cauchy. (French) [[On the general definition of analytic functions, after Cauchy]] Trans. Amer. Math. Soc. 1 (1900), no. 1, 14–16. MR1500519.

The second one simplifies some of the arguments of the above paper:
Moore, Eliakim Hastings . A simple proof of the fundamental Cauchy-Goursat theorem. Trans. Amer. Math. Soc. 1 (1900), no. 4, 499–506, MR1500551

But note the following typos:
P. 501, I.5. For $< \epsilon$, read $\leq \epsilon$
P. 504, II.3,7,11 up, for $<$, read $\leq$.

Both of these articles are reachable through Bilkent network.