Due Date: 12 December 2017, Tuesday NAME:..

STUDENT NO:..

Math 503 Complex Analysis - Midterm 2 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

General Rules for Take-Home Assignments

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you **write your answers alone**.

(2) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(3) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you **exhibit your total understanding of the ideas involved**.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

Please sign here:

NAME: STUDENT NO: DEPARTMENT:

Q-1) The original proof of the Riemann Mapping Theorem assumes that the proper, connected and simply connected open set U is bounded. Show that this causes no loss of generality.
(Of course you cannot use the Riemann Mapping Theorem here!)

Solution:

On page 162 of Conway, we showed that if b is a point in U^c, complement of U in $\mathbb{C}$, then $g(z) = \sqrt{z-b}$ is defined and is a one-to-one holomorphic map. Then we showed that the interior of the complement of $g(U)$ is non-empty, (see equation 4.8 on that page).

So we might as well assume from the start that the complement of U has non-empty interior. Let z_0 be a point in the interior of U^c. Then $f(z) = \dfrac{1}{z - z_0}$ is a Mobius transformation so U is biholomorphic to $f(U)$. It is clear that $f(U)$ is bounded even if U is not.

So we can assume U is bounded.

NAME: STUDENT NO: DEPARTMENT:

Q-2) On the internet find the original proof Riemann gave for his mapping theorem and explain the steps of the proof in your own words.

Solution:

Let U be an open, bounded, connected and simply connected proper subset of $\mathbb{C}$ with ∂U smooth.

Fix a point $z_0 \in U$.

By Dirichlet principle there is a harmonic function $u(z)$ on U such that $u(z) = -\ln|z - z_0|$ for $z \in \partial U$.

Let $v(z)$ be a harmonic conjugate of $u(z)$ on U.

Then check that

$$f(z) = (z - z_0)e^{u(z)+iv(z)}$$

is a one-to-one holomorphic map of U onto the unit disc.

NAME: STUDENT NO: DEPARTMENT:

Q-3 Let G be a simply connected region which is not the whole plane and suppose that $\bar{z} \in G$ whenever $z \in G$. Let $a \in G \cap \mathbb{R}$ and suppose that $f\colon G \to D = \{z\colon |z| < 1\}$ is a one-to-one analytic function with $f(a) = 0$, $f'(a) > 0$ and $f(G) = D$. Let $G_+ = \{z \in G\colon \operatorname{Im} z > 0\}$. Show that $f(G_+)$ must lie entirely above or entirely below the real axis.
(There are solutions of this on the Internet. Again use your own wording in your solution in a way to show your understanding.)

Solution:

We first prove that $f(\bar{z}) = \overline{f(z)}$.
Proof of claim: Define $g(z) = \overline{f(\bar{z})}$. If $f(z) = u(x, y) + iv(x, y)$ and $g(z) = U(x, y) + iV(x, y)$, then $U(x, y) = u(x, -y)$ and $V(x, y) = -v(x, -y)$. Checking Cauchy-Riemann conditions we see that g is holomorphic. By definition g is one-to-one and onto D. Check that $g(a) = 0$ and $g'(a) = u_x(a, 0) > 0$, so by the uniquesness claim of Riemann mapping theorem we must have $g(z) = f(z)$. Now it follows that $f(\bar{z}) = \overline{f(z)}$. This proves the claim.

Since G_+ is open, $f(G_+)$ is also open. If $f(G_+)$ does not lie entirely above or below the real line then there exists $w \in f(G_+)$ such that $\bar{w}$ is also in $f(G_+)$, and $w \neq \bar{w}$.

Let α and β in G_+ be such that $f(\alpha) = w$ and $f(\beta) = \bar{w}$. Putting these together we obtain

$$f(\beta) = \bar{w} = \overline{f(\alpha)} = f(\bar{\alpha}).$$

Since f is one-to-one, we must have $\bar{\alpha} = \beta$, but then both of α and β cannot be in G_+. This contradiction shows that $f(G_+)$ must lie totally above or below the real line.

NAME: STUDENT NO: DEPARTMENT:

Q-4) Let $I_n = \int_0^\infty \frac{\log x}{(1+x^2)^n}\,dx$, $n = 2, 3, \ldots$.

Find a formula in terms of residues for I_n and using a software to calculate these residues, write the values of I_n for $n = 2, \ldots, 10$. *(Check privately using the same software that the values of I_n match your residue calculations.)*

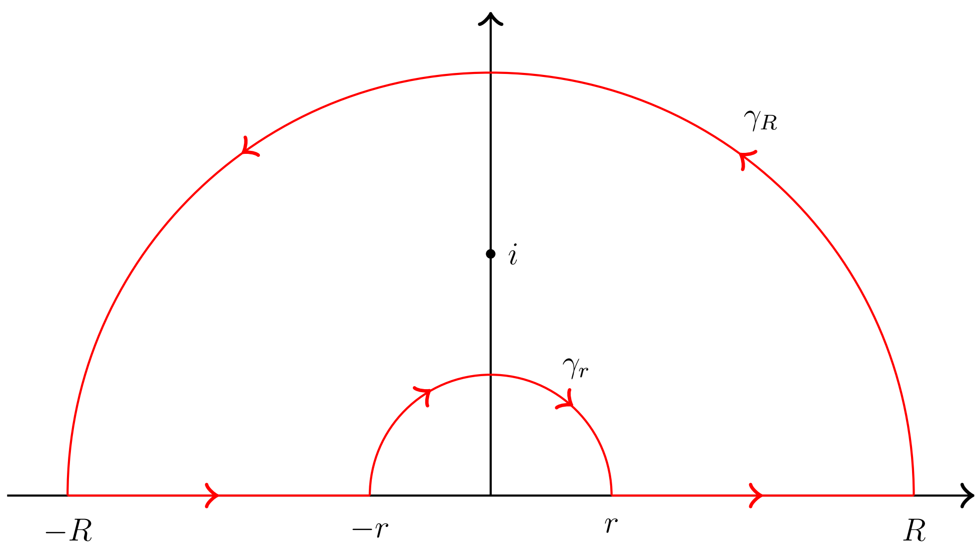
Solution:

First notice that by a change of variables we observe that

$$\int_0^1 \frac{\log x}{(1+x^2)}\,dx = -\int_1^\infty \frac{\log x}{(1+x^2)}\,dx,$$

so $I_1 = 0$

Next let $f(z) = \frac{\log z}{(1+z^2)^n}$ and integrate f around the following path.



Letting $\rho = r$ or R, we see that

$$\left|\int_{\gamma_\rho} f(z)\,dz\right| \le \frac{\pi\rho^2(\log\rho + \pi)}{|1-\rho^2|^n}.$$

The expression on the right goes to zero when $\rho \to 0$ for $n \ge 1$, and it still goes to zero when $\rho \to \infty$ when $n > 1$.

On $[-R, -r]$ we have

$$\int_{-R}^{-r} f(z)\,dz = \int_r^R \frac{\log x}{(1+x^2)^n}\,dx + i\pi\int_r^R \frac{dx}{(1+x^2)^n} = \int_r^R \frac{\log x}{(1+x^2)^n}\,dx + i\pi\,\frac{\pi}{2^{2n-1}}\,\frac{(2n-2)!}{[(n-1)!]^2},$$

where the last equality is derived in class.

Letting $\gamma_{r,R}$ be the above path with $0 < r < 1 < R$, we have

$$\int_{\gamma_{r,R}} f(z) = 2\pi i \operatorname{Res}(f, i).$$

Putting these together we have

$$I_n = \pi i \operatorname{Res}(f, i) - i\frac{\pi^2}{2^{2n}} \frac{(2n-2)!}{[(n-1)!]^2}.$$

To calculate the residue let $g(z, n) = \dfrac{\log z}{(z+i)^n}$. Then

$$\operatorname{Res}(f, i) = \frac{1}{(n-1)!} \left(\frac{\partial^{n-1}}{\partial z^{n-1}} g(z, n) \bigg|_{z=i} \right).$$

Finally we have

$$I_n = i\frac{\pi}{(n-1)!)} \left(\frac{\partial^{n-1}}{\partial z^{n-1}} g(z, n) \bigg|_{z=i} \right) - i\frac{\pi^2}{2^{2n}} \frac{(2n-2)!}{[(n-1)!]^2}, \text{ for } n > 1.$$

The right hand side does give a real number and in fact a negative real number!

$$I_2 = -\frac{1}{4}\pi \qquad I_3 = -\frac{1}{4}\pi \qquad I_4 = -\frac{23}{96}\pi \qquad I_5 = -\frac{11}{48}\pi \qquad I_6 = -\frac{563}{2560}\pi$$

$$I_7 = -\frac{1627}{768}\pi \quad I_8 = -\frac{88069}{430080}\pi \quad I_9 = -\frac{1423}{7168}\pi \quad I_{10} = -\frac{1593269}{8257536}\pi \quad I_{11} = -\frac{7759469}{41287680}\pi$$

Recall that $I_1 = 0$.