Bilkent University

Homework # 01
Math 503 Complex Analysis I
Due: 11 October 2020
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) On page 43 of Conway's book we have the theorem:

> **2.30 Theorem.** *Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u\colon G \to \mathbb{R}$ is a harmonic function then u has a harmonic conjugate.*

After the proof the section ends with the following paragraph:

> Where was the fact that G is a disk or $\mathbb{C}$ used? Why can't this method of proof be doctored sufficiently that it holds for general regions G? Where does the proof break down when $G = \mathbb{C} - \{0\}$ and $u(z) = \log |z|$?

Discuss your answers to these questions.

Answer:

At the last line of the proof we define $v(x, y)$ as the sum of two integrals. Let (x_0, y_0) be a point in the region G. Then according to the definition of v we have

$$v(x_0, y_0) = \int_0^{y_0} u_x(x, t)\, dt - \int_0^{x_0} u_y(s, 0)\, ds.$$

In particular for all $(x, y) \in \mathbb{C}$ with $0 \le x \le x_0$ and $0 \le y \le y_0$, we must have

$$v(x, y) = \int_0^{y} u_x(x, t)\, dt - \int_0^{x} u_y(s, 0)\, ds.$$

For the integrals of this last line to make sense, the whole rectangle with corners $(0, 0)$, $(x_0, 0)$, (x_0, y_0) and $(0, y_0)$ must be in the domain G of u. This generally requires that G is a convex set.

If u is defined on a general region G which is not necessarily convex, then around each point $p \in G$ we can consider a small ball $B_p \subset G$, and define a harmonic conjugate $v[p]$ of u on B_p. If on all possible non-empty intersections $B_p \cap B_q$ the functions $v[p]$ and $v[q]$ agree, then we have a harmonic conjugate for u on G.

However if G is not simply connected, then this process may fail for some functions. In particular for $G = \mathbb{C} - \{0\}$ and $u(z) = \log |z|$, this is precisel what happens. If you choose the points p on a circle around the origin, and move p on this circle counterclockwise, you will see that when you return to your original starting point the new v you find does not agree with the v you found at the start. This is because the complex log function picks up an extra 2π imaginary part when you turn around the origin once.