Bilkent University

Homework # 03
Math 503 Complex Analysis I
Due: 3 December 2020 Thursday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) Prove the folowing identities where $a \in \mathbb{C}$ but is not an integer.

(a) $\dfrac{\pi^2}{\sin^2 \pi a} = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}$

(b) $\pi^2 = 8 \displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$

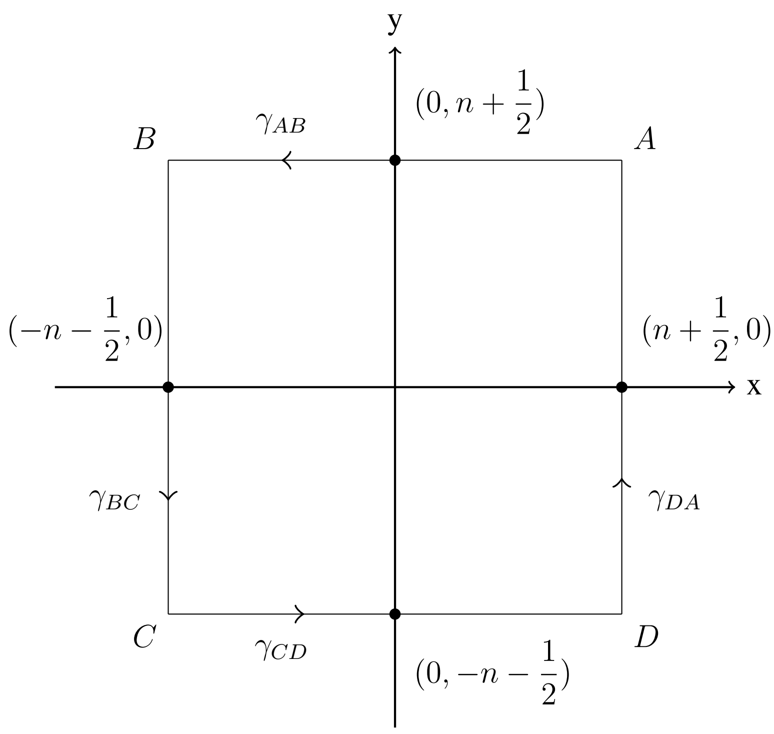
(c) $\pi \cot \pi a = \dfrac{1}{a} + \displaystyle\sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}$

(d) $\dfrac{\pi}{\sin \pi a} = \dfrac{1}{a} + \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2a}{a^2 - n^2}$

Remarks: The result of (c) is crucially used in the factorization of the sine function. All these identities are proved in a very similar manner so they can all be considered as the manifestation of a single idea. All the information needed to attack these identities are explained in detail on page 122 of Conway's book.

Solutions start on next page.

Answer:



$$\gamma_n = \gamma_{AB} + \gamma_{BC} + \gamma_{CD} + \gamma_{DA}$$

We consider the rectangle with the corners at

$$A = (n + \frac{1}{n}, n), B = (-n - \frac{1}{n}, n), C = (-n - \frac{1}{n}, -n), D = (n + \frac{1}{n}, -n),$$

where n is a positive integer. Our path γ_n is traced counterclockwise.

Our first task is to find upper bounds for the moduli of $\sin \pi z$ and $\cos \pi z$ when $z = x + iy \in \gamma_n$. We use the usual identities

$$|\cos(x + iy)\pi|^2 = \cos^2 \pi x + \sinh^2 \pi y, |\sin(x + iy)\pi|^2 = \sin^2 \pi x + \sinh^2 \pi y.$$

On γ_{DA} we have $z = (n + 1/2) + iy$, for $-n \leq y \leq n$. Then

$$|\cos[(n + 1/2)\pi + iy\pi]|^2 = \cos^2(n + 1/2)\pi + \sinh^2 \pi y = \sinh^2 \pi y,$$
$$|\sin[(n + 1/2)\pi + iy\pi]|^2 = \sin^2(n + 1/2)\pi + \sinh^2 \pi y = 1 + \sinh^2 \pi y.$$

Hence for $z \in \gamma_{DA}$ we have

$$|\cot \pi z|^2 = \frac{\sinh^2 \pi y}{1 + \sinh^2 \pi y} \leq 1,$$

and

$$|\csc \pi z|^2 = \frac{1}{1 + \sinh^2 \pi y} \leq 1.$$

On γ_{AB} we have $z = x + in$, for $-n - 1/2 \leq x \leq n + 1/2$. Then for $z \in \gamma_{AB}$ we have

$$|\cot \pi z|^2 = \frac{\cos^2 \pi x + \sinh^2 \pi n}{\sin^2 \pi x + \sinh^2 \pi n} \leq \frac{1 + \sinh^2 \pi n}{\sinh^2 \pi n} \leq 2,$$
$$|\csc \pi z|^2 = \frac{1}{\sin^2 \pi x + \sinh^2 \pi n} \leq \frac{1}{\sinh^2 \pi n} \leq 1.$$

When $z \in \gamma_{BC}$, then $-z \in \gamma_{DA}$, and when $z \in \gamma_{CD}$, then $-z \in \gamma_{AB}$. Hence the upper bounds for $|\cot \pi z|$ and $|\csc \pi z|$ on these parts of the boundary are the same.

Hence for $z \in \gamma_n$ we have

$$|\cot \pi z| \leq 2 \quad \text{and} \quad |\csc \pi z| \leq 1.$$

(a) Let a be a complex number which is not an integer. Let

$$f_a(z) = \frac{\pi \cot \pi z}{(z+a)^2}.$$

For any integer $n > |a|$, let

$$I_{n,a} = \int_{\gamma_n} f_a(z)\, dz.$$

By residue theorem we know that $I_{n,a}$ is equal to $2\pi i$ times the sum of the residues of $f_a(z)$ inside the contour γ_n. The poles of $f_a(z)$ inside thsi contour are $z = a$ and $z = k$, for $k = -n, \ldots, n$. We calculate the residues to be

$$\text{Res}(f_a(z), z = -a) = -\frac{\pi^2}{\sin^2 \pi a}, \quad \text{and} \quad \text{Res}(f_a(z), z = k) = \frac{1}{(k+a)^2}.$$

Hence we have

$$I_{n,a} = 2\pi i \left(-\frac{\pi^2}{\sin^2 \pi a} + \sum_{k=-n}^{n} \frac{1}{(k+a)^2} \right).$$

Now we take the limit of both sides as n goes to infinity. For this first we examine $|I_{n,a}|$.

For this purpose observe that when $z \in \gamma_n$, we have $|z| > |a|$ and $|z| \geq |n|$. Hence

$$|(z+a)^2| \geq (|z| - |a|)^2 \geq |z|^2 \geq n^2,$$

and it then follows that

$$\left| \frac{1}{(z+a)^2} \right| \leq \frac{1}{n^2}.$$

We can know see that

$$|I_{n,a}| \leq \frac{2\pi(8n+2)}{n^2},$$

where $8n+2$ is the length of the contour γ_n. It then follows that

$$\lim_{n \to \infty} I_{n,a} = 0.$$

This gives

$$\lim_{n \to \infty} \left(-\frac{\pi^2}{\sin^2 \pi a} + \sum_{k=-n}^{n} \frac{1}{(k+a)^2} \right) = 0,$$

which is equivalent to what we wanted to establish

$$\frac{\pi^2}{\sin^2 \pi a} = \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}.$$

(b) In the previous result we choose $a = 1/2$. Then we have

$$\pi^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(n+1/2)^2} = \sum_{-\infty}^{\infty} \frac{4}{(2n+1)^2} = 8 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},$$

as claimed.

(c) We again use the contours γ_n for $n > |a|$, but this time we set

$$f_a(z) = \frac{\pi \cot \pi z}{z^2 - a^2}.$$

For $z \in \gamma_n$ we again have

$$\left| \frac{1}{z^2 - a^2} \right| \leq \frac{1}{n^2}, \quad \text{and} \quad |\cot \pi z| \leq 2.$$

The poles of $f_a(z)$ inside the contour γ_n are $z = \pm a$ and $z = k$, for $k = -n, \ldots, n$. Then the residues are

$$\operatorname{Res}(f_a(z), z = \pm a) = \frac{\pi \cot \pi a}{2a}, \quad \operatorname{Res}(f_a(z), z = k) = \frac{1}{k^2 - a^2}.$$

We argue as in (a) above and find that the sum of the residues as n goes to infinity is zero. This gives

$$\frac{\pi \cot \pi a}{a} = \sum_{k=-\infty}^{\infty} \frac{1}{a^2 - k^2}.$$

Multiplying both sides by a, taking out the $k = 0$ case and observing that k and $-k$ give the same summand we get

$$\pi \cot \pi a = \frac{1}{a} + \sum_{k=1}^{\infty} \frac{2a}{a^2 - k^2},$$

as claimed.

(d) We again use γ_n for $n > |a|$, but this time we set

$$f_a(z) = \frac{\pi \csc \pi z}{z^2 - a^2}.$$

Repeating the above arguments we see that

$$\operatorname{Res}(f_a(z), z = \pm a) = \frac{\pi}{2a} \frac{1}{\sin \pi a}, \quad \operatorname{Res}(f_a(z), z = k) = \frac{(-1)^k}{k^2 - a^2}.$$

Since we showed that

$$|\csc \pi z| \leq 1 \quad \text{for} \quad z \in \gamma_n,$$

we have, as above,

$$\lim_{n \to \infty} \left(\frac{\pi}{a \sin \pi a} + \sum_{k=-n}^{n} \frac{(-1)^k}{k^2 - a^2} \right) = 0.$$

Rearranging this we get

$$\frac{\pi}{\sin \pi a} = \frac{1}{a} + \sum_{k=1}^{\infty} (-1)^k \frac{2a}{a^2 - k^2},$$

as claimed.