Bilkent University

Homework # 04
Math 503 Complex Analysis I
Due: 11 December 2020 Friday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) Find an infinite product factorization for

$$\cos\left(\frac{\pi z}{4}\right) - \sin\left(\frac{\pi z}{4}\right).$$

Q-2) Show that

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}.$$

Answer-1: Let $\phi(z) = \cos\left(\frac{\pi z}{4}\right) - \sin\left(\frac{\pi z}{4}\right)$. The zeros of this function are $1 - 4k$ where $k \in \mathbb{Z}$. Using the Weierstrass factorization theorem it is easy to write an infinite product which has exactly the same zeros. Then the tricky part is to determine the exact ratio of these two functions, which can be done by some ingenious calculations.

Another way to approach this factorization is to rewrite $\phi(z)$ in terms of sine function and use the factorization of sine.

$$
\begin{aligned}
\phi(z) &= \cos\left(\frac{\pi z}{4}\right) - \sin\left(\frac{\pi z}{4}\right) \\
&= \sin\left(\frac{\pi}{2} - \frac{\pi z}{4}\right) - \sin\left(\frac{\pi z}{4}\right) \\
&= 2\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4} - \frac{\pi z}{4}\right)
\end{aligned}
$$

where we used

$$
\begin{aligned}
&\sin(A+B) = \sin A\cos B + \cos A\sin B \\
&\sin(A-B) = \sin A\cos B - \cos A\sin B \\
&\sin(A+B) - \sin(A-B) = 2\cos A\sin B
\end{aligned}
$$

where $A = \frac{\pi}{4}$ and $B = \frac{\pi}{4} - \frac{\pi z}{4}$.

Continuing from where we left off, we have

$$
\begin{aligned}
\phi(z) &= \sqrt{2}\sin\pi\left(\frac{1-z}{4}\right) = \frac{\sin\pi\left(\frac{1-z}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} \\
&= \frac{\left(\frac{1-z}{4}\right)\pi\prod_{n=1}^{\infty}\left(1 - \frac{\left(\frac{1-z}{4}\right)^2}{n^2}\right)}{\frac{\pi}{4}\prod_{n=1}^{\infty}\left(1 - \frac{\left(\frac{1}{4}\right)^2}{n^2}\right)} \\
&= (1-z)\prod_{n=1}^{\infty}\frac{1 - \left(\frac{1-z}{4n}\right)^2}{1 - \left(\frac{1}{4n}\right)^2} \\
&= (1-z)\prod_{n=1}^{\infty}\frac{(4n-1+z)(4n+1-z)}{(4n-1)(4n+1)} \\
&= (1-z)\prod_{n=1}^{\infty}\left(1 + \frac{z}{4n-1}\right)\left(1 - \frac{z}{4n+1}\right) \\
&= (1-z)\left(1 - \frac{z}{5}\right)\left(1 - \frac{z}{9}\right)\cdots\left(1 + \frac{z}{3}\right)\left(1 + \frac{z}{7}\right)\cdots \\
&= \prod_{n=1}^{\infty}\left(1 + (-1)^n\frac{z}{2n-1}\right).
\end{aligned}
$$

Answer-2: We recall that

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Here we put $z = 1/2$ to obtain

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right).$$

We note that

$$\left(1 - \frac{1}{4n^2}\right) = \left(1 - \frac{1}{2n}\right)\left(1 + \frac{1}{2n}\right) = \frac{(2n-1)(2n+1)}{(2n)^2}.$$

This then gives the required equality

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}.$$