Bilkent University

Midterm # 01
Math 503 Complex Analysis I
Due: 18 November 2020
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Solution Key

Q-1) Let $f(z) = u(x, y) + iv(x, y)$ be a C^1-function on $\mathbb{C}$. Here as usual u and v are real valued C^1-function of the real variables x and y, and $z = x + iy$. Assume that f is conformal. Show that f is complex analytic.

Answer: We will show that f satisfies the Cauchy-Riemann equations.

Let z_0 be an arbitrary point in $\mathbb{C}$ and $z(t) = x(t) + iy(t)$ be a C^1-curve passing through z_0. Assume without loss of generality that $z(0) = z_0$. Also assume that $z(t)$ is smooth in the sense that $z'(t) \neq 0$ for any t in its domain.

$z'(0)$ is the tangent vector to $z(t)$ at $t = 0$.

Define the image of $z(t)$ under f as $w(t) = f(z(t))$. Since $z(t)$ and $f(x, y)$ are C^1-functions, $w'(t)$ exists.

To say that f is conformal at $z(0)$ means that the f rotates $z'(0)$ by a fixed angle regardless of what $z'(0)$ is. In other words f is conformal at z_0 if the difference $\arg w'(0) - \arg z'(0)$ is independent of $z'(0)$. We note here that $\arg w'(0) - \arg z'(0) = \arg \dfrac{w'(0)}{z'(0)}$. Now we want to explicitly write $\dfrac{w'(0)}{z'(0)}$.

Since we can write $w(t) = f(z(t)) = f(x(t), y(t)) = u(x(t), y(t)) + iv(x(t), y(t))$, using the chain rule for real variables we have

$$\begin{aligned} w'(t) &= f_x x' + f_y y' \\ &= f_x \frac{1}{2}(z' + \bar{z}') + f_y \frac{1}{2i}(z' - \bar{z}') \\ &= \frac{1}{2}(f_x - if_y)\ z' + \frac{1}{2}(f_x + if_y)\ \bar{z}'. \end{aligned}$$

It then follows that

$$\frac{w'}{z'} = \frac{1}{2}(f_x - if_y) + \frac{1}{2}(f_x + if_y)\ \frac{\bar{z}'}{z'}.$$

For all possible choices of curves $z(t)$ with $z(0) = z_0$, this expression describes points on a circle of radius $\dfrac{1}{2}(f_x + if_y)$ and center $\frac{1}{2}(f_x - if_y)$. The argument on this circle certainly changes and depends on $z(t)$ unless of course the the circle is a point, i.e. the radius is zero.

Hence if f is conformal we have

$$0 = f_x + if_y = u_x + iv_x + i(u_y + iv_y) = (u_x - v_y) + i(u_y + v_x),$$

which are precisely the Cauchy-Riemann equations. Hence f is analytic.

See: Ahlfors, Complex Analysis, Second Edition, McGraw-Hill, 1966, page 74.

Q-2) If $f(z)$ is analytic on a region G and is zero on a non-empty open subset U of G, then $f(z) \equiv 0$ on G. This is in stark contrast with what is possible in real analysis. To see this wide difference between these two worlds construct a real valued, non-negative C^∞-function $f(x)$ of the real variable x with the property that $f(x) = 1$ on the open interval $(-1, 1)$, and is zero outside the interval $(-2, 2)$.

Answer: First consider the function

$$g(x) = \begin{cases} e^{-1/x} & \text{when } x > 0, \\ 0 & \text{when } x \leq 0. \end{cases}$$

In Calculus courses we proved that this function is C^∞.

Our second auxiliary function is defined as

$$h(x) = \frac{g(x)}{g(x) + g(1-x)}.$$

Check that h is non-negative, C^∞, and $h(x) = 1$ when $x \geq 1$, and is zero when $x \leq 0$.

We finally define our required function as

$$f(x) = h(x+2)h(2-x).$$

Check that f satisfies our expectations.

See: Warner, Foundations of Differentiable Manifolds and Lie Groups, Springer-Verlag, 1983, page 10.

Q-3) Let $m \geq 1$ be an integer, and define

$$F(m) = \int_{|z|=1} \frac{\sin z}{z^m}\, dz,$$

where the integration is taken counterclockwise. Find an explicit formula for $F(m)$.

Answer:

For any analytic function $f(z)$ we have, by Cauchy Integral Formula (Corollary 2.13, page 73)

$$\int_{|z|=1} \frac{f(z)}{z^m}\, dz = \frac{2\pi i}{(m-1)!}\, f^{(m-1)}(0).$$

Define

$$\epsilon_m = \begin{cases} 0 & m \equiv 1 \mod 4, \\ 1 & m \equiv 2 \mod 4, \\ 0 & m \equiv 3 \mod 4, \\ -1 & m \equiv 0 \mod 4. \end{cases}$$

Notice that if $f(z) = \sin z$, then $f^{(m-1)}(0) = \epsilon_m$. Then we have

$$F(m) = \frac{2\pi i}{(m-1)!}\, \epsilon_m, \quad m \geq 1.$$

Q-4) Show that

$$\tan z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}\ z^{2n-1}, \quad |z| < \frac{\pi}{2},$$

where B_n are Bernoulli numbers with the convention that $B_0 = 1$ and $\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0$, for $n \geq 1$.

Answer:

There are several ways to derive this formula but they all revolve around the same idea, $e^x = \cos x + i \sin x$, where x is real. Of course the Bernoulli numbers also play a crucial role.

We will follow Euler for Bernoulli numbers.

We let the following equation define the constants B_n. Letting z be a complex parameter we write

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,\ |z| < 2\pi.$$

Note that the left hand side can be extended to $z = 0$, and this sets

$$B_0 = 1.$$

Also observe that the series converges for $|z| < 2\pi$ since the nearest pole to zero of the function $z/(e^z - 1)$ is $z = 2\pi i$.

Now we determine the coefficients B_n.

$$1 = \left(\frac{e^z - 1}{z}\right)\left(\frac{z}{e^z - 1}\right) = \left(\sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)!}\right)\left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_k \binom{n+1}{k}\right) \frac{z^n}{(n+1)!}.$$

Thus the coefficients B_n are determined by the following recursive relation.

$$B_0 = 1, \quad \sum_{k=0}^{n} B_k \binom{n+1}{k} = 0 \text{ for } n \geq 1.$$

These are precisely the defining conditions of Bernoulli numbers.

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, \ldots$$

Let

$$f(z) = \frac{z}{e^z - 1} + \frac{z}{2} = 1 + \sum_{n=2}^{\infty} \frac{b_n}{n!} z^n.$$

Direct calculation shows that $f(z) = f(-z)$, so f is an even function and hence the coefficients of odd powers of z are zero.

$$B_{2n+1} = 0 \text{ for } n \geq 1.$$

Next we can verify by straightforward simplification the following identities.

$$\begin{aligned} f(z) &= \frac{z}{2} \coth \frac{z}{2} \\ \tanh z &= 2 \coth 2z - \coth z \\ z \tanh z &= f(4z) - f(2z) \\ \tan z &= -i \tanh iz \end{aligned}$$

Hence we have

$$\begin{aligned}
\tan z &= -i\tanh iz \\
&= -i\left(\frac{1}{iz}[f(4iz)-f(2iz)]\right) \\
&= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}\; z^{2n-1}
\end{aligned}$$

where the series converges for $|z| < \pi/2$ since this is where $f(4iz)$ converges.