Due Date: 15 March 2013, Friday
NAME:.................................................................
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STUDENT NO:.................................................................

Math 504 Complex Analysis II – Take-Home Exam 02 – Solutions

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Please do not write anything inside the above boxes!

Check that there are 4 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail.

For each question I will post the best student solution on the web. If there are more than one interesting solutions, I will post them all. Having your solution posted on the web will get you extra 10 points for each solution posted. These will be added to your total exam grades before an average is taken at the end of the semester.
Q-1) Find the type and multiplier of each of the following transformations:

(i) \( \frac{z + 1}{z + 3} \),
(ii) \( \frac{iz + 1}{z + 3i} \),
(iii) \( iz + 1 \),
(iv) \( \frac{-z}{z + 4} \).

[page 53, Exercise 2B]

Solution:

For a general Möbius transformation \( T(z) = \frac{az + b}{cz + d} \), let

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} \frac{a}{\sqrt{\det A}} & \frac{b}{\sqrt{\det A}} \\ \frac{c}{\sqrt{\det A}} & \frac{d}{\sqrt{\det A}} \end{pmatrix}.
\]

(i) \( A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \), \( \det A = 2 \), \( \sqrt{\det A} = \sqrt{2} \), \( B = \begin{pmatrix} 1/\sqrt{2} & 1/(2i) \\ 1/(2i) & 3/2 \end{pmatrix} \), \( \text{tr} B = 2\sqrt{2} \).

Since \( \text{tr}^2(B) = 8 > 4 \), \( T \) is hyperbolic. Its multipliers are the roots of the equation

\[
z^2 + (2 - \text{tr}^2(B))z + 1 = z^2 - 6z + 1 = 0,
\]

hence the multipliers are

\( \{\lambda, \frac{1}{\lambda}\} = \{3 + 2\sqrt{2}, 3 - 2\sqrt{2}\} \).

(ii) \( A = \begin{pmatrix} i & 1 \\ 1 & 3i \end{pmatrix} \), \( \det A = -4 \), \( \sqrt{\det A} = 2i \), \( B = \begin{pmatrix} 1/2 & 1/(2i) \\ 1/(2i) & 3/2 \end{pmatrix} \), \( \text{tr} B = 2 \).

Since \( \text{tr}^2(B) = 4 \), \( T \) is parabolic. Its multipliers are the roots of the equation

\[
z^2 + (2 - \text{tr}^2(B))z + 1 = z^2 - 2z + 1 = 0,
\]

hence the multipliers are

\( \{\lambda, \frac{1}{\lambda}\} = \{1, 1\} \).

(iii) \( A = \begin{pmatrix} i & 1 \\ 0 & 1 \end{pmatrix} \), \( \det A = i \), \( \sqrt{\det A} = 1/\sqrt{2} + i/\sqrt{2} \), \( B = \begin{pmatrix} 1/\sqrt{2} + i/\sqrt{2} & 1/\sqrt{2} - i/\sqrt{2} \\ 0 & 1/\sqrt{2} - i/\sqrt{2} \end{pmatrix} \), \( \text{tr} B = \sqrt{2} \).

Since \( 0 \leq \text{tr}^2(B) = 2 < 4 \), \( T \) is elliptic. Its multipliers are the roots of the equation

\[
z^2 + (2 - \text{tr}^2(B))z + 1 = z^2 + 1 = 0,
\]

hence the multipliers are

\( \{\lambda, \frac{1}{\lambda}\} = \{i, -i\} \).

(iv) \( A = \begin{pmatrix} -1 & 0 \\ 1 & 4 \end{pmatrix} \), \( \det A = -4 \), \( \sqrt{\det A} = 2i \), \( B = \begin{pmatrix} -1/(2i) & 0 \\ 1/(2i) & 4/(2i) \end{pmatrix} \), \( \text{tr} B = 3/(2i) \).

Since \( \text{tr}^2(B) = -9/4 < 0 \), \( T \) is loxodromic. Its multipliers are the roots of the equation

\[
z^2 + (2 - \text{tr}^2(B))z + 1 = z^2 + \frac{17}{4}z + 1 = 0,
\]

hence the multipliers are

\( \{\lambda, \frac{1}{\lambda}\} = \{-4, -\frac{1}{4}\} \).
Q-2) Let $C$ be a circle in $\Sigma$ passing through $z_1, z_2, z_3$. Prove that $I_C(z) = w$ if and only if

$$(w, z_1; z_2, z_3) = (z, z_1; z_2, z_3).$$

[page 55, Exercise 2M]

Solution:

Let $p$ be the center and $r$ the radius of $C$. We then have the following sequence of equations.

$$(w, z_1; z_2, z_3) \overset{(1)}{=} (p + \frac{r^2}{\bar{z} - p}, z_1; z_2, z_3)$$

$$\overset{(2)}{=} \left( \frac{r^2}{\bar{z} - p}, z_1 - p; z_2 - p, z_3 - p \right)$$

$$\overset{(3)}{=} \left( \bar{z} - \bar{p}, \frac{r^2}{z_1 - p}; \frac{r^2}{z_2 - p}, \frac{r^2}{z_3 - p} \right)$$

$$\overset{(4)}{=} \left( \bar{z} - \bar{p}, z_1 - p; z_2 - p, z_3 - p \right)$$

$$\overset{(5)}{=} (\bar{z}, \bar{z}_1; \bar{z}_2, \bar{z}_3),$$

where the equalities are justified as follows:

1. This can be written if and only if $I_C(z) = w$ by definition of $I_C$.
2. Apply the Mobius transformation $T(z) = z - p$.
3. Apply the Mobius transformation $T(z) = r^2/z$.
4. Each point $z_i$ on the circle $C$ satisfies $|z_i - p|^2 = (\bar{z}_i - \bar{p})(z_i - p) = r^2$.
5. Apply the Mobius transformation $T(z) = z + p$. 


Q-3) Show that $PSL(2, \mathbb{C})$ is generated by parabolic elements.

[page 55, Exercise 2N]

Solution:

Choose any non-parabolic element $T \in PSL(2, \mathbb{C})$. Then there exist a $\lambda \in \mathbb{C} \setminus \{0, 1\}$ and $A \in PSL(2, \mathbb{C})$ such that

$$T = AU_\lambda A^{-1},$$

where

$$U_\lambda = \pm \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}.$$

Let $t = \sqrt{\lambda}$. If $T$ is non-parabolic, then $\lambda \neq 1$, so $t \neq \pm 1$.

Now let

$$P_1 = \begin{pmatrix} p & q \\ -q & 2-p \end{pmatrix}, \text{ where } p = \frac{2}{1+t}, \ q = \pm \frac{t-1}{t+1},$$

and define

$$P_2 = U_\lambda P_1.$$

Note that $P_1$ and $P_2$ are parabolic elements of $PSL(2, \mathbb{C})$ and we have $U_\lambda = P_2 P_1^{-1}$. It now follows that

$$T = AU_\lambda A^{-1} = AP_2 P_1^{-1} A^{-1} = (AP_2 A^{-1})(AP_1^{-1} A^{-1}).$$

Since $AP_1 A^{-1}$ and $AP_2 A^{-1}$ are clearly parabolic elements of $PSL(2, \mathbb{C})$, we showed that every element of $PSL(2, \mathbb{C})$ is a product of two parabolic elements. This shows that $PSL(2, \mathbb{C})$ is generated by parabolic elements, as claimed.
Q-4) Let $F_\theta$ be a rotation of $S^2$ by an angle $\theta$ around $x_1$-axis. This induces a Mobius transformation on $\Sigma$. Write this transformation explicitly. Can you generalize this by replacing $x_1$-axis by some other diagonal of $S^2$?

[Öğuz Gezmüş]

Solution:

Let $f_\theta$ be the Mobius transformation induced by $F_\theta$. Let $\pi : S^2 \to \mathbb{C}$ be the stereographic projection from North pole,

$$\pi(x_1, x_2, x_3) = \left( \frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right).$$

If $\theta = \pm \pi$, then $f_\theta(0) = \infty$ and $f_\theta(i) = -i$, forcing $f_\theta(z) = 1/z$.

Now let $-\pi < \theta < \pi$. We know that $f_\theta$ is of the form

$$f_\theta(z) = \frac{az + b}{-bz + a}, \quad |a|^2 + |b|^2 = 1.$$

From the above observation we know that for this case $a \neq 0$.

Note that

$$F_\theta(0, 0, -1) = (0, \sin \theta, -\cos \theta) \text{ and } F_\theta(0, -\sin \theta, -\cos \theta) = (0, 0, -1).$$

Since

$$\pi(0, 0, -1) = (0, 0), \pi(0, \sin \theta, -\cos \theta) = (0, \tan \frac{\theta}{2}) \text{ and } \pi(0, -\sin \theta, -\cos \theta) = (0, -\tan \frac{\theta}{2}),$$

we must have

$$f_\theta(0) = i \tan \frac{\theta}{2} \text{ and } f_\theta(-i \tan \frac{\theta}{2}) = 0.$$

Using these to solve for $a, b \in \mathbb{C}$, we find that

$$f_\theta(z) = \frac{(\cos \frac{\theta}{2})z + i \sin \frac{\theta}{2}}{(i \sin \frac{\theta}{2})z + \cos \frac{\theta}{2}}.$$