Due Date: 6 May 2013, Monday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 504 Complex Analysis II – Take-Home Exam 07 – Solutions

1	2	3	4	5	TOTAL
25	25	25	25	0	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail.

For each question I will post the best student solution on the web. If there are more than one interesting solutions, I will post them all. Having your solution posted on the web will get you extra 10 points for each solution posted. These will be added to your total exam grades before an average is taken at the end of the semester.

NAME: STUDENT NO:

Q-1) Let F be a Dirichlet region for a Fuchsian group Λ and let s be a side of F. If $T \in \Lambda$ and $T(s)$ is a side of F prove that

$$F \cap T(F) = T(s).$$

Moreover show that if there is $T' \in \Lambda \setminus \{I\}$ such that $T'(s)$ is also a side of F, then $T = T'$, hence no three sides of F can be congruent.

[page 269, Exercise 5R]

Solution:

It is clear that $T(s) \subset F \cap T(s)$. In a tessellation, two fundamental regions either touch each other at vertices or along a single edge. So the intersection here must be a whole edge and is $T(s)$.

For the second part see page 247.

NAME: STUDENT NO:

Q-2) On page 265, Corollary 5.11.4 says "*A Hurwitz group of smallest order is simple.*"
Explain what is meant by *smallest order*, and prove the corollary.

Solution:

If H is a Hurwitz group and if no homomorphic image of H with smaller order is Hurwitz, then H is a Hurwitz group of smallest order. In other words assume that for every group homomorphism $\phi : H \to G$ onto some group G with non-trivial kernel, we know that G is not Hurwitz. Then H is a Hurwitz group of smallest degree.

Suppose now H is a Hurwitz group of smallest degree and assume that it has a non-trivial normal subgroup N. Then there is a natural homomorphism $\phi : H \to H/N$, with non-trivial kernel N. By Theorem 5.11.3, the quotient group H/N is Hurwitz with smaller order than H. This violates the assumption on H. Hence H must be simple.

NAME: STUDENT NO:

Q-3) Let $p = g - 1 > 84$ be a prime number. Show that there is no compact Riemann surface of genus g admitting $84(g - 1)$ automorphisms.

[page 269, Exercise 5U]

Solution:

Suppose such a Riemann surface exists with an automorphism group H of $84p$ elements. Then H is a Hurwitz group by definition. Let N be a normal subgroup of H. Then by Theorem 5.11.3, the quotient group H/N must also be Hurwitz so its order is $84q$ for some integer $q > 1$ since by Theorem 5.11.5(i) there is no Hurwitz group of order 84. Since $84q|84p$, we must have $q = 1$ or $q = p$ since p is prime. Since $q > 1$, it must be p, in which case $N = \{id\}$, so H is simple.

By Sylow theorem H contains $pk + 1$ p-Sylow subgroups for some $k \geq 1$. But we must also have $(pk + 1)|84p$, so $(pk + 1)m = 84p$. Since p does not divide $pk + 1$, we must have $m = pn$ for some integer $n \geq 1$. But then we have $(pk+1)n = 84$ which is impossible with $k > 0$ since $p > 84$. Hence $k = 0$ and a p-Sylow subgroup is normal in H contradicting the above finding about H.

Hence no compact Riemann surface exists with $84p$ automorphisms.

NAME: STUDENT NO:

Q-4) Summarize the ideas involved in showing that there are infinitely many compact Riemann surfaces whose automorphism groups are Hurwitz groups.

Solution:

Either follow the ideas of Exercises 5W, 5X and 5Y or simply summarize
Macbeath, A. M., On a theorem of Hurwitz., Proc. Glasgow Math. Assoc. 5 1961 9096 (1961).

Here is a way of how we can follow the ideas of the exercises in the book, inspired by the solution of Burak.

Note that combining the proofs of Theorem 5.10.9 and Theorem 5.11.1, every Hurwitz group is a quotient of the a triangle group of signature $(0; 2, 3, 7)$ with a non-trivial normal subgroup.

Let Γ be a triangle group with signature $(0; 2, 3, 7)$. Let Λ be a normal subgroup of signature $(g; -)$. Existence of at least one such subgroup of signature $(3; -)$ is guaranteed by by Theorem 5.11.5. Now Γ/Λ is a Hurwitz group of order $84(g-1)$ being the automorphism group of the compact Riemann surface $\mathcal{U}/\Lambda$ of genus g.

Let $K_m = \Lambda^m[\Lambda, \Lambda]$, where m is a positive integer. By Exercise 5W(ii), K_m is characteristic in Λ, and by 5W(i) is normal in Γ. By Exercise 5X, Λ/K_m is a finite abelian group of order m^{2g}.

On the other hand Γ/K_m, being a quotient of the triangle group Γ by a normal subgroup, is a Hurwitz group. Therefore its order is $84(g'-1)$ for some integer $g' > 2$.

Now we have two ways of calculating the order of Γ/K_m.

$$84(g'-1) = |\Gamma/K_m| = |\Gamma/\Lambda| \cdot |\Lambda/K_m| = 84(g-1) \cdot m^{2g},$$

from where we solve for g' to find

$$g' = (g-1)m^{2g} + 1.$$

Since we know that we can choose Λ to be $g = 3$, and since $m > 0$, this shows that there are infinitely many Hurwitz groups.