Due Date: 13 May 2013, Monday | NAME:..

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Math 504 Complex Analysis II – Take-Home Exam 08 – Solutions

1	2	3	4	5	TOTAL
25	25	25	25	0	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail.

For each question I will post the best student solution on the web. If there are more than one interesting solutions, I will post them all. Having your solution posted on the web will get you extra 10 points for each solution posted. These will be added to your total exam grades before an average is taken at the end of the semester.

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Q-1) On page 278 it says "Calculations similar to above give ...". Prove one of these identities.

[page 315, Exercise 6A]

Solution:

This is straightforward calculation. However as you mentioned there is a typo in these formulas. The correct formulations should be as follows.

$$\begin{aligned} g_2(T(\tau)) &= (c\bar{\tau}+d)^4\,\overline{g_2(\tau)}, \\ g_3(T(\tau)) &= (c\bar{\tau}+d)^6\,\overline{g_3(\tau)}, \\ \Delta(T(\tau)) &= (c\bar{\tau}+d)^{12}\,\overline{\Delta(\tau)}, \\ J(T(\tau)) &= \overline{J(\tau)}. \end{aligned}$$

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Q-2) Prove the claim on page 289 that the Riemann surface of $w = \sqrt{p(z)}$, where p is a polynomial of degree 4 with distinct roots, is conformally equivalent to $\mathbb{C}/\Omega$ for some lattice Ω.

[page 317, Exercise 6Q]

Solution:

The hints given in the exercise solves the problem.

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Q-3) Find the genus of the Riemann surface $\bar{\mathcal{U}}/\Gamma(n)$.

[page 317, Exercise 6M]

Solution:

Being defined as the kernel of some group homomorphism, the group $\Gamma(n)$ is normal in Γ, see page 300. The level of $\Gamma(n)$ is n, as both definition of level coincide in this case, see pages 301 and 310.

Now use Theorem 6.10.9 on page 310 to conclude that the required genus is $g = 1 + \frac{N}{2}\left(\frac{1}{6} - \frac{1}{n}\right)$ where $N = |\Gamma : \Gamma(n)|$ and is given in exercise 6L on page 317.

We see that $g = 0$ if $n = 2$, and for $n > 2$ we have

$$g = 1 + \frac{n-6}{24} \prod_{p|n} \left[(p^2 - 1) \left(\frac{n}{p}\right)^2 \right],$$

where the product is over all distinct primes dividing n.

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Q-4) The proof of Theorem 6.9.3 on page 301 is wrong! Find the error and give a correct proof.

Solution:

First of all observe that the conclusion of the arguments are wrong. For this let

$$a = 60, b = 350, c = 59, d = 350, n = 349, k = 1.$$

Now we have

$$(a, b) = 10, \frac{a}{(a,b)} = 6, b' = b + \frac{an}{(a,b)} = 244, \text{ and } (a, b') = 4,$$

whereas in the proof it is claimed that $(a, b') = 1$. The error occurs because it is not enough to divide a by (a, b). We define an integer m as follows:

$$m = p_1^{s_1} \cdots p_r^{s_r},$$

where $p_1, \ldots, p_r$ are the distinct prime divisors of (a, b) and s_i is the largest power of p_i dividing a. Clearly $(a, b)|m$ but they are not necessarily equal as demonstrated in the above counterexample. Now defining $b' = b + \frac{an}{m}$, the proof proceeds as intended.

Note that in the above counterexample if we define $b' = b + \frac{an}{m}$, then $b' = 1397$ and we get $(60, 1397) = 1$ as desired.