Due on November 15, 2006, Wednesday.

MATH 591 Homework 2

1: For an affine variety $X \subset \mathbb{A}^N$ of dimension n let

$$P(z) = \frac{1}{n!}\left(c_n z^n + \cdots + c_0\right)$$

be its Hilbert polynomial. What is known about the geometric significance of the coefficients c_j, $j = 0, \ldots, n$?

2: Show that any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

3: Show that any two curves in $\mathbb{P}^2$ intersect.

4: Solve Exercise 7.2, Hartshorne page 54:
Let Y be a variety of dimension r in $\mathbb{P}^n$, with Hilbert polynomial P_Y. We define the *arithmetic genus* of Y to be $p_a(Y) = (-1)^r(P_Y(0) - 1)$. (This is an invariant of Y independent of its projective embedding.)

(a) Show that $p_a(\mathbb{P}^n) = 0$.

(b) If Y is a plane curve of degree d, show that $p_a(Y) = \frac{1}{2}(d-1)(d-2)$.

(c) More generally, if H is a hypersurface of degree d in $\mathbb{P}^n$, show that $p_a(H) = \binom{d-1}{n}$.

(d) If Y is a complete intersection of surfaces of degrees a, b in $\mathbb{P}^3$, then show that $p_a(Y) = \frac{1}{2}ab(a+b-4)+1$.

(e) Let $Y \subset \mathbb{P}^n$, $Z \subset \mathbb{P}^m$ be projective varieties of dimensions r and s respectively, and embed $Y \times X \subset \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$ by the Segre embedding. Show that

$$p_a(X \times Z) = p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z).$$

5: Solve Exercise 5.6.1, Karen page 79:
Assume that the variety $V \subset \mathbb{P}^n$ has the Hilbert polynomial $P(n)$. Calculate the Hilbert polynomial of the image variety $\nu_d(V) \subset \mathbb{P}^N$ of the Veronese map, where $N = \binom{n+d}{d} - 1$.