Ex I.1.8 **p 8** Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r-1$.

Let B be the coordinate ring of Y and let $\wp$ be the prime ideal corresponding to the hypersurface H. By Theorem 1.11A (p7), $\wp$ has height 1. The coordinate ring of $Y \cap H$ is isomorphic to $B/\wp$, and by Theorem 1.8A (p6), height $\wp + \dim B/\wp = \dim B$. from here it follows that $\dim Y \cap H = r-1$.

Ex I.2.14 **p 13** *The Segre Embedding.* Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_i b_j, \ldots)$ in lexicographic order, where $N = rs+r+s$. Note that ψ is well defined and injective. It is called the *Segre embedding*. Show that the image of ψ is a *subvariety* of $\mathbb{P}^N$. [*Hint*: Let the homogeneous coordinates of $\mathbb{P}^N$ be $\{z_{ij} | i = 0, \ldots, r,\ j = 0, \ldots, s\}$, and $\mathfrak{a}$ be the kernel of the homomorphism $k[\{z_{ij}\}] \to k[x_0, \ldots, x_r, y_0, \ldots, y_s]$ which sends z_{ij} to $x_i y_j$. Then show that $\mathrm{Im}\psi = Z(\mathfrak{a})$.]

ψ is well defined since $a_i b_j$ is nonzero when a_i and b_j are nonzero, and λa_i and δb_j map to $\lambda\delta a_i b_j$ which is the same point as $a_i b_j$. To show that ψ is injective, suppose that $(c_0, \ldots, c_N)$ is in the image of ψ. Assume without loss of generality that c_0 is nonzero. Then a_0 and b_0 are nonzero. Again assume without loss of generality that $c_0 = a_0 = b_0 = 1$. Then $c_j = b_j$ for $j = 0, \ldots, s$, and $c_{i(s+1)} = a_i$ for $i = 0, \ldots, r$. Thus the preimage is uniquely defined and ψ is injective. To show that the image is a subvariety first observe that if $p \in \mathfrak{a}$ is any polynomial then it vanishes identically on $\mathrm{Im}\psi$, so $\mathrm{Im}\psi \subseteq Z(\mathfrak{a})$. For the converse observe that the polynomials $z_{ij}z_{uv} - z_{iv}z_{uj}$ are all in $\mathfrak{a}$. If $c \in Z(\mathfrak{a})$, then $(z_{ij}z_{uv} - z_{iv}z_{uj})(c) = 0$. We can assume for convenience of notation that $c_{00} = 1$. Then $c_{uv} = c_{0v}c_{u0}$ for $u = 0, \ldots, r$ and $v = 0, \ldots, s$. This shows $\psi\left((c_{00}, c_{10}, \ldots, c_{r0}) \times (c_{00}, c_{01}, \ldots, c_{0s})\right) = c$, so $c \in \mathrm{Im}\psi$. Thus $Z(\mathfrak{a}) \subseteq \mathrm{Im}\psi$, and the equality holds.

Ex I.5.1 **p 35** Locate the singular points and sketch the following curves in $\mathbb{A}^2$. (assume char k$\neq 2$). Which is which in Figure 4?
(a) $x^2 = x^4 + y^4$;
(b) $xy = x^6 + y^6$;
(c) $x^3 = y^2 + x^4 + y^4$;
(d) $x^2y + xy^2 = x^4 + y^4$.

We recognize these curves through their tangent cones. The tangent cone is the zero set of the lowest degree homogeneous part of the defining equation for the curve. We match the tangent cone of the given equation with the curves in Figure 4:
(a) $x^2 = 0$ is the double y-axis. **Tachnode.**
(b) $xy = 0$ is the x and y axes. **Node.**
(c) $y^2 = 0$ is the double x-axis. **Cusp.**
(d) $x^2y + xy^2 = 0$ gives the x-axis, the y-axis and the line $y = -x$. **Triple point.**

Ex I.5.2 p 36 Locate the singular points and describe the singularities of the following surfaces in $\mathbb{A}^3$ (assume char k $\neq 2$). Which is which in Figure 5?
(a) $xy^2 = z^2$;
(b) $x^2 + y^2 = z^2$;
(c) $xy + x^3 + y^3 = 0$.

Let the Jacobian matrix be defined as $\theta(f)(P) = \left(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P), \frac{\partial f}{\partial z}(P)\right)$. Then $P \in \mathbb{A}^3$ is a singular point of $Z(f)$ if $f(P) = 0$ and $\theta(f)(P) = 0$.

(a) $f = xy^2 - z^2$. $\theta(f) = (y^2, 2xy, -2z) = 0 \implies y = 0, z = 0$ and x is free. This is on the surface. Hence the surface is singular along the x-axis. **Pinch point**.

(b) $f = x^2 + y^2 - z^2$. $\theta(f) = (2x, 2y, -2z) = 0 \implies x = 0, y = 0, z = 0$ which is also a point on the surface. Hence the surface has an isolated singularity at the origin. **Conical double point**.

(c) $f = xy + x^3 + y^3$. $\theta(f) = (y + 3x^2, x + 3y^2, 0) = 0$ This gives two points $(0, 0, z)$ which is on the surface, and $((-1/27)^{(1/3)}, -3(-1/27)^{(2/3)}, z)$ which is not on the surface. Hence the surface is singular along the z-axis. **Double line**.

Ex I.5.3 p 36 *Multiplicities.* Let $Y \subseteq \mathbb{A}^2$ be the curve defined by the equation $f(x, y) = 0$. Let $P = (a, b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0, 0)$. Then write f as a sum $f = f_0 + f_1 + \cdots + f_d$, where f_i is a homogeneous polynomial of degree i in x and y. Then we define the *multiplicity* of P on Y, denoted by $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note that $P \in Y \iff \mu_P(Y) > 0$.) The linear factors of f_r are called the *tangent directions* at P.
(a) Show that $\mu_P(f) = 1 \iff P$ is a nonsingular point of Y.
(b) Find the multiplicity of each of the singular points in (Ex. 5.1) above.

The Jacobian matrix at the origin is $\theta(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ evaluated at $(x, y) = (0, 0)$.

(a) $P = (0, 0)$ is a singular point of the affine plane curve Y if and only if $\theta = (a, b) \neq (0, 0)$ if and only if $f_1 = ax + by$ with $(a, b) \neq (0, 0)$ if and only if $\mu_P(Y) = 1$.

(b) In all these examples the singularity is the origin so to apply the definition we will check the degree of the smallest nonzero homogeneous part of the given polynomials: For 5.1-a, $\mu_p(Y) = 2$, for 5.1-b, $\mu_p(Y) = 2$, for 5.1-c, $\mu_p(Y) = 3$ and for 5.1-d, $\mu_p(Y) = 3$.

Ex I.7.2 p 54 Let Y be a variety of dimension r in $\mathbb{P}^n$, with Hilbert polynomial P_Y. We define the *arithmetic genus* of Y to be $p_a(Y) = (-1)^r(P_Y(0) - 1)$. This is an important invariant which (as we will see later in (III, Ex. 5.3)) is independent of the projective embedding of Y.

(a) Show that $p_a(\mathbb{P}^n) = 0$.

(b) If Y is a plane curve of degree d, show that $p_a(Y) = \frac{1}{2}(d-1)(d-2)$.

(c) More generally, if H is a hypersurface of degree d in $\mathbb{P}^n$, then $p_a(H) = \binom{d-1}{n}$.

(d) If Y is a complete intersection (EX. 2.17) of surfaces of degrees a, b in $\mathbb{P}^3$, then $p_a(Y) = \frac{1}{2}ab(a+b-4)+1$.

(e) Let $Y^r \subseteq \mathbb{P}^n$, $Z^s \subseteq \mathbb{P}^m$ be projective varieties, and embed $Y \times Z \subseteq \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ by the Segre embedding. Show that

$$p_a(Y \times Z) = p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z).$$

(a) Hilbert polynomial of $\mathbb{P}^n$ is $P(z) = \binom{z+n}{n}$, where $P(n)$ is the number of distinct homogeneous forms in the variables $x_0, \ldots, x_n$ (see p 52). $P(z) = \frac{1}{n!}(z+1)\cdots(z+n)$, and $P(0) = 1$. Then $p_a(\mathbb{P}^n) = 0$, regardless of what n is.

(b) Y is a hypersurface of degree d in $\mathbb{P}^2$, so we can use the formula derived on page 52, $P_Y(z) = \binom{z+2}{2} - \binom{z-d+2}{2} = \frac{1}{2}((z+1)(z+2) - (z-d+1)(z-d+2))$. Putting $z = 0$, $P_Y(0) = 1 - \frac{1}{2}(d-1)(d-2)$. The dimension r of Y is 1. Hence $p_a(Y) = \frac{1}{2}(d-1)(d-2)$.

(c) Using the formula on page 52 again we have $P_Y(z) = \binom{z+n}{n} - \binom{z-d+n}{n}$, and $P_Y(0) = 1 - (-1)^n \frac{1}{n!}(d-n)(d-n+1)\cdots(d-1) = 1 - (-1)^n\binom{d-1}{n}$. It follows that $p_a(Y) = \binom{d-1}{n}$.

(d) Let $Y = Z(f, g)$ in $\mathbb{P}^3$ where f and g are polynomials of degrees a and b respectively. We already know from Proposition 7.6 on page 52 that $\phi_{S/(f)}(\ell) = \binom{\ell+3}{3} - \binom{\ell-a+3}{3}$, where $S = k[x_0, \ldots, x_3]$. Consider the short exact sequence of grades S-modules

$$0 \to S(-b) \xrightarrow{g} S/(f) \to S/(f,g) \to 0.$$

Then

$$\begin{aligned} \phi_{S/(f,g)}(\ell) &= \phi_{S/(f)}(\ell) - \phi_{S/(f)}(\ell - b) \\ &= \binom{\ell+3}{3} - \binom{\ell-a+3}{3} - \binom{\ell-b+3}{3} + \binom{\ell-a-b+3}{3}. \end{aligned}$$

And putting in $\ell = 0$ we get

$$\phi_{S/(f,g)}(0) = 1 - (1 + \frac{1}{2}ab(a+b-4)),$$

from where it follows that $p_a(Y) = 1 + \frac{1}{2}ab(a+b-4))$ since the dimension r of Y is 1.

(e) Let x_i, y_j and z_{ij} for $i = 0, \ldots, n$, $j = 0, \ldots, m$ be the homogeneous coordinates of $\mathbb{P}^n$, $\mathbb{P}^m$ and $\mathbb{P}^{mn+m+n}$ respectively. As in (Ex. I.2.14) $z_{ij} = x_i y_j$ when restricted to $\mathbb{P}^n \times \mathbb{P}^m$. In particular any homogeneous form of degree d in z_{ij} restricted to $\mathbb{P}^n \times \mathbb{P}^m$ is the product of a form of degree d in x_i and a form of degree d in y_j. If P denotes the Hilbert polynomial, then we have $P_{Y\times Z}(d) = P_Y(d) \cdot P_Z(d)$. In particular if $\dim Y = r$ and $\dim Z = s$, then we have

$$\begin{aligned}
p_a(Y \times Z) &= (-1)^{r+s}(P_{Y\times Z}(0) - 1) \\
&= (-1)^{r+s}(P_Y(0)P_Z(0) - 1) \\
&= (-1)^r(P_Y(0) - 1) \cdot (-1)^s(P_Z(0) - 1) + (-1)^s[(-1)^r(P_Y(0) - 1)] \\
&\quad +(-1)^r[(-1)^s(P_Z(0) - 1)] \\
&= p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z)
\end{aligned}$$

as required.