Let $Y$ be an affine variety of dimension $r$ in $\mathbb{A}^n$. Let $H$ be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subsetneq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

Let $B$ be the coordinate ring of $Y$ and let $\varphi$ be the prime ideal corresponding to the hypersurface $H$. By Theorem 1.11A (p7), $\varphi$ has height 1. The coordinate ring of $Y \cap H$ is isomorphic to $B/\varphi$, and by Theorem 1.8A (p6), height $\varphi + \dim B/\varphi = \dim B$. From here it follows that $\dim Y \cap H = r - 1$.

The Segre Embedding. Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \rightarrow \mathbb{P}^n$ be the map defined by sending the ordered pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_ib_j, \ldots)$ in lexicographic order, where $N = rs + r + s$. Note that $\psi$ is well defined and injective. It is called the Segre embedding. Show that the image of $\psi$ is a subvariety of $\mathbb{P}^N$. [Hint: Let the homogeneous coordinates of $\mathbb{P}^N$ be $\{z_{ij} | i = 0, \ldots, r, j = 0, \ldots, s\}$, and $a$ be the kernel of the homomorphism $k[[z_{ij}]] \rightarrow k[x_0, \ldots, x_r, y_0, \ldots, y_s]$ which sends $z_{ij}$ to $x_iy_j$. Then show that $\text{Im} \psi = Z(a)$.]
Ex 1.5.2 Locate the singular points and describe the singularities of the following surfaces in \( \mathbb{A}^3 \) (assume char \( k \neq 2 \)). Which is which in Figure 5?
(a) \( xy^2 = z^2 \);
(b) \( x^2 + y^2 = z^2 \);
(c) \( xy + x^3 + y^3 = 0 \).

Let the Jacobian matrix be defined as \( \theta(f)(P) = \left( \frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P), \frac{\partial f}{\partial z}(P) \right) \). Then \( P \in \mathbb{A}^3 \) is a singular point of \( Z(f) \) if \( f(P) = 0 \) and \( \theta(f)(P) = 0 \).

(a) \( f = xy^2 - z^2 \). \( \theta(f) = (y^2, 2xy, -2z) = 0 \implies y = 0, z = 0 \) and \( x \) is free. This is on the surface. Hence the surface is singular along the \( x \)-axis. **Pinch point.**

(b) \( f = x^2 + y^2 - z^2 \). \( \theta(f) = (2x, 2y, -2z) = 0 \implies x = 0, y = 0, z = 0 \) which is also a point on the surface. Hence the surface has an isolated singularity at the origin. **Conical double point.**

(c) \( f = xy + x^3 + y^3 \). \( \theta(f) = (y + 3x^2, x + 3y^2, 0) = 0 \) This gives two points \( (0, 0, z) \) which is on the surface, and \( ((-1/27)^{(1/3)}, -3(-1/27)^{(2/3)}, z) \) which is not on the surface. Hence the surface is singular along the \( z \)-axis. **Double line.**

Ex 1.5.3 Multiplicities. Let \( Y \subseteq \mathbb{A}^2 \) be the curve defined by the equation \( f(x, y) = 0 \). Let \( P = (a, b) \) be a point of \( \mathbb{A}^2 \). Make a linear change of coordinates so that \( P \) becomes the point \( (0, 0) \). Then write \( f \) as a sum \( f = f_0 + f_1 + \cdots + f_d \), where \( f_i \) is a homogeneous polynomial of degree \( i \) in \( x \) and \( y \). Then we define the multiplicity of \( P \) on \( Y \), denoted by \( \mu_P(Y) \), to be the least \( r \) such that \( f_r \neq 0 \). (Note that \( P \in Y \iff \mu_P(Y) > 0 \).) The linear factors of \( f_r \) are called the tangent directions at \( P \).
(a) Show that \( \mu_P(f) = 1 \iff P \) is a nonsingular point of \( Y \).
(b) Find the multiplicity of each of the singular points in (Ex. 5.1) above.

The Jacobian matrix at the origin is \( \theta(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \) evaluated at \( (x, y) = (0, 0) \).

(a) \( P = (0, 0) \) is a singular point of the affine plane curve \( Y \) if and only if \( \theta = (a, b) \neq (0, 0) \) if and only if \( f_1 = ax + by \) with \( (a, b) \neq (0, 0) \) if and only if \( \mu_P(Y) = 1 \).

(b) In all these examples the singularity is the origin so to apply the definition we will check the degree of the smallest nonzero homogeneous part of the given polynomials: For 5.1-a, \( \mu_p(Y) = 2 \), for 5.1-b, \( \mu_p(Y) = 2 \), for 5.1-c, \( \mu_p(Y) = 3 \) and for 5.1-d, \( \mu_p(Y) = 3 \).
Ex 1.7.2  Let Y be a variety of dimension r in \( \mathbb{P}^n \), with Hilbert polynomial \( P_Y \). We define the \textit{arithmetic genus} of Y to be \( p_a(Y) = (-1)^r (P_Y(0) - 1) \). This is an important invariant which (as we will see later in (III, Ex. 5.3)) is independent of the projective embedding of Y.

(a) Show that \( p_a(\mathbb{P}^n) = 0 \).

(b) If Y is a plane curve of degree d, show that \( p_a(Y) = \frac{1}{2} (d-1)(d-2) \).

(c) Using the formula on page 52 again we have

\[
p_a(Y) = \frac{1}{m} (z + 1) \cdots (z + n),
\]

and \( P(0) = 1 \).

Then \( p_a(\mathbb{P}^n) = 0 \), regardless of what n is.

(b) Y is a hypersurface of degree d in \( \mathbb{P}^2 \), so we can use the formula derived on page 52,

\[
P_Y(z) = \binom{z+2}{2} - \binom{z-d+2}{2} = \frac{1}{2} ((z+1)(z+2) - (z-d+1)(z-d+2)).
\]

Putting \( z = 0 \),

\[
P_Y(0) = 1 - \frac{1}{2} (d-1)(d-2).
\]

The dimension \( r \) of Y is 1. Hence \( p_a(Y) = \frac{1}{2} (d-1)(d-2) \).

(c) Using the formula on page 52 again we have \( P_Y(z) = \binom{z+n}{n} - \binom{z-d+n}{n} \), and \( P_Y(0) = 1 - (-1)^n \frac{1}{m} (d-n)(d-n+1) \cdots (d-1) = 1 - (-1)^n \binom{d-1}{n} \). It follows that \( p_a(Y) = \binom{d-1}{n} \).

(d) Let \( Y = Z(f, g) \) in \( \mathbb{P}^3 \) where f and g are polynomials of degrees a and b respectively. We already know from Proposition 7.6 on page 52 that \( \phi_{S/(f)}(\ell) = \binom{\ell+3}{3} - \binom{\ell-a+3}{3} \), where \( S = k[x_0, \ldots, x_3] \). Consider the short exact sequence of grades S-modules

\[
0 \to S(-b) \xrightarrow{g} S/(f) \to S/(f, g) \to 0.
\]

Then

\[
\phi_{S/(f,g)}(\ell) = \phi_{S/(f)}(\ell) - \phi_{S/(f)}(\ell - b) = \binom{\ell+3}{3} - \binom{\ell-a+3}{3} - \binom{\ell-b+3}{3} + \binom{\ell-a-b+3}{3}.
\]

And putting in \( \ell = 0 \) we get

\[
\phi_{S/(f,g)}(0) = 1 - (1 + \frac{1}{2} ab(a + b - 4)),
\]

from where it follows that \( p_a(Y) = 1 + \frac{1}{2} ab(a + b - 4) \) since the dimension \( r \) of Y is 1.

(e) Let \( x_i, y_j \) and \( z_{ij} \) for \( i = 0, \ldots, n, \ j = 0, \ldots, m \) be the homogeneous coordinates of \( \mathbb{P}^n \), \( \mathbb{P}^m \) and \( \mathbb{P}^{mn+m+n} \) respectively. As in (Ex. I.2.14) \( z_{ij} = x_i y_j \) when restricted to \( \mathbb{P}^n \times \mathbb{P}^m \). In
particular any homogeneous form of degree \( d \) in \( z_{ij} \) restricted to \( \mathbb{P}^n \times \mathbb{P}^m \) is the product of a form of degree \( d \) in \( x_i \) and a form of degree \( d \) in \( y_j \). If \( P \) denotes the Hilbert polynomial, then we have \( P_{Y \times Z}(d) = P_Y(d) \cdot P_Z(d) \). In particular if \( \dim Y = r \) and \( \dim Z = s \), then we have

\[
p_a(Y \times Z) = (-1)^{r+s}(P_{Y \times Z}(0) - 1)
\]
\[
= (-1)^{r+s}(P_Y(0)P_Z(0) - 1)
\]
\[
= (-1)^r(P_Y(0) - 1) \cdot (-1)^s(P_Z(0) - 1) + (-1)^s[(-1)^r(P_Y(0) - 1)]
\]
\[
= p_a(Y)p_a(Z) + (-1)^sp_a(Y) + (-1)^rp_a(Z)
\]
as required.