Ex II.1.18 p 68 *Adjoint Property of f^{-1}*. Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathcal{F}$ on X there is a natural map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$, and for any sheaf $\mathcal{G}$ on Y there is a natural map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y,

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Hence we say that f^{-1} is a *left adjoint* of f_*, and that f_* is a *right adjoint* of f^{-1}.

First we show that there is a natural map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$:
Let $\mathcal{H}$ be the presheaf on X defined as

$$\mathcal{H}(U) = \lim_{V \supseteq f(U)} f_*\mathcal{F}(V) = \lim_{V \supseteq f(U)} \mathcal{F}(f^{-1}(V)).$$

$\mathcal{H}(U)$ consists of equivalence classes of the form $[\alpha]$ where $\alpha \in \mathcal{F}(f^{-1}(V))$ for some open V in Y with $V \supseteq f(U)$. Here for $\alpha_1 \in \mathcal{F}(f^{-1}(V_1))$ and $\alpha_2 \in \mathcal{F}(f^{-1}(V_2))$ we have $[\alpha_1] = [\alpha_2]$ if there is an open set $W \subseteq V_1 \cap V_2$ in Y with $W \supseteq f(U)$ such that

$$\rho_{f^{-1}(V_1)f^{-1}(W)}(\alpha_1) = \rho_{f^{-1}(V_2)f^{-1}(W)}(\alpha_2).$$

We can define a map $\phi : \mathcal{H} \to \mathcal{F}$ by $\phi([\alpha]) = \rho_{f^{-1}(V)U}(\alpha)$ if $\alpha \in \mathcal{F}(f^{-1}(V))$. Since $f^{-1}f_*\mathcal{F}$ is the sheaf associated to the presheaf $\mathcal{H}$, there is a unique map $\psi : f^{-1}f_*\mathcal{F} \to \mathcal{F}$ such that $\phi = \psi \circ \theta$ where $\theta : \mathcal{H} \to \mathcal{H}^+ = f^{-1}f_*\mathcal{F}$ is the natural morphism associated to the sheafification, see (1.2, p 64).

Next we show that there is a natural map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$:
Let $\mathcal{H}$ be the presheaf on Y defined by

$$\begin{aligned} V &\mapsto \lim_{W \supseteq f(f^{-1}(V))} \mathcal{G}(W) \\ &= \lim_{W \supseteq V} \mathcal{G}(W) \\ &\cong \mathcal{G}(V). \end{aligned}$$

Let $\phi : \mathcal{G} \to \mathcal{H}$ be the map taking $\alpha \in \mathcal{G}(V)$ into $\mathcal{G}(V)$ via the above isomorphism. Let $\theta : \mathcal{H} \to \mathcal{H}^+ = f_*f^{-1}\mathcal{G}$ be the map associated with the sheafification as above. Then $\theta \circ \phi$ is the natural map from $\mathcal{G}$ to $f_*f^{-1}\mathcal{G}$.

For a sheaf $\mathcal{A}$ on Y, let us look closely at the sheaf $f^{-1}\mathcal{A}$ on X:
$f^{-1}\mathcal{A}$ is the sheaf associated to the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathcal{A}(V)$. Then the stalk of this presheaf at p is the stalk $\mathcal{A}_{f(p)}$ of $\mathcal{A}$ at $f(p)$. An element s of $(f^{-1}\mathcal{A})(U)$ can be considered as a map

$$s : U \mapsto \coprod_{p \in U} \mathcal{A}_{f(p)}$$

such that for each $q \in U$, $s(q) \in \mathcal{A}_{f(q)}$ and furthermore for each $p \in U$ there is an open neighbourhood V of p contained in U and an open W in Y with $W \supseteq f(V)$ such that there exists an $\alpha \in \mathcal{A}(W)$ with $s(q) = \alpha_{f(q)}$ for each $q \in V$, where $\alpha_{f(q)}$ denotes the restriction of α to the stalk at $f(q)$. So we can consider $f^{-1}\mathcal{A}$ as the disjoint union of stalks of $\mathcal{A}$ satisfying the above obvious conditions.

Finally we come to the bijection between the sets $\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F})$ and $\mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$.

Let $F \in \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F})$, V an open set in Y and $\alpha \in \mathcal{G}(V)$. Since $f^{-1}(V)$ is open in X, it can be seen immediately that $(f^{-1}\mathcal{G})(f^{-1}(V)) = \mathcal{G}(V)$. So α can be considered as an element in $(f^{-1}\mathcal{G})(f^{-1}(V))$ and $F(\alpha)$ is in $\mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V)$. Thus we can define a map

$$\begin{array}{rcl} \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) & \rightarrow & \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) \\ F & \rightarrow & T(F) \end{array}$$

where

$$\begin{array}{rcl} \mathcal{G}(V) & \rightarrow & f_*\mathcal{F}(V) \\ \alpha & \rightarrow & T(F)(\alpha) = F(\alpha) \end{array}$$

for every open set V in Y.

Conversely let $G \in \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$, U an open set in X, and $\beta \in (f^{-1}\mathcal{G})(U)$. For every $p \in U$, there is an open neighbourhood U' of p in U, an open set W in Y containing $f(U')$ and a $\beta_W \in \mathcal{G}(W)$ such that $\beta|_{U'}$ is β_W (this follows from the above discussion of $f^{-1}\mathcal{A}$). Now $G(\beta_W)$ is in $f_*\mathcal{F}(W) = \mathcal{F}(f^{-1}(W))$. Using the restriction map from $f^{-1}(W)$ to U' we can consider $G(\beta_W)$ in $\mathcal{F}(U')$. Since the sheaf morphism G commutes with the restriction maps, $G(\beta_W)$ is a well defined element in $\mathcal{F}(U')$, independent of W. Moreover U is covered by such U' so by the sheaf property of $\mathcal{F}$ these local elements patch together to give a unique element in $\mathcal{F}(U)$ which we denote by $\bigcup' G(\beta_W)$. So by this notation $G(\beta) = \bigcup' G(\beta_W)$. Thus we can define a map

$$\begin{array}{rcl} \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) & \rightarrow & \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \\ G & \rightarrow & S(G) \end{array}$$

where

$$\begin{array}{rcl} f^{-1}\mathcal{G}(U) & \rightarrow & \mathcal{F}(U) \\ \beta & \rightarrow & S(G)(\beta) = \bigcup^{\prime} G(\beta_W) = G(\beta) \end{array}$$

for every open set U in X.

Finally we check that $S \circ T$ and $T \circ S$ are the identity maps on their respective domains:
$S(T(F)(\beta) = \bigcup' T(F)(\beta_W) = \bigcup' F(\beta_W) = F(\beta)$ and
$T(S(G))(\alpha) = S(G)(\alpha) = \bigcup' G(\alpha_W) = G(\alpha)$.
This completes the solution.

Ex II.1.19 p 68 *Extending a Sheaf by Zero.* Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X - Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.

(a) Let $\mathcal{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathcal{F})_P$ of the direct image sheaf on X is $\mathcal{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathcal{F}$ the sheaf obtained by extending $\mathcal{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathcal{F}$ instead of $i_*\mathcal{F}$, and say "consider $\mathcal{F}$ as a sheaf on X," when we mean "consider $i_*\mathcal{F}$."

(b) Now let $\mathcal{F}$ be a sheaf on U. Let $j_!(\mathcal{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathcal{F}))_P$ is equal to $\mathcal{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathcal{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathcal{F}$. We call $j_!\mathcal{F}$ the sheaf obtained by *extending $\mathcal{F}$ by zero* outside U.

(c) Now let $\mathcal{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0.$$

(a) If $p \in X - Z$, then there is an open neighbourhood V of p disjoint from Z and it follows from

$$(i_*\mathcal{F})(V) = \mathcal{F}(i^{-1}(V)) = \mathcal{F}(V \cap Z) = \mathcal{F}(\emptyset) = 0$$

that $\mathcal{F}_p = 0$. Using the induced topology on Z, it follows by the same argument that when $p \in Z$, then for any open neighbourhood V of p

$$(i_*\mathcal{F})(V) = \mathcal{F}(i^{-1}(V)) = \mathcal{F}(V \cap Z)$$

and hence $(i_*\mathcal{F})_p = \mathcal{F}_p$.

(b) If $p \in U$, then there is an open neighbourhood V of p lying totally in U. The presheaf $V \mapsto \mathcal{F}(V)$ gives $(j_!\mathcal{F})_p = \mathcal{F}_p$. If however p is not in U, then no open neighbourhood V of p can possibly lie in U, so the presheaf $V \mapsto 0$ gives $(j_!\mathcal{F})_p = 0$. Since for any open set V in X, the sections $(j_!\mathcal{F})(V)$ are explicitly defined as the maps $s \to \coprod_{p \in V}(j_!\mathcal{F})_p$, the sheaf $j_!\mathcal{F}$ is the only sheaf satisfying the given conditions (see 1.2).

(c) When $p \in U$, the sequence becomes

$$0 \to \mathcal{F}_p \to \mathcal{F}_p \to 0 \to 0,$$

and when $p \notin U$, the sequence becomes

$$0 \to 0 \to \mathcal{F}_p \to \mathcal{F}_p \to 0.$$

So the given short sequence is exact (see 1.2.1).

Ex II.1.21 p 68 *Some Examples of Sheaves on Varieties.* Let X be a variety over an algebraically closed field k, as in Ch. I. Let $\mathcal{O}_X$ be the ring of regular functions on X (1.0.1).

(a) Let Y be a closed subvariety of X. For each open set $U \subseteq X$, let $\mathcal{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathcal{I}_Y(U)$ is a sheaf. It is called the *sheaf of ideals* $\mathcal{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.

(b) Show that the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i : Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

(c) Now let $X = \mathbb{P}^1$, and let Y be the union of two distinct points $P, Q \in X$. Thus by (b) we have an exact sequence of sheaves on X

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0.$$

Show however that the induced map on the global sections $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, i_*\mathcal{O}_Y)$ is not surjective. This shows that the global section functor $\Gamma(X, \cdot)$ is not exact (cf. (Ex. 1.8) which shows that it is left exact).

(d) Again let $X = \mathbb{P}^1$, let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathcal{K}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathcal{K}$. Show that the quotient sheaf $\mathcal{K}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\sum_{P \in X} i_P(I_P)$, where I_P is the group $K/\mathcal{O}_P$, and $i_P(I_P)$ denotes the skyscraper sheaf (Ex. 1.17) given by I_P at the point P.

(e) Finally show that in the case of (d) the sequence

$$0 \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{K}/\mathcal{O}) \to 0$$

is exact. (This is an analogue of what is called the "first Cousin problem" in several complex variables. See Gunning and Rossi [1, p. 248].)

(a) Let U be an open subset of X covered by the open subsets V_i. Let $s \in \mathcal{I}_Y(U)$ be such that $s|_{V_i} = 0$ for all i. Then considering s as an element of $\mathcal{O}_X(U)$, we know that $s = 0$. Moreover if for each i we have an $s_i \in \mathcal{I}_Y(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all i and j, then there is an element in $\mathcal{O}_X(U)$ such that $s|_{V_i} = s_i$. Since each s_i vanishes on Y, s also vanishes on Y and is in $\mathcal{I}_Y(U)$. Hence $\mathcal{I}_Y$ is a sheaf.

(b) At each point $p \in Y$, the map $(\mathcal{O}_X)_p \to (i_*(\mathcal{O}_Y))_p$ is the restriction map and is surjective, the kernel being $(\mathcal{I}_Y)_p$. Thus the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to the sheaf $\mathcal{O}_Y$.

(c) Global regular functions on $\mathbb{P}^1$ are constants. Restriction of this to Y gives a function whose value at P and Q are the same. However we can clearly have functions on Y with different values at P and Q. Hence the global section functor is not exact.

(d) Let $\mathcal{S}$ denote the sheaf $\sum_{p\in X} i_P(I_P)$. We construct a short sequence

$$0 \to \mathcal{O} \to \mathcal{K} \xrightarrow{\phi} \mathcal{S} \to 0$$

and show that on the germ level it is an exact sequence, which establishes the required isomorphism.

Let $q \in X$ and f a germ in the stalk $\mathcal{K}_q$. $\phi_q(f) = 0$ if and only if f is regular at q. Hence it remains to show that ϕ_q is surjective. An element of I_q is an equivalence class of elements of the form h/z^n for some positive integer n, where z is a local coordinate centered at q and h is a rational function of z regular at q. If $c_n = h(q)$, then there are a polynomial h_1, h_2 such that $h/z^n = c_n/z^n + h_1/(z^m h_2)$ for some $m < n$, where $h_2(q) \neq 0$. Proceeding inductively there are constants $c_n, \ldots, c_1$ and a polynomial h_0 such that $h/z^n = c_n/z^n + \cdots + c_1/z + h_0/h_2$. But h_0/h_2 is regular at q, so h/z^n in I_q is in the same equivalence class of $g = c_n/z^n + \cdots + c_1/z$. This function g then represents an element of the stalk $\mathcal{K}_q$, and note that the only pole of g is at q. This shows that ϕ_q is surjective (we actually showed more than necessary but we will use this in (e)). This completes the proof that the above short sequence is exact and the required isomorphism now follows.

(e) The global sections functor is left exact, see (Ex. 1.8). So it suffices to show surjectivity on the right.

Let $f \in \Gamma(X, \mathcal{K}/\mathcal{O})$. From part (d) we can consider f as an element of $\Gamma(X, \sum_{p\in X} i_P(I_P))$. $f = f_1 + \cdots + f_r$ where each f_i is equivalent to a principal part of the form $c_n/z^n + \cdots + c_1/z$, as in (d), where z is a local coordinate centered at some q_i. For each such principal part there is a rational function g_i as in (d) which is regular everywhere except at p_i and differs from f by a regular function at p_i. Let $g = g_1 + \cdots + g_r$. Then g is an element of $\Gamma(X, \mathcal{K})$ which maps to f, giving the right exactness.

This is a Cousin problem in the sense that given a finite number of points in $\mathbb{P}^1$ and principal parts at those points, then there exists a rational function which is regular everywhere except the given points and has precisely the assigned principal parts at those points.

Ex II.6.5 p 147 *Quadric Hypersurfaces.* Let char $k \neq 2$, and let X be the affine quadric hypersurface Spec $k[x_0, \ldots, x_n]/(x_0^2 + x_1^2 \cdots + x_r^2)$-cf. (I, Ex. 5.12).

(a) Show that X is normal if $r \geq 2$ (use (Ex. 6.4)).

(b) Show by a suitable linear change of coordinates that the equation of X could be written as $x_0x_1 = x_2^2 + \cdots + x_r^2$. Now imitate the method of (6.5.2) to show that:

(1) If $r = 2$, then Cl $X \cong \mathbb{Z}/2\mathbb{Z}$;

(2) If $r = 3$, then Cl $X \cong \mathbb{Z}$ (use (6.6.1) and (Ex. 6.3) above);

(3) If $r \geq 4$, then Cl $X = 0$.

(c) Now let Q be the projective quadric hypersurface in $\mathbb{P}^n$ defined by the same equations. Show that:

(1) If $r = 2$, then Cl $Q \cong \mathbb{Z}$, and the class of a hyperplane section $Q.H$ is twice the generator;

(2) If $r = 3$, then Cl $Q \cong \mathbb{Z} \oplus \mathbb{Z}$;

(3) If $r \geq 4$, then Cl $Q \cong \mathbb{Z}$, generated by $Q.H$.

(d) Prove Klein's theorem, which says that if $r \geq 4$, and if Y is an irreducible subvariety of codimension 1 on Q, then there is an irreducible hypersurface $V \subseteq \mathbb{P}^n$ such that $V \cap Q = Y$, with multiplicity one. In other words, Y is a complete intersection. (First show that for $r \geq 4$, the homogeneous coordinate ring $S(Q) = k[x_0, \ldots, x_n]/(x_0^2 + \cdots + x_r^2)$ is a UFD.)

(a) If $r \geq 2$, and the characteristic of k is $\neq 2$, then $f = -(x_1^2 + \cdots + x_r^2)$ is square free. The quotient ring $k[x_0, \ldots, x_n]/(x_0^2 - f)$ is now integrally closed by (Ex. 6.4) page 147.

(b) Replacing $x_0 + ix_1$ by x_0 and $-x_0 + ix_1$ by x_1, we can write $-x_0x_1$ for $x_0^2 + x_1^2$. Then $x_0^2 + \cdots + x_r^2 = 0$ becomes $x_0x_1 = x_2^2 + \cdots + x_r^2$.

(b-1) This is basically Example 6.5.2 on page 133; Let $A = k[x_0, \ldots, x_n]/(x_0x_1 + x_2^2)$, and set $Y : x_1 = x_2 = 0$. Y is a prime divisor and by 6.5 p.133 there is an exact sequence

$$\mathbb{Z} \to \mathrm{Cl}\ X \to \mathrm{Cl}(X - Y) \to 0,$$

where the first map is $1 \mapsto 1 \cdot Y$. Y can be cut out set theoretically by $x_1 = 0$. The prime ideal corresponding to Y in *Spec* A is generated by x_1 and x_2. Localizing A at this prime ideal, which is the generic point corresponding to Y, we see that x_0 becomes invertible and $x_1 = x_2/x_0$. All x_i with $i > 2$ also becomes invertible. Hence the maximal ideal of this local ring is generated by x_2. Since $x_1 = 0$ gives $x_2^2 = 0$ in this local ring, we have $2 \cdot Y = 0$. Moreover $X - Y$ correspond to *Spec* A_y, and since $x_0 = x_2^2/x_1$ in A_y, we have $A_y \cong k[x_1, x_1^{-1}, x_2, x_3, \ldots, x_n]$ which is a UFD. By 6.2 p.131 $\mathrm{Cl}(X - Y) = 0$. Thus Cl X is generated by Y and $2 \cdot Y = 0$. Hence Cl $X \cong \mathbb{Z}/2\mathbb{Z}$.

(b-2) By a change of variables we can write $x_0^2 + \cdots + x_3^2$ as $x_0x_1 - x_2x_3$. Let V be the projective variety in $\mathbb{P}^3$ defined by this equation. By 6.6.1 p.135, Cl $V \cong \mathbb{Z} \oplus \mathbb{Z}$. Let X' be the affine cone over V in $\mathbb{A}^4$. By (Ex. 6.3.b) p.147, Cl $X' \cong \mathbb{Z}$. Now $X \cong X' \times \mathbb{A}^{n-4}$, and by 6.6 p.134, Cl $X \cong$ Cl $X' \cong \mathbb{Z}$.

(b-3) We first prove a technical result about the irreducibility of $x_0^2 + \cdots + x_r^2$ for $r \geq 2$.

Let $f_r = \alpha_0 x_0^2 + \cdots + \alpha_r x_r^2$, where $\alpha_0 \cdots \alpha_r \neq 0$. If f_r is reducible then it can be written as the product of two linear forms.

Let $r = 2$. We will show that f_2 is irreducible. Then it will follow by induction that f_r is irreducible for all $r \geq 2$, since otherwise putting $x_3 = \cdots = x_r = 0$ would give a decomposition of f_2. To show the irreducibility of f_2 assume to the contrary that

$$\begin{aligned} f_2 &= (a_0x_0 + a_1x_1 + a_2x_2)(b_0x_0 + b_1x_1 + b_2x_2) \\ &= \sum_{i=0}^{2} a_i b_i x_i^2 + \sum_{0 \leq i < j \leq 2} (a_i b_j + a_j b_i) x_i x_j. \end{aligned}$$

Then we have the following two linear equations

$$\begin{aligned} \begin{pmatrix} a_0 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}, \\ \begin{pmatrix} a_1 & a_0 & 0 \\ a_2 & 0 & a_0 \\ 0 & a_2 & a_1 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

The first matrix equation implies that $a_0a_1a_2 \neq 0$ and $b_0b_1b_2 \neq 0$. For the second matrix equation to have a nonzero solution the determinant of the coefficient matrix must be zero but that gives $a_0a_1a_2 = 0$, a contradiction. Thus we conclude, with the above induction argument, that f_r is irreducible for $r \geq 2$.

Now let X be defined by the equation $x_0x_1 + x_2^2 + \cdots + x_r^2 = 0$, and let $A = k[x_0, \ldots, x_r]/(x_0x_1 + x_2^2 + \cdots + x_r^2)$. To calculate Cl X, let Y be the divisor in X cut out by the equation $x_0 = 0$. By 6.5 p.131 we have the usual exact sequence

$$\mathbb{Z} \to \text{Cl } X \to \text{Cl } (X - Y) \to 0$$

where the first map is given by $1 \mapsto 1 \cdot Y$. Since Y is given by $x_0 = 0$ in A, $X - Y = \text{Spec} A_{x_0}$. But in A_{x_0}, $x_1 = -x_0^{-1}(x_2^2 + \cdots + x_r^2)$, so we can eliminate x_1. Then $A_{x_0} \cong k[x_0, x_0^{-1}, x_2, x_3, \ldots, x_r]$ which is a UFD, and consequently Cl $(X - Y) =$Cl Spec$A_{x_0} = 0$.

From the above exact sequence we now conclude that Cl X is generated by Y. Depending on the nature of Y we have two cases:

- If Y is principal, then Cl $X = 0$.
- If Y is not principal then

– If $d \cdot Y = 0$ for some positive integer d, then Cl $X = \mathbb{Z}/d\mathbb{Z}$.
– If $d \cdot Y \neq 0$ for any positive integer d, then Cl $X = \mathbb{Z}$.

The prime ideal $\wp$ of Y in A is generated by x_0 and by $g_{r-2} = x_2^2 + \cdots + x_r^2$. The divisor class of X now depends on the value of r as follows:

- If $r \geq 4$, then g_{r-2} is irreducible as proved at the beginning. In A, $x_0 = -g_{r-2}$ so $\wp$ is principal being generated by only x_0. In this case Cl $X = 0$.
- If $r = 3$, then by a change of variables not affecting x_0 we can write $g_{r-2} = x_2x_3$. Since $\wp$ is prime, both x_2 and x_3 must be in $\wp$. Now $\wp = (x_2, x_3)$ is not principal and since there is no multiplicity, Cl $X = \mathbb{Z}$.
- If $r = 2$, then $x_2^2 \in \wp$ implies $x_2 \in \wp$ and as argued above x_2 generates the maximal ideal of the local ring at the generic point of Y. In this case Cl $X = 2 \cdot Y$.

Note that by 6.5 p.131 we have shown that $k[x_0, \ldots, x_n]/(x_0^2 + \cdots + x_r^2)$ is a UFD when $r \geq 4$.

(c) We use (Ex. 6.3) p.146. What is called V and X in (Ex. 6.3) is Q and X respectively in this problem. We use the short exact sequence

$$0 \to \mathbb{Z} \to \text{CL } Q \to \text{CL } X \to 0$$

of (Ex. 6.3.b).

(c-1) When $r = 2$, Cl $X = \mathbb{Z}/2\mathbb{Z}$. From (Ex. 6.3.b) we know that the first map of the above exact sequence sends 1 to a hyperplane section of Q. Since this must be in the kernel, it is twice the generator. Moreover since the first map is injective, we have Cl $Q = \mathbb{Z}$.

(c-2) When $r = 3$, Cl $X = \mathbb{Z}$, and from the above exact sequence we have Cl $Q = \mathbb{Z} \oplus \mathbb{Z}$.

(c-3) When $r \geq 4$, Cl $X = 0$, and using the above exact sequence one last time we conclude that Cl $Q = \mathbb{Z}$. Here again from the nature of the first map we know that 1 is send to a hyperplane section which generates Cl Q.

(d) The homogeneous coordinate ring $S(Q) = k[x_0, \ldots, x_n]/(x_0^2 + \cdots + x_r^2)$ is also the affine coordinate ring of X. We showed in b-3 above that Cl $X = 0$. By (Ex. 6.4) p.147, it is integrally closed so by 6.2 p.131 it is a UFD. By 1.12A p.7 every prime ideal of height 1 is principal in $S(Q)$. Hence the prime ideal corresponding to Y, being of height 1, is principal. The generator then gives the hypersurface V.