Bilkent University

Take-Home Exam # 01
Math 633 Algebraic Geometry
Due on: 5 November 2019 Tuesday - Class Time
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Solution Key

Q-1) Let X be the projective twisted cubic in $\mathbb{P}^3_k$, where k is an algebraically closed field.

(i) Find an ideal J in $k[x, y, z, w]$, where we take $[x\colon y\colon z\colon w]$ as homogeneous coordinates for $\mathbb{P}^3$, such that $Z(J) = X$.

(ii) Calculate $I(X)$ in $k[x, y, z, w]$.

Solution:

(i) The twisted cubic is given as

$$X = \{[s^3 : s^2t : st^2 : t^3] \in \mathbb{P}^3 \mid [s : t] \in \mathbb{P}^2\}.$$

Now consider the polynomials

$$f_1 = xz - y^2, f_2 = yw - z^2, f_3 = xw - yz,$$

and let $J = (f_1, f_2, f_3)$ be the ideal generated by these polynomials.

We claim that

$$Z(J) = X,$$

where $Z(J)$ denotes the zero set of the ideal J.

By the direct substitution of $x = s^3, y = s^2t, z = st^2, w = t^3$ into the generators of J, we see that $X \subseteq Z(J)$.

For the other inclusion let $[x : y : z : w] \in Z(J)$.

If $x = 0$, then $f_1 = 0$ gives $y = 0$, and $f_2 = 0$ gives $z = 0$. Now $f_3 = 0$ already holds and we see that the only point in $Z(J)$ with $x = 0$ is $[0 : 0 : 0 : 1]$ and this point is clearly in X.

Next assume, without loss of generality that $x = 1$. Now the generators of J can be written as

$$f_1 = z - y^2, f_2 = yw - z^2, f_3 = w - yz.$$

Setting y free, we see that $f_1 = 0$ gives $z = y^2$, and $f_3 = 0$ gives $w = yz$ or equivalently $w = y^3$. These make $f_2 = 0$. So all points in $Z(J)$ of the form $x \neq 0$ are of the form $[1 : y : y^2 : y^3]$, and all such points are in X. Thus $Z(J) \subseteq X$ and the equality follows.

(ii) We claim that $I(X) = J$.

Since $Z(J) = X$, by Hilbert's nullstellensatz we have $J \subseteq \sqrt{J} = I(X)$. Therefore we need to show that $I(X) \subseteq J$. For this we first observe that there is no linear polynomial in $I(X)$.

To continue we need to use a common trick.

We claim that any homogeneous polynomial $h \in k[x, y, z, w]$ of degree $n \geq 2$ can be written as

$$a_n(x, w) + b_{n-1}(x, w)y + c_{n-1}(x, w)z + g_n, \tag{$*$}$$

where a_ℓ, b_ℓ, c_ℓ are homogeneous polynomials of degree ℓ, and g_n is a homogeneous polynomial in J of degree n.

For a moment let us assume this statement and assume that this is a polynomial in $I(X)$. Then putting in $x = s^3, y = s^2t, z = st^2, w = t^3$, we get

$$a_n(s^3, t^3) + b_{n-1}(s^3, t^3)s^2t + c_{n-1}(s^3, t^3)st^2 = 0.$$

Now observe that the s-degree of $a_n(s^3, t^3)$ is either 0 or $3n$, that of $b_{n-1}(s^3, t^3)s^2t$ is either 0 or $3n - 1$, and that of $c_{n-1}(s^3, t^3)st^2$ is either 0 or $3n - 2$. Then the only way for these to add up to zero is to have their s-degrees equal to zero, i.e. these are polynomials only in t. Then we compare their t-degrees which similarly forces all t-degrees to be zero. Then these are constant polynomials which add up to zero. This says that the only homogeneous polynomial of degree n inside $I(X)$ is of the form g_n, where $g_n \in J$. This shows $I(X) \subseteq J$, and completes the proof of equality.

Now we prove ($*$) by induction on n.

When $n = 2$, any homogeneous polynomial f in $k[x, y, z, w]$ of degree 2 can be written as

$$\begin{aligned} f &= \alpha_1 x^2 + \alpha_2 xy + \alpha_3 xz + \alpha_4 xw + \alpha_5 y^2 + \alpha_6 yz + \alpha_7 yw + \alpha_8 z^2 + \alpha_{zw} + \alpha_{10} w^2 \\ &= [\alpha_1 x^2 + (\alpha_4 + \alpha_6)xw + \alpha_{10} w^2] + [\alpha_2 x + (\alpha_7 + \alpha_8)w]y + [(\alpha_3 + \alpha_5)x + \alpha_9 w]z \\ &\quad + \alpha_5(y^2 - xz) + \alpha_6(yz - xw) + \alpha_8(z^2 - yw), \end{aligned}$$

which is of the claimed form.

Now assume that ($*$) holds for n. Any homogeneous polynomial f in $k[x, y, z, w]$ of degree $n+1$ can be written as

$$f = u_n x + u'_n y + u''_n z + u'''_n w,$$

where u_n, u'_n, u''_n, u'''_n are homogeneous polynomial of degree n in the variables x, y, z, w. By the induction hypothesis each of these polynomials can be written as claimed in ($*$). Hence we have, notation being self-explanatory,

$$\begin{aligned} f = &(a_n + b_{n-1}y + c_{n-1}z + g_n)x + (a'_n + b'_{n-1}y + c'_{n-1}z + g'_n)y \\ &+ (a''_n + b''_{n-1}y + c''_{n-1}z + g''_n)z + (a'''_n + b'''_{n-1}y + c'''_{n-1}z + g'''_n)w, \end{aligned}$$

where g_n, g'_n, g''_n, g'''_n are in J. By rearranging terms, we can rewrite f as

$$\begin{aligned} f = &[a_n x + c'_{n-1}xw + b''_{n-1}xw + a'''_n w] + [b_{n-1}x + a'_n + c''_{n-1}w + b'''_{n-1}w]y \\ &+ [c_{n-1}x + b'_{n-1}x + a''_n + c'''_{n-1}w]z \\ &+ b'_{n-1}(y^2 - xz) + c'_{n-1}(yz - xw) + b''_{n-1}(yz - xw) + c''_{n-1}(z^2 - yw) \\ &+ g_n x + g'_n y + g''_n z + g'''_n w, \end{aligned}$$

which is of the required form, and this completes the induction.