Bilkent University

Take-Home Exam # 02
Math 633 Algebraic Geometry
Due on: 7 November 2019 Thursday - Class Time
Instructor: Ali Sinan Sertöz
**Solution Key**

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**Q-1)** Show that the $d$-uple embedding of $\mathbb{P}^n$ is an isomorphism onto its image.

**Solution:**

We first show that $\rho_d$ is one-to-one. Let $c = [c_0 : \cdots : c_n]$ and $e = [e_0 : \cdots : e_n]$ are points of $\mathbb{P}^n$ such that $\rho_d(c) = \rho_d(e)$. At least one of the homogeneous coordinates $e_i$ is different than zero. For convenience of notation assume $e_0 \neq 0$. Since the images of $c$ and $e$ agree under $\rho_d$, we must have $c_0^d = e_0^d$ and $c_0^{d-1}c_i = e_0^{d-1}e_i$ for $i = 1, \ldots, n$. From $c_0^d = e_0^d$, we get $c_0 = \omega e_0$, where $\omega^d = 1$. From the other equations we get $c_i = \omega e_i$ for $i = 1, \ldots, n$. This shows that $c = e$ and hence $\rho_d$ is one-to-one.

Let $\mathfrak{a}$ be the prime ideal with $\rho_d(\mathbb{P}^n) = Z(\mathfrak{a})$ in $\mathbb{P}^N$.

Next we show that $\rho_d$ is continuous. In fact if $Z(f_1, \ldots, f_m)$ is a closed subset of $Z(\mathfrak{a})$, where $f_1, \ldots, f_m$ are homogeneous polynomials in $y_0, \ldots, y_N$, then $\rho_d^{-1}(Z(f_1, \ldots, f_m)) = Z(f_1 \circ \rho_d, \ldots, f_m \circ \rho_d)$ is closed. Thus $\rho_d$ is continuous.

We now have a continuous isomorphism from $\rho_d(\mathbb{P}^n)$ onto $Z(\mathfrak{a})$. It is well known that a continuous isomorphism is not necessarily a homeomorphism. Therefore we have to check separately if $\rho_d$ is a homeomorphism in this case.

What remains to be shown for $\rho_d$ to be a homeomorphism is that it maps closed sets to closed sets. Since every ideal in $k[x_0, \ldots, x_n]$ is finitely generated, every closed set is an intersection of finitely many hypersurfaces. It therefore suffices to show that a hypersurface is mapped under $\rho_d$ to a closed set in $Z(\mathfrak{a})$. And for this we need to show that for any homogeneous polynomial $g \in k[x_0, \ldots, x_n]$, there corresponds a homogeneous polynomial $G \in k[y_0, \ldots, y_N]$ such that $\rho_d(Z(g)) = Z(\mathfrak{a}) \cap Z(G)$. We now describe a way of obtaining $G$ from $g$.

Let $g \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree $m$. Let

$$w_t = x_0^{i_{0t}} x_1^{i_{1t}} \cdots x_n^{i_{nt}},\ t = 1, \ldots, d.$$

be homogeneous monomials, not necessarily distinct, occurring in $g$ with non-zero coefficients, where $i_{0t}, \ldots, i_{nt}$ are non-negative integers with $i_{0t} + \cdots + i_{nt} = m$ for $t = 1, \ldots, d$. A typical monomial occurring in the polynomial $g^d$ is of the form

$$w = w_1 \cdots w_d = x^{i_{01} + \cdots + i_{0d}} \cdots x^{i_{n1} + \cdots + i_{nd}}.$$

For $s = 0, \ldots, n$, let

$$i_{s1} + \cdots + i_{sd} = u_s d + v_s,$$

where $0 \le u_s$ and $0 \le v_s < d$. Then we can write

$$w = (x_0^d)^{u_0} \cdots (x_n^d)^{u_n} (x_0^{v_0} \cdots x_n^{v_n}).$$

Since the degree of the monomial $w$ is $dm$, we see that $v_0 + \cdots + v_n = \ell d$, where $\ell \ge 0$ is a non-negative integer. Since each $v_s < d$, we have that $0 \le \ell \le n$. Therefore there exist integers $0 \le j_0 < \cdots < j_\ell \le n$ such that we can write

$$v_{j_e} = v'_{j_e} + v''_{j_e},\ e = 0, \ldots, \ell$$

in such a way that

$$v_0 + \cdots + v_n = (v_0 + \cdots + v_{j_0 - 1} + v'_{j_0}) + (v''_{j_0} + v_{j_0+1} + \cdots + v_{j_1 - 1} + v'_{j_1}) + \cdots + (v''_{j_\ell} + v_{j_\ell + 1} + \cdots + v_n)$$

in such a way that each parenthesis adds up to $d$. Then

$$x_0^{v_0} \cdots x_n^{v_n} = (x_0^{v_0} \cdots x_{j_0}^{v'_{j_0}}) \cdots (x_{j_\ell}^{v''_{j_\ell}} \cdots x_n^{v_n}).$$

This proves that the monomial $w$ of degree $dm$ can be written as a product of monomials of degree $d$ in the variables $x_0, \ldots, x_n$. In fact let $M_0, \ldots, M_N$ be a list of monomials of degree $d$ in $x_0, \ldots, x_n$, and let $\phi$ be the map

$$\phi : k[M_0, \ldots, M_N] \to k[y_0, \ldots, y_N],$$

sending each $M_i$ to $y_i$, then $\phi(w)$ is a monomial of degree $m$ in the variables $y_0, \ldots, y_N$. Define the polynomial $G$ as

$$G(y_0, \ldots, y_N) = \phi(g^d(x_0, \ldots, x_n) \in k[y_0, \ldots, y_N],$$

where the above process of writing $g^d$ as a polynomial in $M_i$ is understood, before $\phi$ is applied. Then $G$ is uniquely defined and is homogeneous of degree $m$.

It is now clear from the description of the maps that for any point $a = [a_0 : \cdots : a_n] \in \mathbb{P}^n$,

$$G(\rho_d(a)) = \phi(g^d(a)).$$

It follows from this that $a \in Z(g)$ if and only if $\rho_d(a) \in Z(G) \cap Z(\mathfrak{a})$. This shows that the closed set $Z(g)$ is mapped onto the closed set $Z(G) \cap Z(\mathfrak{a})$, which completes the proof that $\rho_d$ is a closed map.

Thus $\rho_d$ is a homeomorphism from $\mathbb{P}^n$ onto $Z(\mathfrak{a})$