Bilkent University

Take-Home Exam # 03
Math 633 Algebraic Geometry
Due on: 5 December 2019 Thursday - Class Time
Instructor: Ali Sinan Sertöz

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Q-1) Hartshorne Exercise II.1.21 pages 68-69

Some Examples of Sheaves on Varieties. Let X be a variety over an algebraically closed field k, as in Ch. I. Let $\mathcal{O}_X$ be the ring of regular functions on X (1.0.1).

(a) Let Y be a closed subvariety of X. For each open set $U \subseteq X$, let $\mathcal{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathcal{I}_Y(U)$ is a sheaf. It is called the *sheaf of ideals* $\mathcal{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.

(b) Show that the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i : Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

(c) Now let $X = \mathbb{P}^1$, and let Y be the union of two distinct points $P, Q \in X$. Thus by (b) we have an exact sequence of sheaves on X

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0.$$

Show however that the induced map on the global sections $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, i_*\mathcal{O}_Y)$ is not surjective. This shows that the global section functor $\Gamma(X, \cdot)$ is not exact (cf. (Ex. 1.8) which shows that it is left exact).

(d) Again let $X = \mathbb{P}^1$, let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathcal{K}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathcal{K}$. Show that the quotient sheaf $\mathcal{K}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\sum_{P \in X} i_P(I_P)$, where I_P is the group $K/\mathcal{O}_P$, and $i_P(I_P)$ denotes the skyscraper sheaf (Ex. 1.17) given by I_P at the point P.

(e) Finally show that in the case of (d) the sequence

$$0 \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{K}/\mathcal{O}) \to 0$$

is exact. (This is an analogue of what is called the "first Cousin problem" in several complex variables. See Gunning and Rossi [1, p. 248].)

Solution: