Bilkent University

Take-Home Exam # 05
Math 633 Algebraic Geometry
Due on: 12 December 2019 Thursday - Class Time
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Solution Key

Q-1) Hartshorne, Example II.3.0.1, page 82.
After reading the three definitions on page 82 of Hartshorne's book, prove the following statements which are quoted in the above mentioned example without proof. Here $X = \operatorname{Spec} A$ is an affine scheme.

(i) X is irreducible if and only if the nilradical $\operatorname{nil} A$ of A is prime.

(ii) X is reduced if and only if $\operatorname{nil} A = 0$.

(iii) X is integral if and only if A is an integral domain.

Solution:

Throughout the following solutions $X = \operatorname{Spec} A$ is an affine scheme where A is a ring whose properties we will investigate.

(i)
($\Rightarrow$) X is irreducible. Assume that the nil radical of A, $\operatorname{nil} A = \sqrt{(0)} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$ is not prime.

Then there exis elements a_1, a_2 in A such that $a_1 a_2 \in \operatorname{nil} A$ but neither a_1 nor a_2 is in $\operatorname{nil} A$. Define two closed sunsets of $\operatorname{Spec} A$ as follows.

$$V(a_i) = \{\mathfrak{p} \in \operatorname{Spec} A \mid a_i \in \mathfrak{p}\}, \quad i = 1, 2.$$

For any $\mathfrak{p} \in \operatorname{Spec} A$, since $a_1 a_2 \in \operatorname{nil} A \subset \mathfrak{p}$, and since $\mathfrak{p}$ is prime, either $a_1 \in \mathfrak{p}$ or $a_2 \in \mathfrak{p}$. Thus $\mathfrak{p}$ is either in $V(a_1)$ or in $V(a_2)$. This shows that

$$\operatorname{Spec} A = V(a_1) \cup V(a_2).$$

But since a_1 is not in $\operatorname{nil} A$, there exists a $\mathfrak{p}$ such that $a_1 \notin \mathfrak{p}$. This shows that $\mathfrak{p} \notin V(a_1)$, and hence that $V(a_1)$ is a proper closed subset of $\operatorname{Spec} A$. Similarly $V(a_2)$ is also proper. But this violates the fact that $\operatorname{Spec} A$ was taken as irreducible.

Hence $\operatorname{nil} A$ must be prime.

($\Leftarrow$) X is not irreducible. Assume that the nil radical of A, $\operatorname{nil} A = \sqrt{(0)} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$ is prime.

Then there exist ideals $\mathfrak{a}_1, \mathfrak{a}_2$ in A such that the closed sets $V(\mathfrak{a}_1)$ and $V(\mathfrak{a}_2)$ of $\operatorname{Spec} A$ are proper, and $\operatorname{Spec} A = V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2)$. Hence there exist $\mathfrak{p}_1 \in V(\mathfrak{a}_1)$ which does not contain $\mathfrak{a}_2$, and $\mathfrak{p}_2 \in V(\mathfrak{a}_2)$ which does not contain $\mathfrak{a}_1$.

Thus there exist elements $a_1 \in \mathfrak{a}_1 \setminus \mathfrak{p}_2$, and $a_2 \in \mathfrak{a}_2 \setminus \mathfrak{p}_1$.

But $a_1 a_2 \in \mathfrak{a}_1 \cap \mathfrak{a}_2 \subset \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$. Hence $a_1 a_2 \in \operatorname{nil} A$. Since $\operatorname{nil} A$ is assumed to be prime either a_1 or a_2 is in $\operatorname{nil} A$. Say $a_1 \in \operatorname{nil} A \subset \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$. But this contradicts the choice of a_1.

Hence $\operatorname{nil} A$ is not prime.

(ii)
($\Rightarrow$) X is reduced. Assume $\operatorname{nil} A$ is not zero.

Then there exist a non-zero a in $\operatorname{nil} A$ and $a^n = 0$ for some integer $n > 1$. This means a belongs to every prime ideal $\mathfrak{p}$ of A. Now a has a natural image in $A_{\mathfrak{p}}$, but then its image is nilpotent, contrary to the fact that X is reduced.

Hence $\operatorname{nil} A = 0$.

($\Leftarrow$) X is not reduced.

Then there exist $\mathfrak{p} \in \operatorname{Spec} A$ such that $A_{\mathfrak{p}}$ has a nilpotent element. Let a/b be a nilpotent element in $A_{\mathfrak{p}}$. First a/b is not zero in $A_{\mathfrak{p}}$ means that $ra \neq 0$ for any $r \in A \setminus \mathfrak{p}$. To say that a/b is nilpotent means also that a^n/b^n is zero in $A_{\mathfrak{p}}$ for some integer $n > 1$. This means that $ra^n = 0$ for some $r \in A \setminus \mathfrak{p}$. Then clearly we also have $(ra)^n = 0$, which implies that the non-zero element ra is in $\operatorname{nil} A$.

Hence $\operatorname{nil} A$ is not zero.

(iii)
($\Rightarrow$) X is integral.

Then in particular $\mathcal{O}(X) = A$ is an integral domain.

($\Leftarrow$) A is an integral domain.

Consider the open set $D(f) \subset \operatorname{Spec} A$ for some non-zero f in A.

We claim that $\mathcal{O}(D(f)) = A_f$ is an integral domain. If not, there would be elements a/f^n and b/f^m such that $a^n b^m / f^{n+m}$ would be zero in A_f. But this means there is an integer ℓ such that $f^\ell a^n b^m = 0$ in A. This is not possible since A is an integral domain. Hence every $\mathcal{O}(D(f))$ is an integral domain.

Now for any open subset $U \subset \operatorname{Spec} A$ let $D(f) \subset U$ be an open subset. Any zero divisor in $\mathcal{O}(U)$ would be restricted to a zero divisor in $\mathcal{O}(D(f))$. But we just showed that no such zero divisors exist in the latter. Hence $\mathcal{O}(U)$ must be an integral domain too.

Hence X is integral.