MONOMIAL CURVES AND THE COHEN-MACAULAYNESS OF THEIR TANGENT CONES

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By

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February, 1999

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

MONOMIAL CURVES AND THE COHEN-MACAULAYNESS OF THEIR TANGENT CONES

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Ph. D. in Mathematics
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February, 1999

In this thesis, we show that in affine l-space with $l \geq 4$, there are monomial curves with arbitrarily large minimal number of generators of the tangent cone and still having Cohen-Macaulay tangent cone. In order to prove this result, we give complete descriptions of the defining ideals of infinitely many families of monomial curves. We determine the tangent cones of these families of curves and check the Cohen-Macaulayness of their tangent cones by using Gröbner theory. Also, we compute the Hilbert functions of these families of monomial curves. Finally, we make some genus computations by using the Hilbert polynomials for complete intersections in projective case and by using Riemann-Hurwitz formula for complete intersection curves of superelliptic type.

Keywords : Monomial curves, tangent cone, Cohen-Macaulay, Gröbner basis, Hilbert function, genus.

ÖZET

TEKTERİMLİ EĞRİLER VE TEĞET KONİLERİNİN COHEN-MACAULAY OLMA PROBLEMİ

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Bu tezde, $l \geq 4$ için her afin l-uzayında orijindeki teğet konileri Cohen-Macaulay olan ve bu teğet konilerinin minimum üreteç sayısı istenildiği kadar büyük olabilen tekterimli eğriler olduğunu gösteriyoruz. Bu sonuca ulaşmak için, sonsuz sayıda tekterimli eğri ailelerinin ideallerinin tam bir betimlemesini veriyoruz. Bu tekterimli eğri ailelerinin teğet konilerini belirlemek ve bunların Cohen-Macaulay olduklarını incelemek için Gröbner teorisini kullanıyoruz. Ayrıca, bu tekterimli eğri ailelerinin Hilbert fonksiyonlarını hesaplıyoruz. Son olarak, projektif uzayda eksiksiz kesişimlerin cinslerini Hilbert polinomlarını kullanarak, bazı süperelliptik eğrilerin cinslerini de Riemann-Hurwitz formülünden yararlanarak hesaplıyoruz.

Anahtar Kelimeler : Tekterimli eğriler, teğet koni, Cohen-Macaulay, Gröbner bazları, Hilbert fonksiyonu, cins.

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> L'algèbre n'est qu'une géométrie écrite; la géométrie n'est qu'une algèbre figurée.
>
> (Algebra is but written geometry; geometry is but drawn algebra.)
>
> Sophie Germain (1776-1831)

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Chapter 1

Introduction

Classification of singularities of varieties is an important problem in algebraic geometry. The tangent cone of a variety at a point and Cohen-Macaulayness are both important for the purpose of classifying singularities. Tangent cone of a variety at a point, which gives local information by approximating the variety at this point, is especially useful when the point is singular. Cohen-Macaulayness, which is a local property, also gives information about the singularity. Vasconcelos gives a beautiful characterization of Cohen-Macaulayness by expressing that although most of the Cohen-Macaulay rings are singular, their singularities may be said to be regular [43, p311]. Also, Cohen-Macaulayness makes it possible to have connections between geometry, algebra, combinatorics and homology, and this is a very rich ground for being able to do computations. Thus, our principal aim is to check the Cohen-Macaulayness of the tangent cone of a variety at the origin.

Let V be a variety in $\mathbb{A}^l$ and $I(V) \subset k[x_1, x_2, \cdots, x_l]$ be the defining ideal of the variety V. Let $P = (0, \cdots, 0)$ be a point of the variety and $\mathcal{O}_P$ be the local ring of the variety at P. We have the isomorphism

$$gr_{\mathfrak{m}}(\mathcal{O}_P) \cong k[x_1, x_2, \cdots, x_l]/I(V)_* \tag{1.1}$$

where $I(V)_*$ is the ideal generated by the polynomials f_* and f_* is the homogeneous summand of $f \in I(V)$ of least degree. Thus, checking the Cohen-Macaulayness of the tangent cone of a variety at the origin is checking the Cohen-Macaulayness of the associated graded ring of the local ring of the variety at the origin with respect to the maximal ideal.

It is an important problem to discover, whether the associated graded ring of a local ring $(R, \mathfrak{m})$ with respect to its maximal ideal $\mathfrak{m}$ is Cohen-Macaulay, since this property assures a better control on the blow-up of $Spec(R)$ along $V(\mathfrak{m})$. Moreover, the Cohen-Macaulaynes of the associated graded ring of a local ring with respect to the maximal ideal reduces the computation of the Hilbert function of a local ring to a computation of the Hilbert function of an Artin local ring [40]. The computation of the Hilbert function of an Artin ring is trivial, because it has a finite number of nonzero values.

We will study this problem for monomial curves. Our main interest is to check the Cohen-Macaulayness of the tangent cone of a monomial curve C, having parameterization

$$x_1 = t^{n_1},\ x_2 = t^{n_2},\ \cdots,\ x_l = t^{n_l} \tag{1.2}$$

where $n_1, n_2, \cdots, n_l$ are positive integers. In other words, we are interested in the Cohen-Macaulayness of $gr_{\mathfrak{m}}(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ or $k[x_1, x_2, \cdots, x_l]/I(C)_*$. The semigroup ring $k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]]$ shows the connection between a monomial curve and the additive semigroup generated by $n_1, n_2, \cdots, n_l$, which is denoted by $< n_1, n_2, \cdots, n_l >$ and is defined as

$$< n_1, n_2, \cdots, n_l >= \{n \mid n = \sum_{i=1}^{l} a_i n_i,\ a_i \in \mathbb{Z}_{\geq 0}\} \tag{1.3}$$

where $\mathbb{Z}_{\geq 0}$ denotes the nonnegative integers. This makes monomial curves a meeting ground for geometric, algebraic, and arithmetical techniques. In literature, there are many results concerning the Cohen-Macaulayness of the tangent cone of a monomial curve, which depend on studying the semigroup ring $< n_1, n_2, \cdots, n_l >$. We prefer to study the problem by using the ring $k[x_1, x_2, \cdots, x_l]/I(C)_*$, since we have the tools to find the generators of $I(C)_*$ and to check the regularity of an element by using Gröbner theory.

Our main result is to show that in affine l-space with $l \geq 4$, the minimal number of generators $\mu(I(C)_*)$ of a Cohen-Macaulay tangent cone of a monomial curve can be arbitrarily large. In order to prove this result, we determine the generators of the defining ideals of infinitely many families of monomial curves which have Cohen-Macaulay tangent cones.

The associated graded ring with respect to the maximal ideal of a local ring $(R, \mathfrak{m})$ gives some measure of the singularity at R [38]. This is a consequence of the fact that $gr_{\mathfrak{m}}(R)$ determines the Hilbert function of R. The Hilbert function of the local ring $(R, \mathfrak{m})$ is $H_R(n) = dim_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Thus, we compute the Hilbert series and polynomials of the families of monomial curves.

We are also interested in genus computations by using the Hilbert polynomials for complete intersections in projective case and by using Riemann-Hurwitz formula for complete intersection curves of superelliptic type.

In Chapter 2, we give the theory of monomial curves and mention the literature about monomial curves. We give the results about the generators of the defining ideals of monomial curves. We mention the connection between the semigroup $< n_1, n_2, \cdots, n_l >$ and a monomial curve, and naturally the famous Frobenius problem. Then we recall some open problems related with monomial curves. We also define tangent cone and prove some preparatory results.

In Chapter 3, we define the Cohen-Macaulayness and the significance of this property. We give two important checking criteria for the Cohen-Macaulayness of a graded ring.

In Chapter 4, we mention the importance of the problem of Cohen-Macaulayness of the tangent cone of a monomial curve, and discuss some entries from the vast literature about this problem. We first give a checking criteria for Cohen-Macaulayness of the tangent cone of a monomial curve (Theorem 4.4). We determine exactly the defining ideals of families of monomial curves (Proposition 4.10) and compute the generators of their tangent cones (Proposition 4.12). Our main theorem shows that all of these families of monomial curves have Cohen-Macaulay tangent cone at the origin (Theorem 4.7). This then proves our main claim.

In Chapter 5, we first find the Hilbert series and Hilbert polynomials of the families of monomial curves found in Chapter 4 by using the Cohen-Macaulayness of the tangent cone, see (5.2). We also make some genus computations by using Hilbert polynomials for complete intersections in the projective case (Theorem 5.2). Lastly, we make genus computations by using Riemann-Hurwitz formula for complete intersection curves of superelliptic type in the affine case (Theorem 5.10 and Corollary 5.11).

Chapter 2

Monomial Curves

The main geometric objects we are interested in are monomial curves. These curves are important since they provide a link between geometry, algebra and arithmetic. This is a consequence of the relationship between the monomial curves and semigroups generated by integers. The additive semigroup generated by $n_1, n_2, \cdots, n_l$ is denoted by $< n_1, n_2, \cdots, n_l >$ and is defined as

$$< n_1, n_2, \cdots, n_l >= \{n \mid n = \sum_{i=1}^{l} a_i n_i,\ a_i \in \mathbb{Z}_{\geq 0}\} \tag{2.1}$$

where $\mathbb{Z}_{\geq 0}$ denotes the nonnegative integers. A monomial curve C in affine l-space $\mathbb{A}^l$ has parameterization

$$x_1 = t^{n_1},\ x_2 = t^{n_2},\ \cdots,\ x_l = t^{n_l} \tag{2.2}$$

where $n_1, n_2, \cdots, n_l$ are positive integers with $gcd(n_1, n_2, \cdots, n_l) = 1$ and $\{n_1, n_2, \cdots, n_l\}$ is a minimal generator set for $< n_1, n_2, \cdots, n_l >$. The defining ideal $I(C) \subset k[x_1, x_2, \cdots, x_l]$ (where k is a field) is the prime ideal defined as

$$I(C) = \{f(x_1, x_2, \cdots, x_l) \in k[x_1, x_2, \cdots, x_l] \mid f(t^{n_1}, t^{n_2}, \cdots, t^{n_l}) = 0\} \tag{2.3}$$

where t is transcendental over k. The obvious isomorphism with x_i mapped to t^{n_i} for $1 \leq i \leq l$

$$k[x_1, x_2, \cdots, x_l]/I(C) \cong k[t^{n_1}, t^{n_2}, \cdots, t^{n_l}] \tag{2.4}$$

shows the relationship between the monomial curve and the semigroup. This isomorhism leads to isomorphism of local rings,

$$(k[x_1, x_2, \cdots, x_l]/I(C))_{(x_1, x_2, \cdots, x_l)} \cong k[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]_{(t^{n_1}, t^{n_2}, \cdots, t^{n_l})}$$

and the completions of the local rings give

$$k[[x_1, x_2, \cdots, x_l]]/I(C) \cong k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]]. \qquad (2.5)$$

2.1 Generators of $I(C)$

Herzog, in his paper [21] on generators and relations of abelian semigroups and semigroup rings studies the relations of finitely generated abelian semigroups and he shows that $I(C)$ is generated by binomials $F(\nu, \mu)$ of the form

$$F(\nu, \mu) = x_1^{\nu_1} x_2^{\nu_2} \cdots x_l^{\nu_l} - x_1^{\mu_1} x_2^{\mu_2} \cdots x_l^{\mu_l}, \quad \sum_{i=1}^{l} \nu_i n_i = \sum_{i=1}^{l} \mu_i n_i \qquad (2.6)$$

with $\nu_i \mu_i = 0$, $1 \leq i \leq l$. Herzog's proof is as follows with some slight modification.

Proposition 2.1 [21, Proposition 1.4] $I(C) = (\{F(\nu, \mu)\})$.

Proof: Let $J = (\{F(\nu, \mu)\})$. $J \subset I(C)$ is trivial. To prove the converse part, we grade the polynomial ring $k[x_1, x_2, \cdots, x_l]$ with *deg* $x_i = n_i$ so that the map $\varphi : k[x_1, x_2, \cdots, x_l] \to k[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]$ satisfying $\varphi(x_i) = t^{n_i}$ is a homogeneous homomorphism of degree 0. Let $f \in I(C)$ be a polynomial of degree d with respect to the defined grading. Then $f = \sum_{i=1}^{m} k_i x_1^{\nu_{i1}} x_2^{\nu_{i2}} \cdots x_l^{\nu_{il}}$ such that $n_1\nu_{i1} + n_2\nu_{i2} + \cdots + n_l\nu_{il} = d$ and since $f \in I(C)$, $\varphi(f) = \sum_{i=1}^{m} k_i t^d = 0$ and $\sum_{i=1}^{m} k_i = 0$. Thus,

$$\begin{aligned} f &= (\textstyle\sum_{i=1}^{m-1} k_i x_1^{\nu_{i1}} x_2^{\nu_{i2}} \cdots x_l^{\nu_{il}}) + k_m x_1^{\nu_{m1}} x_2^{\nu_{m2}} \cdots x_l^{\nu_{ml}} \quad (k_m = -\textstyle\sum_{i=1}^{m-1} k_i) \\ &= \textstyle\sum_{i=1}^{m-1} k_i (x_1^{\nu_{i1}} x_2^{\nu_{i2}} \cdots x_l^{\nu_{il}} - x_1^{\nu_{m1}} x_2^{\nu_{m2}} \cdots x_l^{\nu_{ml}}) \end{aligned}$$

This proves that every $f \in I(C)$ is generated by $F(\nu, \mu)$'s. $\square$

By using this proposition Bresinsky gives the following method for checking whether a given set of polynomials $\{f_1, f_2, \cdots, f_n\}$ generates $I(C)$ [8]. If it can be shown that for all $F(\nu, \mu) \in I(C)$, there is an element $f \in (f_1, f_2, \cdots, f_n)$ such that $F(\nu, \mu) - f = (\prod_{i=1}^{l} x_i^{a_i}) g$ with $g = 0$ or $g = F(\nu', \mu')$ with $\partial(F(\nu', \mu')) < \partial(F(\nu, \mu))$, then $\{f_1, f_2, \cdots, f_n\}$ generate $I(C)$. Here $\partial(F(\nu, \mu))$ is defined to be $\partial(F(\nu, \mu)) = \sum_{i=1}^{l} \nu_i n_i = \sum_{i=1}^{l} \mu_i n_i$. This proves that any binomial $F(\nu, \mu)$ can be generated by $\{f_1, f_2, \cdots, f_n\}$. Thus, $\{f_1, f_2, \cdots, f_n\}$

is a generator set for $I(C)$, since $F(\nu,\mu)$'s also generate $I(C)$. Bresinsky uses this technique to show that in affine l-space with $l \geq 4$, there are monomial curves having arbitrary large finite minimal sets of generators for the defining ideals [8]. He works with the monomial curves in affine 4-space with $n_1 = q_1q_2$, $n_2 = q_1d_1$, $n_3 = q_1q_2 + d_1$, $n_4 = q_2d_1$ where q_2 is even and $q_2 \geq 4$, $q_1 = q_2 + 1$ and $d_1 = q_2 - 1$. He shows that the number of the generators of the defining ideal of a monomial curve satisfying these conditions is greater than or equal to q_2. Thus, for arbitrary large q_2, we have arbitrary large number of generators. He also extends this result to higher dimensions.

Before we finish this section, we want to mention the relation between the symmetric semigroups and the number of generators of the defining ideals of corresponding monomial curves in affine 3-space and 4-space. Thus, we need more information about semigroups. It is well known that for a semigroup $< n_1, n_2, \cdots, n_l >$ with $gcd(n_1, n_2, \cdots, n_l) = 1$, there is an integer c not contained in the semigroup such that every integer greater than c is in the semigroup. This number $c = max\{\mathbb{Z}- < n_1, n_2, \cdots, n_l >\}$ is also known as the Frobenius number. An integer $n \in < n_1, n_2, \cdots, n_l >$, $0 \leq n < c$ is called a nongap, and an integer $n \notin < n_1, n_2, \cdots, n_l >$, $0 \leq n \leq c$ is called a gap [10]. The semigroup $< n_1, n_2, \cdots, n_l >$ is symmetric if and only if the number of gaps is equal to the number of nongaps. In [25], Kunz gives a beautiful algebraic characterization of symmetric semigroups by showing that $< n_1, n_2, \cdots, n_l >$ is symmetric if and only if $k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]]$ is Gorenstein. By using the notions of system of parameters and irreducible ideal, a quick definition of a Gorenstein local ring can be given as follows.

Definition 2.2 [4] *Let $(R, \mathfrak{m})$ be a local ring of dimension d. Any d-element set of generators of an $\mathfrak{m}$-primary ideal is called a system of parameters of the local ring $(R, \mathfrak{m})$.*

Definition 2.3 [4] *A proper ideal which cannot be expressed as an intersection of two ideals properly containing it is called as an irreducible ideal.*

Definition 2.4 *A local ring $(R, \mathfrak{m})$ is Gorenstein if and only if every system of parameters of the ring R generates an irreducible ideal.*

In our case, $R = k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]]$ and it has dimension 1. Thus, R is Gorenstein, if every principal ideal (r) generated by an element $r \in R$ with $\sqrt{(r)} = (t^{n_1}, t^{n_2}, \cdots, t^{n_l})$ is irreducible. In fact, we can define a Gorenstein ring as a Cohen-Macaulay ring, which has a set of parameters generating an irreducible ideal, and Cohen-Macaulayness is the subject of the next chapter.

Herzog shows that for a monomial curve C in (4.4) with $l = 3$, the defining ideal $I(C)$ has 2 generators if and only if the semigroup $< n_1, n_2, n_3 >$ is symmetric [21]. Bresinsky shows that for a monomial curve C in (4.4) with $l = 4$, if $< n_1, n_2, n_3, n_4 >$ is symmetric, then $I(C)$ is generated by 3 or 5 elements [9]. For higher dimensions, it is still an open question whether symmetry always implies the existence of a finite upper bound for the number of generators of the defining ideal of a monomial curve. Bresinsky has some results for the monomial curves in affine 5-space [10].

2.2 Frobenius Problem and Monomial Curves

For a semigroup $< n_1, n_2, \cdots, n_l >$ with $gcd(n_1, n_2, \cdots, n_l) = 1$, finding the Frobenius number c (largest integer that is not contained in the semigroup) is a very important problem. It is also known as *Frobenius's Money Change Problem* or *the Coin Problem.* The Frobenius problem has a solution in closed form for $l = 2$, $c = n_1n_2 - n_1 - n_2$. For $n > 2$, there are no known solutions in closed form. There is a vast literature about this problem. Heap and Lynn were the first to give a general algorithm [19]. In [41], Sertöz and Özlük, and in [28], Lewin proposed algorithms with different approaches. For more information about the literature, see [1]. Curtis showed that no "reasonable" closed formula is possible [14].

Morales gives an algorithmic algebraic solution for the Frobenius problem [34]. He first makes the observation that the Frobenius number of the semigroup $< n_1, n_2, \cdots, n_l >$ is the index of regularity of the Hilbert function of the ring $A = k[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]$. Hilbert function of the ring $A = k[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]$ is $H(n) = dim_k A_n$, where A_n denotes the set of homogeneous elements of A of degree n and thus $H(n)$ is either 0 or 1. Considering $A \cong k[x_1, x_2, \cdots, x_l]/I(C)$ as a quotient of the weighted polynomial ring $R = k[x_1, x_2, \cdots, x_l]$ with *deg* $x_i = n_i$, as an R-module A has syzgies (i.e. free resolution)

$$0 \to \oplus_i R[-n_{l-1,i}] \to \oplus_i R[-n_{l-2,i}] \to \cdots \to R \to A \to 0 \qquad (2.7)$$

where $R[-d]$ is called a twist of R, and $R[-d]_j = R_{j-d}$. Morales gives the formula for the Frobenius problem by using this resolution,

$$c = max_i\{n_{l-1,i}\} - \sum_{i=1}^{l} n_i. \qquad (2.8)$$

Example 2.5 *Let C be the monomial curve*

$$x_1 = t^6,\ x_2 = t^7,\ x_3 = t^8,\ x_4 = t^9.$$

From our computations with Macaulay [6], *the defining ideal $I(C) = (x_3^2 - x_2x_4, x_2x_3 - x_1x_4, x_2^2 - x_1x_3, x_1^3 - x_4^2)$ and $R/I(C) = k[x_1, x_2, x_3, x_4]/I(C)$ with deg $x_1 = 6$, deg $x_2 = 7$, deg $x_3 = 8$ and deg $x_4 = 9$ has syzgies*

$$0 \to R[-40] \oplus R[-41] \to R[-22] \oplus R[-23] \oplus R[-32] \oplus R[-33] \oplus R[-34] \to \\ R[-14] \oplus R[-15] \oplus R[-16] \oplus R[-18] \to R \to R/I(C) \to 0.$$

Thus, from the given formula

$$c = 41 - (6 + 7 + 8 + 9) = 11$$

Indeed, $< 6, 7, 8, 9 >= \{0, 6, 7, 8, 9, 12 + \mathbb{Z}_{\geq 0}\}$ and the largest integer not contained in $< 6, 7, 8, 9 >$ is 11.

2.3 Tangent Cone of a Monomial Curve at the Origin

Tangent cone of a variety at a point is a very important geometric object, which approximates the variety at this point. This gives local information especially when the point is singular. Thus, tangent cones are studied for the purpose of classifying singularities. The monomial curve given by (4.4) has a singular point at the origin if $n_i > 1$ for all $1 \leq i \leq l$. Thus, the tangent cone of a monomial curve at the origin is important for understanding monomial curves.

Let $V = Z(I)$ be a variety in affine l-space $\mathbb{A}^l$, where I is a radical ideal, and let $P = (0, \cdots, 0)$ be a point of the variety. We denote by f_* the homogeneous summand of f of least degree. For example, for the polynomial $f = x^2 - y^2 + x^3 + x^2y$, we have $f_* = x^2 - y^2$.

Definition 2.6 [31] *Let I_* be the ideal generated by the polynomials f_* for $f \in I$. The geometric tangent cone $C_P(V)$ at P is $V(I_*)$, and the tangent cone is the pair $(V(I_*), k[x_1, \cdots, x_l]/I_*)$.*

Definition 2.7 *The minimal number of generators of I_* which is denoted by $\mu(I_*)$ is called the minimal number of generators of the tangent cone at the origin.*

The associated graded ring of the coordinate ring $k[x_1, x_2, \cdots, x_l]/I(V)$ of a variety V with respect to the maximal ideal $\mathfrak{m}$ makes it possible to study the tangent cone of the variety V at the origin in a different manner. The definition of the associated graded ring with respect to any ideal is as follows.

Definition 2.8 *Let A be a ring and I be any ideal of A. The associated graded ring with respect to the ideal I is*

$$gr_I(A) = \oplus_{i=0}^{\infty} I^i/I^{i+1} = (A/I) \oplus (I/I^2) \oplus \cdots \tag{2.9}$$

We generally work with the associated graded ring of a local ring with respect to its maximal ideal. If a local ring is obtained from a ring by localizing it at one of its maximal ideals, then the associated graded ring of the ring with respect to this maximal ideal and the associated graded ring of the local ring with respect to its maximal ideal are isomorphic and this is the following proposition.

Proposition 2.9 [31, p72] *Let A be any ring and $\mathfrak{m}$ be any maximal ideal of A. If $B = A_{\mathfrak{m}}$ and $\mathfrak{n} = \mathfrak{m}B$, then $gr_{\mathfrak{n}}(B) = \oplus_{i=0}^{\infty} \mathfrak{n}^i/\mathfrak{n}^{i+1} \cong \oplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$*

Proof: We first prove that there is an isomorphism between $\mathfrak{m}^r/\mathfrak{m}^k$ and $\mathfrak{n}^r/\mathfrak{n}^k$ for all integers r, k, with $0 \leq r < k$, from which the proposition follows immediately. Let $\varphi_k : A \to A_{\mathfrak{m}}/\mathfrak{n}^k$ be the natural map such that for any $a \in A$, $\varphi_k(a)$ is the residue class of $\frac{a}{1}$ in $A_{\mathfrak{m}}/\mathfrak{n}^k$. Let us show that the map is surjective. Let $\frac{a}{s}$ be any element in $A_{\mathfrak{m}}$. Since $\mathfrak{m}$ is maximal and $s \notin \mathfrak{m}$, we have $(s) + \mathfrak{m} = A$. Thus, $(s) + \mathfrak{m}^k = A$ because no maximal ideal contains both s and $\mathfrak{m}^k$. Then there exist $b \in A$ and $m \in \mathfrak{m}^k$ such that $bs + m = 1$. This means that $\varphi_k(b)$ is $\frac{1}{s}$ and $\varphi_k(ba) = \frac{a}{s}$, which proves the surjectivity. Now it is time to find the kernel of this map. If $\varphi_k(a)$ is 0 in $A_{\mathfrak{m}}/\mathfrak{n}^k$, then $\frac{a}{1} \in \mathfrak{n}^k$, so that we have $a \in \mathfrak{m}^k$ and the kernel of the map is $\mathfrak{m}^k$. Thus, for all $k \in \mathbb{Z}_{\geq 0}$, the map

$$\overline{\varphi_k} : A/\mathfrak{m}^k \to A_{\mathfrak{m}}/\mathfrak{n}^k$$

is an isomorphism. By using this isomorphism and the exact commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathfrak{m}^r/\mathfrak{m}^k & \longrightarrow & A/\mathfrak{m}^k & \longrightarrow & A/\mathfrak{m}^r & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathfrak{n}^r/\mathfrak{n}^k & \longrightarrow & A/\mathfrak{n}^k & \longrightarrow & A/\mathfrak{n}^r & \longrightarrow 0,
\end{array}$$

we obtain the isomorphism between $\mathfrak{m}^r/\mathfrak{m}^k$ and $\mathfrak{n}^r/\mathfrak{n}^k$ for all integers r, k, with $0 \leq r < k$. This isomorphism proves the proposition. □

Thus, if $V = Z(I)$ is a variety in affine l-space $\mathbb{A}^l$, where I is a radical ideal, and $P = (0, \cdots, 0)$ is a point of the variety, then $\mathcal{O}_P = (k[x_1, x_2, \cdots, x_l]/I)_{(x_1, x_2, \cdots, x_l)}$ and from Proposition 2.9 $gr_{\mathfrak{n}}(\mathcal{O}_P) = \oplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$, where $\mathfrak{m}$ is the maximal ideal in $k[x_1, \cdots, x_l]/I$ corresponding to P and $\mathfrak{n} = \mathfrak{m}\mathcal{O}_P$. With this notation, the following proposition gives the relationship between tangent cone and the associated graded ring with respect to the maximal ideal of the local ring of V at P.

Proposition 2.10 [31] *The map $k[x_1, x_2, \cdots, x_l]/I_* \to gr_{\mathfrak{n}}(\mathcal{O}_P)$ sending the class of x_i in $k[x_1, x_2, \cdots, x_l]/I_*$ to the class of x_i in $gr_{\mathfrak{n}}(\mathcal{O}_P)$ is an isomorphism.*

Proof: $\mathfrak{m}$ is the maximal ideal in $k[x_1, \cdots, x_n]/I$ corresponding to $P = (0, 0, \cdots, 0)$. Then from Proposition 2.9,

$$\begin{aligned}
gr_{\mathfrak{n}}(\mathcal{O}_P) &= \sum_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1} \\
&= \sum_{i=0}^{\infty} (x_1, x_2, \cdots, x_l)^i/(x_1, x_2, \cdots, x_l)^{i+1} + I \cap (x_1, x_2, \cdots, x_l)^i \\
&= \sum_{i=0}^{\infty} (x_1, x_2, \cdots, x_l)^i/(x_1, x_2, \cdots, x_l)^{i+1} + I_i
\end{aligned}$$

where I_i is the homogeneous piece of I_* of degree i (namely, the subspace of I_* consisting of homogeneous polynomials of degree i). But

$$(x_1, x_2, \cdots, x_l)^i/(x_1, x_2, \cdots, x_l)^{i+1} + I_i = i^{th} \text{ homogeneous piece of } k[x_1, x_2, \cdots, x_l]/I_*.$$

□

Let C be the monomial curve given in (4.4). From (2.4), we have $k[x_1, x_2, \cdots, x_l]/I(C) \cong k[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]$, and if $\mathcal{O}_P$ is the local ring at the origin, then from (2.5) $\widehat{\mathcal{O}_P} \cong k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]]$. Let $\mathfrak{m}$ denote both the maximal ideal of the local ring $\mathcal{O}_P$ and the maximal ideal of the local ring $\widehat{\mathcal{O}_P}$.

By using the properties of completion [17, p195] and proposition (2.10)

$$gr_{\mathfrak{m}}(\mathcal{O}_P) \cong gr_{\mathfrak{m}}(\widehat{\mathcal{O}_P}) \cong gr_{\mathfrak{m}}(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]]) \cong k[x_1, x_2, \cdots, x_l]/I(C)_*. \tag{2.10}$$

This isomorphism shows that the tangent cone of a monomial curve at the origin can both be studied by using the ring $gr_{\mathfrak{m}}(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ or the ring $k[x_1, x_2, \cdots, x_l]/I(C)_*$.

Chapter 3

Cohen-Macaulayness

Cohen-Macaulayness is a very important property, which makes it possible to have connections between geometry, algebra, combinatorics and homology. In general, it is important to know whether the local ring of a variety at a point is Cohen-Macaulay, because these properties can give some rough classification of singularities (Gorenstein singularities, normal singularities, etc.) and also varieties all of whose local rings are Cohen-Macaulay have some special properties [26, p190]. To support our interest in Cohen-Macaulay rings, we can quote Eisenbud [17, p447]:

> "*These rings are important because they provide a natural context, broad enough to include the rings associated to many interesting classes of singular varieties and schemes, to which many results about regular rings can be generalized.*"

Vasconcelos makes a similar comment by expressing that although most of the Cohen-Macaulay rings are singular, their singularities may be said to be regular [43, p311].

Geometrically, Cohen-Macaulayness is also an important condition; if a local ring of a point P on a variety X is Cohen-Macaulay, then P cannot lie on two components of different dimensions, [17, p454].

Reminding that Cohen-Macaulay rings include rings of polynomials over a field, rings of formal power series over fields and convergent power series, Vasconcelos considers the Cohen-Macaulay rings as a meeting ground for algebraic, analytic and geometric techniques [43, p311]. Thus, Hochster is quite right when he says "life is really worth living" in a Cohen-Macaulay ring [11, p56].

3.1 Definition and Significance

Cohen-Macaulay rings can be characterized in many different ways with different approaches. Vasconcelos mentions a theorem of Paul Roberts as one of the fastest definitions of a Cohen-Macaulay local ring, which says that a Noetherian local ring R is Cohen-Macaulay if and only if it admits a nonzero finitely generated module E of finite injective dimension [43, p311]. We prefer another definition which depends on depth and height of ideals in the ring. Thus, we need some definitions.

Definition 3.1 *Let R be a ring. A regular sequence on R (or an R-sequence) is a set $\{a_1, a_2, \cdots, a_n\}$ of elements of R with the following properties:*

i) $R \neq (a_1, a_2, \cdots, a_n)R$,

ii) The jth element a_j is not a zero-divisor on the ring $R/(a_1, a_2, \cdots, a_{j-1})R$ for $j = 1, 2, \cdots, n$, where for $j = 1$, we set $(a_1, a_2, \cdots, a_{j-1})$ to be the zero ideal.

Remark 3.2 *For a ring R, every definition and theorem in this section can be generalized to an R-module M, where $M = R$ is a special case, but we prefer giving the definitions and theorems only for R, since we are interested in rings.*

The lengths of all the maximal R-sequences (where R is Noetherian) in an ideal I are the same, which is a result of the following theorem. The theorem uses the Koszul complex and homology of the Koszul complex. Thus, before the theorem, we recall the construction of Koszul complex.

Definition 3.3 [27, 852] *Let R be a commutative ring and let $a_1, a_2, \cdots, a_n \in R$. The Koszul complex $K(\underline{a}; R) = K(a_1, a_2, \cdots, a_n)$ is defined as follows:*

$K_0(a_1, a_2, \cdots, a_n) = R$;

$K_1(a_1, a_2, \cdots, a_n) =$ *the free R-module E with basis $\{e_1, e_2, \cdots, e_n\}$;*

$K_p(a_1, a_2, \cdots, a_n)$ = the free R-module $\bigwedge^p E$ with basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p}\},\ i_1 < \cdots < i_p$;

$K_n(a_1, a_2, \cdots, a_n)$ = the free R-module $\bigwedge^n E$ of rank 1 with basis $e_1 \wedge \cdots \wedge e_r$.

The boundary maps are defined by $d_1(e_i) = a_i$ and in general

$$d_p : K_p(a_1, a_2, \cdots, a_n) \to K_{p-1}(a_1, a_2, \cdots, a_n)$$

by

$$d_p(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \textstyle\sum_{j=1}^{p} (-1)^{j-1} a_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p}.$$

Since $d_{p-1}d_p = 0$, we have a complex

$$0 \to K_n(\underline{a}; R) \to \cdots \to K_p(\underline{a}; R) \to \cdots \to K_1(\underline{a}; R) \to R \to 0. \qquad (3.1)$$

The $p-th$ homology of the Koszul complex is $H^p(K(\underline{a}; R) = (Kerd_p)/(Imd_{p+1})$.

Theorem 3.4 [43, p304] *Let R be a Noetherian ring and $a_1, a_2, \cdots, a_n$ be elements in R. Let $K(a_1, \cdots, a_n)$ be the corresponding Koszul complex and let p be the largest integer for which $H_p(K(a_1, \cdots, a_n)) \neq 0$. Then every maximal R-sequence in $I = (a_1, \cdots, a_n) \subset R$ has length $n - p$.*

Proof: See [43, p304]. □

This theorem gives us the opportunity to define the depth of an ideal of a Noetherian ring.

Definition 3.5 *Let R be a Noetherian ring. The* depth *of an ideal I is the length of any maximal R-sequence in I.*

Some mathematicians prefer to use the term "grade" instead of the depth of an ideal I, and they reserve the term "depth" for the depth of the maximal ideal of a local ring. We prefer to use "depth" in all cases.

Definition 3.6 *Let R be a commutative ring, and $\mathfrak{p}$ be a prime ideal. The height of $\mathfrak{p}$ is the supremum of the lenths l of strictly descending chains*

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_l$$

of prime ideals. The height of any ideal I is the infimum of the heights of the prime ideals containing I.

In general, we have the inequalities

$$\text{depth}(I) \leq \text{height}(I) \leq \mu(I) \tag{3.2}$$

where $\mu(I)$ is the minimal number of generators of I. The relation height$(I) \leq \mu(I)$ is a direct consequence of Krull's theorem, see [4, p13]. For the proof of the relation depth$(I) \leq$ height(I), see [4, p108].

We can now define a Cohen-Macaulay ring.

Definition 3.7 *A Noetherian ring R is Cohen-Macaulay if* depth(I) =height(I) *for each ideal I of R.*

Proposition 3.8 [4, 113] *Let R be a Noetherian ring. The following properties are equivalent.*

i) R is a Cohen-Macaulay ring,

ii) for every maximal ideal $\mathfrak{m}$ of R, depth$(\mathfrak{m})$ =height$(\mathfrak{m})$*,*

iii) for every prime ideal p of R, depth$(\mathfrak{p})$ =height$(\mathfrak{p})$*,*

iii) for every ideal I of R, depth(I) =height(I)*.*

Proof: See [4, p114]. □

From this proposition, if R is a local ring, it is sufficient to test the equation depth$(\mathfrak{m})$ =height$(\mathfrak{m})$ for its maximal ideal. On a local ring R with maximal ideal $\mathfrak{m}$, depth$(\mathfrak{m})$=depth(R) and height$(\mathfrak{m})$=dim(R) so that R is Cohen-Macaulay if and only if depth(R)=dim(R). Let R be a Noetherian ring and $\mathfrak{m}$ be any maximal ideal. What makes Cohen-Macaulayness a local property is the equality depth$(\mathfrak{m})$=depth$(R_\mathfrak{m})$, which follows from the properties of Koszul complex.

3.2 Checking Criteria for Graded Rings

Being familiar with the notion of Cohen-Macaulayness, we can give some criteria for checking the Cohen-Macaulayness of graded rings, since in the next chapter, we will be interested in the Cohen-Macaulayness of some graded rings. We need some more definitions.

Definition 3.9 *A graded ring is a ring A together with a direct sum decomposition*

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots \text{ as commutative groups}$$

such that $A_iA_j \subset A_{i+j}$ for $i, j \geq 0$. Elements of A_r are called elements of degree r.

For the rest of this section, let us assume that $A_0 = k$, where k is a field and A is a graded algebra generated over k by elements of degree 1.

Definition 3.10 *The numerical function $H_A(n)$ =$dim_k(A_n)$ for all $n \in \mathbb{Z}_{\geq 0}$ is called the Hilbert function of A, and $H_A(t) = \sum_{n\in\mathbb{Z}_{\geq 0}} H_A(n)t^n$ is called the Hilbert series of A. The polynomial $P_A(n)$ satisfying $P_A(n) = H_A(n)$ for sufficiently large n is the Hilbert polynomial of A.*

The existence of the Hilbert polynomial was shown by Hilbert, and we know more about the Hilbert polynomial.

Theorem 3.11 [43, p342] *Let the graded ring A have dimenson d.*

i) $H_A(t) = h_A(t)/(1-t)^d$, *where $h_A(t)$ is a polynomial,*

ii) *the Hilbert polynomial $P_A(n)$ of A is of degree $d-1$ with leading coefficient $h_A(1)/(d-1)!$.*

Proof: See [43, p342]. □

Definition 3.12 *With this notation the multiplicity of a graded ring A is defined to be $h_A(1)$ and it is denoted by $e(A)$. The polynomial $h_A(t)$ is called the h-polynomial of A.*

Definition 3.13 *Let A be a graded ring of dimension d. A system of parameters for A is a set of homogeneous elements $a_1, \cdots, a_d \in A$ such that* $\dim A/(a_1, \cdots, a_d)$ *is 0.*

First important criterion for checking the Cohen-Macaulayness of a graded ring is the following proposition.

Proposition 3.14 [43, p56] *Suppose that $a_1, \cdots, a_d$ is a homogeneous system of parameters for a graded ring A. Then A is a Cohen-Macaulay if and only if $a_1, \cdots, a_d$ is a regular sequence. Moreover, if $a_1, a_2, \cdots, a_d$ are of degree 1, and if $H_A(t) = (h_0+h_1t+\cdots+h_rt^r)/(1-t)^d$, then the polynomial $h_0+h_1t+\cdots+h_rt^r$ is the Hilbert series of the Artin ring $A/(a_1, \cdots, a_d)$. In particular, $h_i \geq 0$.*

Proof: The first assertion can be proved by using the relation between the notion of flatness and Cohen-Macaulayness. The other assertions can be proved by using the exact sequence induced by an element of degree 1 which is regular on A,

$$0 \to A(-1) \to A \to A/(z) \to 0$$

which gives $H_{A/(z)}(t) = (1-t)H_A(t)$. □

Vasconcelos also remarks that the condition $h_i \geq 0$ can be used as a pretest for Cohen-Macaulayness.

Another useful test for checking the Cohen-Macaulayness of a graded ring of the form $k[x_1, \cdots, x_n]/I$, where I is a homogeneous ideal is the following proposition.

Proposition 3.15 [6, p117] *Let $A = k[x_1, \cdots, x_n]/I$, where I is a homogeneous ideal, and let* $\dim A = d$. *Then A is Cohen-Macaulay if and only if $e(A) = \dim_k A/(a_1, \cdots, a_d)$, for some (and hence all) system of parameters $a_1, \cdots, a_d$ of degree 1.*

Proof: We adapt the proof of a similar condition for a local ring to the graded ring A, see [4, p117]. Let A be Cohen-Macaulay ring and let $a_1, \cdots, a_d$ be a system of parameters of degree 1. It follows from Proposition 3.14 that $a_1, \cdots, a_d$ is a regular sequence. If $a_1, \cdots, a_d$ is a regular sequence, then A is isomorphic to a polynomial ring $R[T_1, \cdots, T_d]$ with variables $T_1, \cdots, T_d$ of degree 1, and $R = A/(a_1, \cdots, a_d)$. This can be shown by considering the map $\varphi : R[T_1, \cdots, T_d] \to A$ with $\varphi(T_i) = a_i$ for $1 \leq i \leq d$. This is a map of homogeneous degree 0 and gives the isomorphism

$$(A/(a_1, \cdots, a_d))[T_1, \cdots, T_d] \cong A.$$

Then

$$\begin{aligned} \dim_k(A_n) &= \textstyle\sum_{i=1}^{\dim_k A/(a_1,\cdots,a_d)} \begin{pmatrix} n - d_i + d - 1 \\ d - 1 \end{pmatrix} \\ &= (\dim_k A/(a_1, \cdots, a_d)) \tfrac{n^{d-1}}{(d-1)!} + \cdots \end{aligned}$$

where d_i's are degrees of the k-basis elements of $A/(a_1, \cdots, a_d)$. Hence, $e(A) = \dim_k A/(a_1, \cdots, a_d)$ follows immediately.

The converse part of the proof can be done with a similar approach. Let $a_1, \cdots, a_d$ be a set of parameters of the ring A and let $q = (a_1, \cdots, a_d)$. We must show that $a_1, \cdots, a_d$ is a regular sequence. Let $\varphi : (A/q)[T_1, \cdots, T_d] \to A$ be the map such that $\varphi(T_i) = a_i$ for $1 \leq i \leq d$. Let $J = Ker(\varphi)$. We will show that if $J \neq 0$, $e(A) < \dim_k A/q$. If $J \neq 0$, then it contains at least one form of degree p. Consequently,

$$\begin{aligned} \dim_k(A_n) &\leq \textstyle\sum_{i=1}^{\dim_k A/q} \begin{pmatrix} n - d_i + d - 1 \\ d - 1 \end{pmatrix} - \begin{pmatrix} n - p + d - 1 \\ d - 1 \end{pmatrix} \\ &= (\dim_k A/q - 1) \tfrac{n^{d-1}}{(d-1)!} + \cdots \end{aligned}$$

From this equation, we obtain $e(A) < \dim_k A/q = \dim_k A/(a_1, \cdots, a_d)$, which is a contradiction. Thus, $J = 0$ and $a_1, \cdots, a_d$ is a regular sequence. Hence, A is a Cohen-Macaulay ring. $\square$

Chapter 4

Cohen-Macaulayness of the Tangent Cone

Our main interest is checking the Cohen-Macaulayness of the tangent cone of a monomial curve. In other words, we are interested in the Cohen-Macaulayness of the associated graded ring of the local ring of a monomial curve at the origin with respect to its maximal ideal. In general, it is an important problem to discover, whether the associated graded ring of a local ring $(R, \mathfrak{m})$ with respect to its maximal ideal $\mathfrak{m}$ is Cohen-Macaulay, since this property assures a better control on the blow-up of $Spec(R)$ along $V(\mathfrak{m})$. The blow-up of $Spec(R)$ along $V(\mathfrak{m})$ is $Proj(R[\mathfrak{m}t])$ and if the associated graded ring of R with respect to the maximal ideal $\mathfrak{m}$ ($gr_{\mathfrak{m}}(R)$) is Cohen-Macaulay, then $R[\mathfrak{m}t]$ is Cohen-Macaulay [20, p86]. Also, the exceptional divisor of the blow-up is nothing but the projective variety associated to the graded ring with respect to the maximal ideal $gr_{\mathfrak{m}}(R)$. For more information on the blow-up algebra, see[17, p148].

The associated graded ring with respect to the maximal ideal of a local ring $(R, \mathfrak{m})$ gives some measure of the singularity at R [38]. This is a consequence of the fact that $gr_{\mathfrak{m}}(R)$ determines the Hilbert function of R. The Hilbert function of the local ring $(R, \mathfrak{m})$ is $H_R(n) = dim_{R/\mathfrak{m}}\mathfrak{m}^n/\mathfrak{m}^{n+1}$, in other words it is the dimension of the n-th component of $gr_{\mathfrak{m}}(R)$ as a vector space over $R/\mathfrak{m}$. The Hilbert function of R measures the deviation from a regular local ring [40]. Cohen-Macaulaynes of the associated graded ring of a local ring with respect to the maximal ideal reduces the computation of the Hilbert function of a local ring to a computation of the Hilbert function of an Artin local ring [40]. The computation of the Hilbert function of an Artin ring is trivial, because it has a finite number of nonzero values. To see how this reduction can be done, let

$gr(\mathfrak{m}) = \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots$ be the maximal ideal of the associated graded ring $gr_{\mathfrak{m}}(R)$. If $gr(\mathfrak{m})$ contains a nonzero divisor, then it contains a homogeneous nonzero divisor $\overline{x} \in \mathfrak{m}^t/\mathfrak{m}^{t+1}$ for some $t \geq 1$ and multiplication by $\overline{x}$ is a one-to-one vector space homomorphism of $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ to $\mathfrak{m}^{n+t}/\mathfrak{m}^{n+t+1}$ for all $n \geq 0$. Thus, if x is any lifting of $\overline{x}$ to R, then $gr_{\mathfrak{m}}(R)/(\overline{x}) \cong gr_{\mathfrak{m}}(R/(x))$, where $dim(R/(x)) = dimR - 1$. For the details of these arguments, see [38, Lemma 0.1]. If $gr_{\mathfrak{m}}(R)$ is Cohen-Macaulay and $dimR = d$, then $gr(\mathfrak{m})$ contains a regular sequence $\overline{x_1}, \cdots, \overline{x_d}$ of length d. By using the argument above, if $x_1, \cdots, x_d$ are liftings of $\overline{x_1}, \cdots, \overline{x_d}$, then $gr_{\mathfrak{m}}(R)/(\overline{x_1}, \cdots, \overline{x_d}) \cong gr_{\mathfrak{m}}(R/(x_1, \cdots, x_d))$. From Theorem 3.14, $H_R(t) = H_{R/(x_1,\cdots,x_d)}(t)/(1-t)^d$ where $H_R(t)$ is the Hilbert series of the ring R and $H_{R/(x_1,\cdots,x_d)}(t)$ is the Hilbert series of the Artin local ring $R/(x_1, \cdots, x_d)$.

Thus, it is an important problem to discover which local rings have Cohen-Macaulay associated graded rings with respect to the maximal ideal. We will consider this problem for monomial curves.

4.1 Literature

In literature, there are some results considering the Cohen-Macaulayness of the associated graded ring $gr_{\mathfrak{m}}(R)$ of a local ring $(R, \mathfrak{m})$ having dimension d. In [37], Sally proves that $gr_{\mathfrak{m}}(R)$ is Cohen-Macaulay, if $\mu(\mathfrak{m}) = d$, $d+1$ and $e(R)+d-1$, where $\mu(\mathfrak{m})$ is the minimal number of the generators of the maximal ideal $\mathfrak{m}$ of R and $e(R)$ is the multiplicity of R. This result can be applied to Arf rings such that for any Arf ring $(R, \mathfrak{m})$ having dimension 1, $gr_{\mathfrak{m}}(R)$ is Cohen-Macaulay because $e(R) = \mu(\mathfrak{m})$ for an Arf ring, [1] and [29]. Sally also shows that if $(R, \mathfrak{m})$ is a d-dimensional local Gorenstein ring and $\mu(\mathfrak{m}) = d$, $d+1$, $e(R)+d-3$ or $e(R)+d-2$, then $gr_{\mathfrak{m}}(R)$ is Cohen-Macaulay, see [39] and [40].

We are interested in the problem of checking the Cohen-Macaulayness of the tangent cone of a monomial curve C having parameterization

$$x_1 = t^{n_1},\ x_2 = t^{n_2},\ \cdots,\ x_l = t^{n_l} \tag{4.1}$$

where $n_1 < n_2 < \cdots < n_l$ are positive integers with $gcd(n_1, n_2, \cdots, n_l) = 1$ and $\{n_1, n_2, \cdots, n_l\}$ is a minimal generator set for $< n_1, n_2, \cdots, n_l >$. Let us recall the notation. $I(C)$ is the defining ideal of C. $I(C)_*$ is the ideal generated by the polynomials f_* for f in $I(C)$, where f_* is the homogeneous summand of f of least degree, and $\mu(I(C)_*)$ is the minimal number of generators of ideal $I(C)_*$ which is also called the tangent cone of the monomial curve C. The isomorphism in (2.10) shown as a consequence of Proposition 2.10 makes it possible to study this problem both by considering the associated graded ring of $R = k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]]$ with respect to the maximal ideal $\mathfrak{m} = (t^{n_1}, t^{n_2}, \cdots, t^{n_l})$ ($gr_{\mathfrak{m}}(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$) or by considering the ring $k[x_1, x_2, \cdots, x_l]/I(C)_*$. In literature, generally $gr_{\mathfrak{m}}(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ is studied, because without the help of Gröbner theory, it is very difficult to find the generators of $I(C)_*$, but we prefer to study the ring $k[x_1, x_2, \cdots, x_l]/I(C)_*$ with the help of Gröbner theory.

Hironaka was the first, who introduced the concept of standard base in his famous paper, [23]. In our case, a set of generators $f_1, \cdots, f_t$ of $I(C)$ is a standard base, if $f_{1*}, \cdots, f_{t*}$ is a set of generators for $I(C)_*$. Herzog gives a characterization of the standard base by using the concept of super-regular sequence, and applies this characterization to monomial curves in order to obtain a checking criterion for the Cohen-Macaulayness of $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ [22]. In [18], Garcia obtains the same checking criterion by studying the semigroup $< n_1, n_2, \cdots, n_l >$. He considers the subsets $\Gamma(k) \subset < n_1, n_2, \cdots, n_l >$ defined as $\Gamma(k) = \{\sum_{i=1}^{l} a_i n_i$ such that $a_i \in \mathbb{Z}_{\geq 0}$ and $\sum_{i=1}^{l} a_i \geq k\}$, and he finds criteria for $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ to be Cohen-Macaulay in terms of the integers $n_1, n_2, \cdots, n_l$.

Cavaliere and Niesi also attack the same problem by studying the semigroup ring $k[S]$ where $S \subset \mathbb{N}^2$ is generated by $(n_1, 0), (n_2, n_2 - n_1), \cdots, (n_l, n_l - n_1), (0, n_1)$, [12]. This is a consequence of a theorem of Hochster which says that $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ is Cohen-Macaulay if and only if the Rees ring $A = \oplus_{i=-\infty}^{\infty} \mathfrak{m}^i$ is Cohen-Macaulay, see [24] and the isomorphism between the Rees ring A and $k[S]$. Cavaliere and Niesi give a simple criterion for the Cohen-Macaulyness of $k[S]$ and thus for the Cohen-Macaulyness of $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ by introducing the notion of standard basis for S. Molinelli and Tamone use this criterion to show that if $n_1, n_2, \cdots, n_l$ are arithmetic sequence, then $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ is Cohen-Macaulay, [32]. Recently, Molinelli, Patil and Tamone give a necessary and sufficient condition for $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ to be Cohen-Macaulay, if $n_1, n_2, \cdots, n_l$ is an almost arithmetic sequence, in other words $n_1, \cdots, n_{l-1}$ is an arithmetic sequence. Thus, for the case of monomial space curves, they determine exactly when $gr_m(k[[t^{n_1}, t^{n_2}, t^{n_3}]])$ is Cohen-Macaulay, [33]. In fact, Robbiano and Valla has determined before exactly when $gr_m(k[[t^{n_1}, t^{n_2}, t^{n_3}]])$ is Cohen-Macaulay by using a more complex approach [36].

In [36], Robbiano and Valla give a characterization of standard bases, which relies on homological methods and is particularly useful while dealing with determinantal ideals. They show that if $I = (f_1, \cdots, f_t)$, then $f_1, \cdots, f_t$ is a standard base if and only if all the homogeneous syzygies of $f_{1*}, \cdots, f_{t*}$ can be lifted through a suitable map to syzygies of $f_1, \cdots, f_t$. By using this theory with Herzog's [21] description of the defining ideals of monomial curves for $l = 3$, they give a classification of these curves by their tangent cones at the origin. **They prove that a monomial curve C having parameterization**

$$x_1 = t^{n_1},\ x_2 = t^{n_2},\ x_3 = t^{n_3} \tag{4.2}$$

has Cohen-Macaulay tangent cone at the origin if and only if minimal number of generators of the tangent cone, that is $\mu(I(C)_*)$ is less than or equal to three.

Our main theorem may be considered as the generalization of Robbiano and Valla's investigation for all the higher dimensions. **We investigate and show that in higher dimensions, minimal number of generators of a Cohen-Macaulay tangent cone of a monomial curve can be arbitrarily large**. In other words, in l-space with $l > 3$, there are monomial curves with arbitrarily large $\mu(I(C)_*)$ and still having Cohen-Macaulay tangent cones [3].

4.2 When is $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ CM?

In this section, we state and prove a theorem, which we use for checking the Cohen-Macaulayness of the tangent cone of a monomial curve C by considering the ideal $I(C)_*$. The theorem checks the Cohen-Macaulayness of the tangent cone of a monomial curve by using a Gröbner basis with respect to a special monomial order. The standard reference for material related to Gröbner theory is [13]. Here, we only give the definitions of leading term and reverse lexicographic order.

Definition 4.1 *Let* $f = \sum_i c_i x_1^{a_{1i}} x_2^{a_{2i}} \cdots x_l^{a_{li}}$ *be a nonzero polynomial in* $k[x_1, x_2, \cdots, x_l]$. *If for* $i = i_m$ *the l-tuple* $(a_{1i_m}, a_{2i_m}, \cdots, a_{li_m})$ *is maximum among the l-tuples* $(a_{1i}, a_{2i}, \cdots, a_{li})$ *with respect to a given monomial order and* $c_{i_m} \neq 0$*, then* $c_{i_m} x_1^{a_{1im}} x_2^{a_{2im}} \cdots x_l^{a_{lim}}$ *is defined as the leading term of f with respect to this monomial order and denoted as* $in(f) = c_{i_m} x_1^{a_{1im}} x_2^{a_{2im}} \cdots x_l^{a_{lim}}$.

Definition 4.2 [13, p57] (Graded Reverse Lex Order) *Let* $\alpha, \beta \in (\mathbb{Z}_{\geq 0})^l$. *We say* $\alpha >_{grevlex} \beta$ *if*

$$\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$$

or if $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ *and in* $(\alpha_1 - \beta_1, \cdots, \alpha_l - \beta_l)$*, the right-most nonzero entry is negative.*

Example 4.3 *The leading term of the polynomial* $f = 2x_1x_2x_3 + 5x_2^2x_1 + 3x_2^2x_3$ *with respect to the graded reverse lexicographic order with* $x_3 > x_2 > x_1$ *is* $3x_3x_2^2$*, because* $x_3x_2x_1 > x_2^2x_1$ *as* $(1-0, 1-2, 1-1) = (1, -1, 0)$ *and* $x_3x_2^2 > x_3x_2x_1$ *as* $(1-1, 2-1, 0-1) = (0, 1, -1)$*.*

Theorem 4.4 [3] *Let* C *be a curve as in* (4.1)*. Let* $g_1, \cdots, g_s$ *be a minimal Gröbner basis for* $I(C)_*$ *with respect to a reverse lexicographic order that makes* x_1 *the lowest variable, then* $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ *is Cohen-Macaulay if and only if* $x_1 \not| \, in(g_i)$ *for* $1 \leq i \leq s$*, where* $in(g_i)$ *is the leading term of* g_i*.*

The proof will be given after the following two lemmas.

Lemma 4.5 [5, Lemma 2.2] *Let* $I \subset k[x_1, \cdots, x_l]$ *be a homogeneous ideal and consider reverse lexicographic order that makes* x_1 *the lowest variable, then*

$$I : x_1 = I \Leftrightarrow in(I) : x_1 = in(I) \tag{4.3}$$

where $in(I)$ *is the ideal generated by* $in(f)$*'s with* $f \in I$*.*

Proof: See [5, Lemma 2.2]. □

Lemma 4.6 $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ *is Cohen-Macaulay if and only if* t^{n_1} *is not a zero divisor in* $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$*.*

Proof: It follows from the isomorphism (2.10)

$$gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]]) \cong k[x_1, x_2, \cdots, x_l]/I(C)_*,$$

that t^{n_1} is not a zero divisor in $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ if and only if x_1 is not a zero divisor in $k[x_1, x_2, \cdots, x_l]/I(C)_*$. For the graded ring $k[x_1, x_2, \cdots, x_l]/I(C)_*$, x_1 is a system of parameters, since the dimension of the ring $k[x_1, x_2, \cdots, x_l]/I(C)_*$ is 1, and the dimension of the ring $k[x_1, x_2, \cdots, x_l]/(x_1, I(C)_*)$ is 0 (because $x_2^{a_2}, \cdots, x_l^{a_l}$ are all elements of $I(C)_*$ for some $a_2, \cdots, a_l$, since we have $x_2^{n_1} - x_1^{n_2}$, $x_3^{n_1} - x_1^{n_3}$ and $x_4^{n_1} - x_1^{n_4}$ in $I(C)$). From Proposition 3.14, $k[x_1, x_2, \cdots, x_l]/I(C)_*$ is Cohen-Macaulay if and only if x_1 is regular, which proves the lemma. □

We can now give the proof of our theorem which gives a checking criterion for the Cohen-Macaulayness of the tangent cone of a monomial curve.

Proof of Theorem 4.4: t^{n_1} is not a zero divisor in $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ if and only if x_1 is not a zero divisor in $k[x_1, x_2, \cdots, x_l]/I(C)_*$. Combining this with Lemma 4.5 and Lemma 4.6, $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]]$ is Cohen-Macaulay $\Leftrightarrow$ $I(C)_* : x_1 = I(C)_* \Leftrightarrow in(I(C)_*) : x_1 = in(I(C)_*)$ with respect to the reverse lexicographic order that makes x_1 the lowest variable. From the definition of a minimal Grobner basis,

$$in(I(C)_*) = (in(g_1), \cdots, in(g_s)) \text{ and } in(g_i) \not| \, in(g_j) \text{ if } i \neq j.$$

Thus, $gr_m(k[[t^{n_1}, t^{n_2}, \cdots, t^{n_l}]])$ is Cohen-Macaulay if and only if x_1 does not divide $in(g_i)$ for $1 \leq i \leq s$. □

4.3 A family of monomial curves in l-space which have CM tangent cones

In this section, we check the Cohen-Macaulayness of the tangent cone of the monomial curves $C_m^{[l]}$ in affine l-space having the parameterization

$$x_1 = t^{a_1},\ x_2 = t^{a_2},\ \cdots,\ x_l = t^{a_l} \tag{4.4}$$

where $a_1 = 2^{l-4}m(m+1)$, $a_2 = 2^{l-4}(m(m+1)+1)$, $a_3 = 2^{l-4}(m+1)^2$, $a_4 = 2^{l-4}((m+1)^2+1)$, $a_5 = 2^{l-4}(m+1)^2 + 2^{l-5}$ and $a_i = 2^{l-4}(m+1)^2 + 2^{l-5} + \sum_{j=6}^{i}(-1)^j 2^{l-j}$ for $i \geq 6$, with $m \geq 2$, $l \geq 4$.

Our main result is the following theorem, which we prove at the end of this section.

Theorem 4.7 [3] *The monomial curve $C_m^{[l]}$ having parameterization as in (4.4) has Cohen-Macaulay tangent cone at the origin, with $\mu(I(C_m^{[l]})_*) = 2m + l - 2$.*

This theorem not only gives infinitely many families of monomial curves having Cohen-Macaulay tangent cone at the origin, but also shows that in each affine l-space with $l \geq 4$, there are monomial curves having Cohen-Macaulay tangent cone with arbitrarily large $\mu(I(C_m^{[l]})_*)$. Our first aim is to give a complete description of the defining ideal $I(C_m^{[4]})$.

Proposition 4.8 [3] *The defining ideal $I(C_m^{[4]})$ of the monomial curve $C_m^{[4]}$ is generated by $G_m^{[4]} = \{g_i = x_1^{m-i}x_3^{i+1} - x_2^{m-i+1}x_4^i$ with $0 \leq i \leq m$, $f_j = x_3^j x_4^{m-j} - x_1^{j+1}x_2^{m-j}$ with $0 \leq j \leq m$ and $h = x_1x_4 - x_2x_3\}$.*

From Proposition 2.1, $I(C_m)$ is generated by binomials $F(\nu, \mu)$ of the form

$$F(\nu, \mu) = x_1^{\nu_1} \cdots x_4^{\nu_4} - x_1^{\mu_1} \cdots x_4^{\mu_4}, \quad \sum_{i=1}^{4} \nu_i n_i = \sum_{i=1}^{4} \mu_i n_i \tag{4.5}$$

with $\nu_i \mu_i = 0, \ \ 1 \leq i \leq l$, $n_1 = m(m+1), n_2 = m(m+1)+1, n_3 = (m+1)^2, n_4 = (m+1)^2 + 1$ and $\partial(F(\nu, \mu))$ is defined to be $\sum_{i=1}^{4} \nu_i n_i = \sum_{i=1}^{4} \mu_i n_i$.

Thus, we can prove the lemma by showing that for all $F(\nu, \mu)$, there is an element $f \in (f_0, f_1, \cdots, f_m, g_0, g_1, \cdots, g_m, h)$ such that $F(\nu, \mu) - f = \prod_{i=1}^{4} x_i^{a_i} g$ with $g = 0$ or $g = F(\nu', \mu')$ with $\partial(F(\nu', \mu')) < \partial(F(\nu, \mu))$, since this proves that any binomial $F(\nu, \mu)$ can be generated by $\{f_0, f_1, \cdots, f_m, g_0, g_1, \cdots, g_m, h\}$.

Thus, the following lemma is crucial for our purpose, since it determines the polynomials $x_{i_1}^{\nu_{i_1}} - x_{i_2}^{\nu_{i_2}} x_{i_3}^{\nu_{i_3}} x_{i_4}^{\nu_{i_4}}$ in $I(C_m^{[4]})$ with $1 \leq i_1, i_2, i_3, i_4 \leq 4$ and ν_{i_1} minimal. These polynomials $x_{i_1}^{\nu_{i_1}} - x_{i_2}^{\nu_{i_2}} x_{i_3}^{\nu_{i_3}} x_{i_4}^{\nu_{i_4}}$ with ν_{i_1} minimal are very useful for finding polynomials f satisfying $f \in (f_0, f_1, \cdots, f_m, g_0, g_1, \cdots, g_m, h)$ such that $F(\nu, \mu) - f = \prod_{i=1}^{4} x_i^{a_i} g$ with $g = 0$ or $g = F(\nu', \mu')$ with $\partial(F(\nu', \mu')) < \partial(F(\nu, \mu))$.

Lemma 4.9 [3] *Let $n_1 = m(m+1), n_2 = m(m+1)+1, n_3 = (m+1)^2, n_4 = (m+1)^2+1$ with $m \geq 2$. If $\nu_{i_1} n_{i_1} \in < n_{i_2}, n_{i_3}, n_{i_4} >$, with $1 \leq i_1, i_2, i_3, i_4 \leq 4$ (all i_k's are distinct), ν_{i_1} strictly positive and minimal, then $\nu_1 = m+1$, $\nu_2 = m+1$, $\nu_3 = m$, $\nu_4 = m$.*

Proof. For $i_1 = 1$, we have the equation

$$\nu_1 m(m+1) = \mu_2(m(m+1)+1) + \mu_3(m+1)^2 + \mu_4((m+1)^2+1) \tag{4.6}$$

which leads to

$$\nu_1 m(m+1) = (m+1)(\mu_2 m + \mu_3(m+1) + \mu_4(m+1)) + (\mu_2 + \mu_4)$$

and $m+1 \mid \mu_2 + \mu_4$ follows immediately. Thus, if either μ_2 or $\mu_4 \neq 0$, then $\mu_2 + \mu_4 \geq m+1$. Also, from (4.6),

$$\nu_1 m(m+1) > \mu_2 m(m+1) + \mu_3 m(m+1) + \mu_4 m(m+1),$$

we have $\nu_1 > \mu_2 + \mu_3 + \mu_4$ and substituting $\mu_2 + \mu_4 \geq m+1$ in this inequality, we obtain $\nu_1 > m+1$. If $\mu_2 = \mu_4 = 0$, then $\mu_3 = m$ and $\nu_1 = m+1$. Thus, the minimal positive value for ν_1 is $m+1$ and we have $(m+1)n_1 = mn_3$.

For $i_1 = 2$, we have the equation

$$\nu_2(m(m+1)+1) = \mu_1 m(m+1) + \mu_3(m+1)^2 + \mu_4((m+1)^2+1) \qquad (4.7)$$

which leads to

$$\nu_2 m(m+1) + \nu_2 - \mu_4 = (m+1)(\mu_1 m + \mu_3(m+1) + \mu_4(m+1))$$

from which, $\nu_2 > \mu_4$ and $m+1 \mid \nu_2 - \mu_4$ follow. Thus, $\nu_2 \geq m+1$. Since $\nu_2 = m+1$, $\mu_1 = m$, $\mu_3 = 1$ and $\mu_4 = 0$ satisfy the equation (4.7), the minimal positive value for ν_2 is $m+1$ and we have $(m+1)n_2 = n_1 m + n_3$.

For $i_1 = 3$, we have the equation

$$\nu_3(m+1)^2 = \mu_1 m(m+1) + \mu_2(m(m+1)+1) + \mu_4((m+1)^2+1) \qquad (4.8)$$

and $m+1 \mid \mu_2 + \mu_4$ follows immediately. If either μ_2 or $\mu_4 \neq 0$, then $\mu_2 + \mu_4 \geq m+1$. Thus,

$$\begin{aligned}
\nu_3(m+1)^2 &\geq \mu_2(m(m+1)+1) + \mu_4((m+1)^2+1) \\
&\geq (\mu_2+\mu_4)(m(m+1)+1) \\
&\geq (m+1)(m(m+1)+1)
\end{aligned}$$

from which we obtain $\nu_3 > m$. If $\mu_2 = \mu_4 = 0$, then $\nu_3 = m$ and $\mu_1 = m+1$. Thus, the minimal positive value for ν_3 is m and we have $mn_3 = (m+1)n_1$.

For $i_1 = 4$, we have the equation

$$\nu_4((m+1)^2+1) = \mu_1 m(m+1) + \mu_2(m(m+1)+1) + \mu_3(m+1)^2 \qquad (4.9)$$

If $\nu_4 > \mu_2$, then $m+1 \mid \nu_4 - \mu_2$ and $\nu_4 \geq m+1$. If $\nu_4 = \mu_2$, then $\nu_4 = \mu_1 m + \mu_3(m+1)$ and $\nu_4 \geq m$. Otherwise, if $\nu_4 < \mu_2$, then by substituting $\mu_2 = \nu_4 + i$ with $i > 0$, we have

$$\nu_4(m+1) = \mu_1 m(m+1) + i(m(m+1)+1) + \mu_3(m+1)^2$$

and $\nu_4 > m$. Since $\nu_4 = m$, $\mu_1 = 1$, $\mu_2 = m$ and $\mu_3 = 0$ satisfy the equation (4.9), the minimal positive value for ν_4 is m and we have $mn_4 = n_1 + mn_2$. $\square$

From the equations $(m+1)n_1 = mn_3$, $(m+1)n_2 = n_1 m + n_3$, $mn_3 = (m+1)n_1$ and $mn_4 = n_1 + mn_2$ found in Lemma 4.9, we obtain the polynomials $x_1^{m+1} - x_3^m$, $x_2^{m+1} - x_1^m x_3$, $x_3^m - x_1^{m+1}$ and $x_4^m - x_1 x_2^m$, which are the polynomials $-f_m, -g_0, f_m$ and f_0 in $G_m^{[4]}$. We can now prove Proposition 4.8.

Proof of Proposition 4.8: For any $F(\nu,\mu)$, if $\nu_4 = \mu_4 = 0$, then $F(\nu,\mu) \in I(C_m) \cap k[x_1,x_2,x_3]$. Since the semigroup $< m(m+1), m(m+1)+1, (m+1)^2 >$ is symmetric, $I(C_m) \cap k[x_1,x_2,x_3] = (g_0, f_m) \subset (f_0, f_1, \cdots, f_m, g_0, g_1, \cdots, g_m, h)$ from [21]. Thus, consider the binomials $F(\nu,\mu)$ with $\nu_4 \neq 0$:

1. If exactly one $\nu_i = 0$: i) $\nu_1 = 0$ then $f = x_1^{\mu_1-(m+1)} f_m$, ii) $\nu_2 = 0$ then $f = x_2^{\mu_2-(m+1)} g_0$, iii) $\nu_3 = 0$ then $f = -x_3^{\mu_3-m} f_m$
2. $\nu_1 = \nu_2 = \nu_3 = 0$ then $\nu_4 \geq m$, i) $\mu_1 = \mu_2 = 0$ then $\mu_3 \geq m$ and $f = x_4^{\nu_4-m} f_0 - x_3^{\mu_3-m} f_m$, ii) μ_1 or $\mu_2 \neq 0$ then $f = x_4^{\nu_4-m} f_0$
3. i) $\nu_2 = \nu_3 = 0$, $\nu_1 \neq 0$ then $f = x_1^{\nu_1-1} x_4^{\nu_4-1} h$

   ii) $\nu_1 = \nu_2 = 0$, $\nu_3 \neq 0$: If $\mu_1 = 0$, then $f = x_2^{\mu_2-(m+1)} g_0$. Otherwise, if $\nu_4 \geq m$, we have $f = x_3^{\nu_3} x_4^{\nu_4-m} f_0$, and if $\nu_3 \geq m$, we have $f = x_3^{\nu_3-m} x_4^{\nu_4} f_m$. The only remaining case is $\nu_4, \nu_3 < m$. Assume that $\nu_4 < \mu_2$. With this assumption, the equation

   $$\nu_3(m+1)^2 + \nu_4((m+1)^2+1) = \mu_1 m(m+1) + \mu_2(m(m+1)+1) \quad (4.10)$$

   gives $\mu_2 = \nu_4 + k(m+1)$ where $k \geq 1$. Substituting this in the equation (5.3) and simplifying, we obtain

   $$\nu_3(m+1) + \nu_4 = \mu_1 m + k(m(m+1)+1) \quad (4.11)$$

   But this equation gives

   $$\begin{aligned} \nu_3 + \nu_4 &= \mu_1 m + k(m(m+1)+1) - \nu_3 m \\ &> m + (m(m+1)+1) - (m-1)m > 2m-2 \end{aligned}$$

   which is a contradiction since $\nu_3, \nu_4 < m$. Thus, $\nu_4 \geq \mu_2$. From equation (5.3), $(m+1) \mid \nu_4 - \mu_2$ so that $\nu_4 = \mu_2$. Substituting $\nu_4 = \mu_2$ in equation (5.3), we obtain

$$\mu_1 m - \nu_3 m = \nu_3 + \nu_4$$

which gives $m \mid \nu_3 + \nu_4$. Thus, $f = f_j$ for some j with $1 \leq j \leq m-1$.

iii) $\nu_1 = \nu_3 = 0$, $\nu_2 \neq 0$ a) If $\nu_4 \geq m$, then there are two cases: If $\mu_1 \neq 0$, $f = x_4^{\nu_4 - m} x_2^{\nu_2} f_0$. If $\mu_1 = 0$, then $\mu_3 \geq m$ and $f = -x_3^{\nu_3-(m+1)}(x_3 f_m + x_1 g_0)$. b) If $\nu_2 \geq m+1$, then $f = -x_4^{\nu_4} x_2^{\nu_2-(m+1)} g_0$. c) If $\nu_4 < m$, $\nu_2 < m+1$, then from the equation

$$\nu_2((m+1)m+1) + \nu_4((m+1)^2+1) = \nu_1 m(m+1) + \nu_3(m+1)^2$$

$m+1 \mid \nu_2 + \nu_4$ and $\nu_2 + \nu_4 = m+1$. Thus, $f = g_i$ for some i with $1 \leq i \leq m-1$.

□

We can now give the description of the ideal $I(C_m^{[l]})$ by induction.

Proposition 4.10 [3] *The defining ideal $I(C_m^{[l]})$ of the monomial curve $C_m^{[l]}$ with $l \geq 4$ is generated by*

$$G_m^{[l]} = \{g_i = x_1^{m-i}x_3^{i+1} - x_2^{m-i+1}x_4^i \text{ with } 0 \leq i \leq m,\ f_j = x_3^j x_4^{m-j} - x_1^{j+1}x_2^{m-j}$$
$$\text{with } 0 \leq j \leq m,\ h = x_1x_4 - x_2x_3,\ x_5^2 - x_4x_3, \cdots, x_l^2 - x_{l-1}x_{l-2}\}$$

We need the following lemma of Morales in the proof.

Lemma 4.11 [35, Lemma 3.2] *Let C be a curve having parameterization*

$$x_1 = \varphi_1(t), \cdots,\ x_{l-1} = \varphi_{l-1}(t),\ x_l = t^a \tag{4.12}$$

where a is a positive integer and $\varphi_i(t) \in k[t]$ for $1 \leq i \leq l-1$. Let β be a positive integer such that $gcd(a, \beta) = 1$, and let $\tilde{C}$ be the curve having parameterization

$$x_1 = \varphi_1(t^\beta), \cdots,\ x_{l-1} = \varphi_{l-1}(t^\beta),\ x_l = t^a. \tag{4.13}$$

For any $f(x_1, \cdots, x_l) \in k[x_1, \cdots, x_l]$, we denote by $\tilde{f}$ the element $f(x_1, \cdots, x_{l-1}, x_l^\beta)$ and let $f_1, \cdots, f_s$ be a set of generators for $I(C)$. Then $\tilde{f}_1, \cdots, \tilde{f}_s$ is a set of generators for $I(\tilde{C})$.

Proof: See [35, Lemma 3.2]. □

Proof of Proposition 4.10. We prove the proposition by induction. The $l = 4$ case is given in Proposition 4.8. Now assume that the proposition is true for some $l \geq 4$ and that $I(C_m^{[l]})$ has the given generator set. By a trivial computation, it is seen that $C_m^{[l+1]}$ has parameterization,

$$x_1 = t^{2a_1},\ x_2 = t^{2a_2}, \cdots, x_l = t^{2a_l},\ x_{l+1} = t^{a_{l-1}+a_l} \tag{4.14}$$

where a_i's are as in 4.4.

Let C' be the curve having the parameterization,

$$x_1 = t^{a_1},\ x_2 = t^{a_2}, \cdots, x_l = t^{a_l},\ x_{l+1} = t^{a_{l-1}+a_l}. \tag{4.15}$$

Let $f \in I(C')$. Then $f(t^{a_1}, t^{a_2}, \cdots, t^{a_l}, t^{a_{l-1}+a_l}) = 0$ and since any $f \in k[x_1, x_2, \cdots, x_l, x_{l+1}]$ can be written as

$$\begin{aligned} f(x_1, \cdots, x_l, x_{l+1}) &= f(x_1, \cdots, x_l, x_{l+1} - x_{l-1}x_l + x_{l-1}x_l) \\ &= (x_{l+1} - x_{l-1}x_l)f_1(x_1, \cdots, x_l) + f_2(x_1, \cdots, x_l), \end{aligned}$$

$f(t^{a_1}, t^{a_2}, \cdots, t^{a_l}, t^{a_{l-1}+a_l}) = 0$ implies $f_2(t^{a_1}, t^{a_2}, \cdots, t^{a_l}) = 0$. Hence, any $f \in I(C')$ can be written as $f = (x_{l+1} - x_{l-1}x_l)f_1 + f_2$ with $f_2 \in I(C_m^{[l]})$. Thus, $I(C')$ is generated by the generator set $G_m^{[l]} \cup \{x_{l+1} - x_{l-1}x_l\}$.

Applying Lemma 4.11 with $C = C'$ in (4.15), $\tilde{C} = C_m^{[l+1]}$ in (4.14) and $\beta = 2$, $I(C_m^{[l+1]})$ is generated by $G_m^{[l+1]} = G_m^{[l]} \cup \{x_{l+1}^2 - x_{l-1}x_l\}$. Thus, the induction is completed. □

Knowing the description of the ideal $I(C_m^{[l]})$, it is possible to to compute a set of generators of $I(C_m^{[l]})_*$ by using the following algorithm, known as the *tangent cone algorithm* [13, p.467]. We first find a generator set of $I(C_m^{[l]})^h \subset k[t, x_1, x_2, \cdots, x_l]$ which is the homogenization of $I(C_m^{[l]})$. It can be found by homogenizing the elements of a Gröbner basis of $I(C_m^{[l]})$ with respect to an any graded monomial order by using the homogenization variable t. From the obtained generator set of $I(C_m^{[l]})^h$, another Gröbner basis $G_1, \cdots, G_s$ is computed with respect to a monomial order, such that among monomials of the same total degree, any monomial involving t is greater than any monomial involving only $x_1, x_2, \cdots, x_l$. For example, lexicographic order with $t > x_1 > x_2 > \cdots > x_l$ is such an order. Then $I(C_m^{[l]})_*$ is generated by the homogeneous summands of the least degree of $G_1(1, x_1, .., x_l), \cdots, G_s(1, x_1, \cdots, x_l)$.

Proposition 4.12 [3] *$I(C_m^{[l]})_*$ is generated by $(G_m^{[l]})_* = \{g_i = x_1^{m-i}x_3^{i+1} - x_2^{m-i+1}x_4^i$ with $0 \le i \le m-1$, $f_j' = x_3^j x_4^{m-j}$ with $0 \le j \le m$, $h = x_1x_4 - x_2x_3$, $x_5^2 - x_4x_3, \cdots, x_l^2 - x_{l-1}x_{l-2}$ $\}$. In particular, $\mu(I(C_m^{[l]})_*) = 2m + l - 2$.*

The proof is a direct application of the tangent cone algorithm with the following lemmas and will be given after the lemmas.

Lemma 4.13 [3] *$G_m^{[l]} = \{g_i = x_1^{m-i}x_3^{i+1} - x_2^{m-i+1}x_4^i$ with $0 \le i \le m$, $f_j = x_3^j x_4^{m-j} - x_1^{j+1}x_2^{m-j}$ with $0 \le j \le m$, $h = x_1x_4 - x_2x_3$, $x_5^2 - x_4x_3, \cdots, x_l^2 - x_{l-1}x_{l-2}\}$ is a Gröbner basis with respect to the graded lexicographic order with $x_l > x_{l-1} > \cdots > x_4 > x_2 > x_3 > x_1$.*

Proof. Let $G_m^{[l]}$ be denoted by G during the proof. For $i < j$, $S(g_i, g_j) = x_4^{j-i}x_3^{i+1}x_1^{m-i} - x_2^{j-i}x_1^{m-j}x_3^{j+1} = x_1^{m-j}x_3^{i+1}(x_1^{j-i}x_4^{j-i} - x_2^{j-i}x_3^{j-i}) = (x_4x_1 - x_2x_3)p_1$ which shows that $S(g_i, g_j) \to_G 0$. $S(g_i, h) = x_1^{m-i+1}x_3^{i+1} - x_2^{m-i+2}x_4^{i-1}x_3 = x_3g_{i-1}$, so that $S(g_i, h) \to_G 0$. Also, $S(f_i, f_j) = x_1^{j-i}x_3^i x_4^{m-i} - x_2^{j-i}x_3^j x_4^{m-j} = x_3^i x_4^{m-j}(x_1^{j-i}x_4^{j-i} - x_2^{j-i}x_3^{j-i}) = (x_4x_1 - x_2x_3)p_2$. Thus, $S(f_i, f_j) \to_G 0$. $S(f_i, h) = x_3^i x_4^{m-i+1} - x_2^{m-i+1}x_1^i x_3 = x_3 f_{i-1}$, and $S(f_i, h) \to_G 0$. For $i < j$, $S(f_i, g_j) = x_3^{i+1}x_1^{j-i}f_m - x_3^j g_{m-j+i}$ which shows that $S(f_i, g_j) \to_G 0$, and the case $i \ge j$ is similar. Let $p_j = x_j^2 - x_{j-1}x_{j-2}$ with $j \ge 5$. Then since $gcd(p_j, f) = 1$ for any $f \in G$, $S(p_j, f) \to_G 0$. □

This lemma gives us the opportunity to obtain $I(C_m^{[l]})^h$ by homogenizing the generators of $G_m^{[l]}$ so that $I(C_m^{[l]})^h$ is generated by $(G_m^{[l]})^h = \{g_i = x_1^{m-i}x_3^{i+1} - x_2^{m-i+1}x_4^i,\ 0 \le i \le m,\ f_j^h = tx_3^j x_4^{m-j} - x_1^{j+1}x_2^{m-j}\ 0 \le j \le m,\ h = x_1x_4 - x_2x_3, x_5^2 - x_4x_3, \cdots, x_l^2 - x_{l-1}x_{l-2}\}$.

Lemma 4.14 [3] *$(G_m^{[l]})^h$ is a Gröbner basis with respect to the lexicographic order with $t > x_l > x_{l-1} > \cdots > x_4 > x_2 > x_3 > x_1$.*

Proof. Let $(G_m^{[l]})^h$ be denoted by G^h during the proof. $S(g_i, g_j)$, $S(g_i, h)$ and $S(f_i^h, f_j^h) = S(f_i, f_j) \to_{G^h} 0$ from Lemma 4.13. $S(f_i^h, g_j) = x_1^{m-j}x_3^{i+j+1-m}f_m^h + x_1^{i+1}g_{i+j-m}$ for $j \ge m-i$. For $j < m-i$, $S(f_i^h, g_j) = x_1^{i+1}x_2^{m-i-j}g_0 + x_1^{i+1}x_3 f_{i+j}^h$. Thus, $S(f_i^h, g_j) \to_{G^h} 0$. For $i \ne m$, $S(f_i^h, h) = x_2 f_{i+1}^h$ and $S(f_i^h, h) \to_{G^h} 0$, while $S(f_m^h, h) \to_{G^h} 0$, since $gcd(in(f_m^h), in(h)) = 1$. Let $p_j = x_j^2 - x_{j-1}x_{j-2}$ with $j \ge 5$. Then since $gcd(p_j, f) = 1$ for any $f \in G^h$, $S(p_j, f) \to_{G^h} 0$ □

Proof of Proposition 4.12: According to the tangent cone algorithm, we must compute a Gröbner basis from $(G_m^{[l]})^h$ with respect to a monomial order, such that among monomials of the same total degree, any monomial involving t is greater than any monomial involving only $x_1, \cdots, x_l$, which is done in Lemma 4.14. Again from the tangent cone algorithm, $I(C_m^{[l]})_*$ is generated by $\{g_i = x_1^{m-i}x_3^{i+1} - x_2^{m-i+1}x_4^i$ with $0 \leq i \leq m$, $f'_j = x_3^j x_4^{m-j}$ with $0 \leq j \leq m$, $h = x_1x_4 - x_2x_3$, $x_5^2 - x_4x_3, \cdots, x_l^2 - x_{l-1}x_{l-2}\}$. Since g_m can be generated by f'_0 and f'_m, we can give a minimal generator set $G_m^{[l]}*$ for $I(C_m)_*$ such that $G_m^{[l]}* = \{g_i = x_1^{m-i}x_3^{i+1} - x_2^{m-i+1}x_4^i$ $0 \leq i \leq m-1$, $f'_j = x_3^j x_4^{m-j}$ $0 \leq j \leq m$, $h = x_1x_4 - x_2x_3$, $x_5^2 - x_4x_3, \cdots, x_l^2 - x_{l-1}x_{l-2}\}$. $\square$

We can now prove Theorem 4.7.

Proof of Theorem 4.7: $I(C_m^{[l]})_*$ is generated by $(G_m^{[l]})_*$ which is also a minimal Gröbner basis with respect to the reverse lexicographic order with $x_l > \cdots > x_4 > x_2 > x_3 > x_1$ (Let $(G_m^{[l]})_*$ be denoted by G_*. $S(f'_i, f'_j) = 0$, $S(f'_j, h) \rightarrow_{G_*} 0$, $S(g_i, h) \rightarrow_{G_*} 0$, $S(g_i, g_j) \rightarrow_{G_*} 0$ and $S(f'_i, g_j) \rightarrow_{G_*} 0$. For any $f \in G_*$ and $p_j = x_j^2 - x_{j-1}x_{j-2}$ with $j \geq 5$, $S(p_j, f) \rightarrow_{G^*} 0$). We can now apply Theorem 4.4. Since x_1 does not divide $in(g_i) = x_2^{m-i+1}x_4^i$, $1 \leq i \leq m$, $in(f'_j) = x_3^j x_4^{m-j}$ $0 \leq j \leq m$, $in(h) = x_2x_3$ and $in(p_j) = x_j^2$ with $j \geq 2$, $k[x_1, \cdots, x_l]/I(C_m^{[l]})_*$ is Cohen-Macaulay. $\square$

Theorem 4.7 shows that the monomial curve $C_m^{[l]}$, for which $\mu(I(C_m^{[l]})_*) = 2m + l - 2$ has Cohen-Macaulay tangent cone, where $m \geq 2$, $l \geq 4$. Thus, there are monomial curves having not only Cohen-Macaulay tangent cones but also arbitrarily large minimal number of generators for the ideal defining the tangent cone in all affine l-spaces with $l \geq 4$.

Remark 4.15 *(a) By the same approach, the monomial curves C_n having the parameterization*

$$x_1 = t^{n(n+1)+1},\ x_2 = t^{n(n+1)+2},\ x_3 = t^{(n+1)^2+1},\ x_4 = t^{(n+1)^2+2} \tag{4.16}$$

with $n \geq 3$, can be shown to have Cohen-Macaulay tangent cones and $\mu(I(C_n)_*) = 2n + 3$.

(b) By a similar approach, Bresinsky curves C_{q_2}, see [8], having the parameterization

$$x_1 = t^{q_1q_2},\ x_2 = t^{q_1d_1},\ x_3 = t^{q_1q_2+d_1},\ x_4 = t^{q_2d_1} \tag{4.17}$$

with $q_1 = q_2 + 1$*,* q_2 *even,* $q_2 \geq 4$*,* $d_1 = q_2 - 1$ *can also be shown to have Cohen-Macaulay tangent cones. The approach depends on checking that* x_4 *is not a zero divisor in the associated graded ring by considering the generators* $F(\nu, \mu)$*, since the homogeneous summands of the least degree of* $F(\nu, \mu)$*'s generate the* $I(C_{q_2})_*$.

Chapter 5

Hilbert Functions and Genus Calculations

In the first section of this chapter, we will find the Hilbert series and Hilbert polynomials of the families of the monomial curves in (4.4). In the second section, we will make some genus computations by using Hilbert polynomials for complete intersections in the projective case, and in the last section we will make genus computations by using Riemann-Hurwitz formula for complete intersection curves of superelliptic type in the affine case.

5.1 Hilbert Series of $I(C_m^{[l]})$

We want to compute the Hilbert series of $I(C_m^{[l]})$ in (4.4). By the Hilbert series of $I(C_m^{[l]})$, we mean the Hilbert series of the local ring $R = k[[t^{a_1}, t^{a_2}, \cdots, t^{a_l}]]$, where a_i's are as in (4.4). The Hilbert function of the local ring $(R, \mathfrak{m})$ is $H_R(n) = dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$, where $\mathfrak{m} = (t^{a_1}, t^{a_2}, \cdots, t^{a_l})$. We have our famous isomorphism

$$gr_{\mathfrak{m}}(R) = gr_{\mathfrak{m}}(k[[t^{a_1}, t^{a_2}, \cdots, t^{a_l}]]) \cong S = k[x_1, x_2, \cdots, x_l]/I(C_m^{[l]})_*$$

so that they have the same Hilbert function and Hilbert series.

From Theorem 4.7, $k[x_1, x_2, \cdots, x_l]/I(C_m^{[l]})_*$ is Cohen-Macaulay for $m \geq 2$, $l \geq 4$. We first compute the Hilbert series of $k[x_1, x_2, x_3, x_4]/I(C_m^{[4]})_*$. Since $S = k[x_1, x_2, x_3, x_4]/I(C_m)_*$ is Cohen-Macaulay, from Proposition 3.14, S and its Artinian reduction $S/(x_1)$ have the same h-polynomial. From Proposition 4.12, the generators of $I(C_m^{[l]})_*$ is known, thus a direct computation shows that the Hilbert series $H_m^{[4]}(t)$ of the monomial curve C_m is given by

$$H_m(t) = \frac{\sum_{i=0}^{m-1}(2i+1)t^i + mt^m}{1-t} \tag{5.1}$$

We can now compute the Hilbert series of $S = k[x_1, \cdots, x_l]/I(C_m^{[l]})_*$ for all $l \geq 5$. Since $G_{m*}^{[l]}$ obtained in Proposition 4.12 is a Gröbner basis with respect to the reverse lexicographic order with $x_l > x_{l-1} > \cdots > x_5 > x_4 > x_2 > x_3 > x_1$, $k[x_1, \cdots, x_l]/I(C_m^{[l]})_*$ and $k[x_1, \cdots, x_l]/in(G_{m*}^{[l]})$ have the same Hilbert series, where $in(G_{m*}^{[l]})$ is the ideal generated by the leading terms of the elements of the generator set $G_{m*}^{[l]}$ with respect to this order. (This is a well known result going back to the famous article of Macaulay [30].) We have $in(G_m^{[l]}*) = (x_2^{m-i+1}x_4^i \ 0 \leq i \leq m-1,\ f_j' = x_3^j x_4^{m-j} \ 0 \leq j \leq m,\ x_2x_3,\ x_5^2, \cdots, x_l^2)$. To compute the Hilbert series of $k[x_1, \cdots, x_l]/in(G_{m*}^{[l]})$, we need the following proposition.

Proposition 5.1 [4, Proposition 2.4] *Let $I \subset A = k[x_1, \cdots, x_l]$ be a monomial ideal. Suppose the variables $x_1, \cdots, x_l$ can be partitioned into disjoint sets $V_1 \cup \cdots \cup V_j$ such that each generator of I belongs to subring $k[V_i]$ for some i. Define $I_i = I \cap k[V_i]$. Then*

$$H_{A/I}(t) = \textstyle\prod_{i=1}^{j} H_{A/I_i}(t).$$

Proof: The proof is a consequence of the tensor product decomposition

$$A/I = k[V_1]/I_1 \otimes_k \cdots \otimes_k k[V_j]/I_j.$$

$\square$

The ideal $in(G_{m*}^{[l]})$ satisfies the assumptions of the above proposition. Thus, the Hilbert series $H_m^{[l]}(t)$ of the associated graded ring of the monomial curve $C_m^{[l]}$ for $l \geq 4$ is given by

$$H_m^{[l]}(t) = \frac{(1+t)^{l-4}(\sum_{i=0}^{m-1}(2i+1)t^i + mt^m)}{1-t} \tag{5.2}$$

From Definition 3.12, the multiplicity is the integer obtained by evaluating the h-polynomial at $t = 1$. Thus, the monomial curve $C_m^{[l]}$ has multiplicity $2^{l-4}m(m+1)$. Moreover, the Hilbert polynomial of the monomial curve $C_m^{[l]}$ is also $2^{l-4}m(m+1)$.

5.2 Hilbert Polynomial of a Projective Complete Intersection

Let S denote the homogeneous coordinate ring, $k[x_0, \cdots, x_n]$ of $\mathbb{P}_k^n$ where k is an algebraically closed field, usually $\mathbb{C}$. We assume that there are hypersurfaces $H_1, \cdots, H_r$ of $\mathbb{P}_k^n$ of degrees $d_1, \cdots, d_r$ respectively such that $X_r = H_1 \cap \cdots \cap H_r$ is a complete intersection. The hypersurfaces $H_1, \cdots, H_r$ correspond to homogeneous polynomials $f_1, \cdots, f_r \in S$ of degrees $d_1, \cdots, d_r$ respectively.

5.2.1 The Hilbert Polynomial of X_r

Theorem 5.2 [2] *The Hilbert polynomial $H_r(z)$ of X_r is given by the following formula*

$$H_r(z) = \varphi(z) + \sum_{m=1}^{r} (-1)^m \sum_{1 \leq i_1 < \cdots < i_m \leq r} \varphi(z - d_{i_1} - \cdots - d_{i_m}) \tag{5.3}$$

where

$$\varphi(z) = \frac{1}{n!}(z+1)(z+2)\cdots(z+n) = \begin{pmatrix} z+n \\ n \end{pmatrix}.$$

Proof: From [15, Theorem 2], the Koszul complex $K(f_1, \cdots, f_r)$ defined in Definition 3.3 is a free resolution of $S/(f_1, \cdots, f_r)$. Namely, we have the following exact sequence

$$0 \to \wedge^r(S^r) \to \cdots \to \wedge^2(S^r) \to S^r \to S \to S/(f_1, \cdots, f_r) \to 0. \tag{5.4}$$

In [16], in order to grade

$$\wedge^m (S^r) = \bigoplus_{1 \leq i_1 < \cdots < i_m \leq r} Se_{i_1} \wedge \cdots \wedge e_{i_m} \;\; (1 \leq m \leq r), \tag{5.5}$$

a degree $d_{i_1} + \cdots + d_{i_m}$ is assigned to a basis element $e_{i_1} \wedge \cdots \wedge e_{i_m}$, so that (5.4) is an exact sequence with maps homogeneous of degree zero. Now imposing the additive property of Hilbert polynomials on the exact sequence (5.4), the formula given in (5.3) is obtained. □

Corollary 5.3 *The arithmetic genus, $g_a(X_r)$, of X_r is given by the formula*

$$g_a(X_r) = \sum_{m=1}^{r} (-1)^{m+n-r} \sum_{1 \leq i_1 < \cdots < i_m \leq r} \varphi(-d_{i_1} - \cdots - d_{i_m}). \tag{5.6}$$

5.3 Genus Computations of Complete Intersection Curves of Superelliptic type

5.4 Affine Case

In this section, we compute the genus of a complete intersection curve C in $\mathbb{A}_{\mathbb{C}}^{n+1}$ given by,

$$\begin{array}{rcl} y_1^{d_1} & = & (x - a_{11}) \cdots (x - a_{1m}) \\ y_2^{d_2} & = & (x - a_{21}) \cdots (x - a_{2m}) \\ \vdots & \vdots & \\ y_n^{d_n} & = & (x - a_{n1}) \cdots (x - a_{nm}) \end{array} \tag{5.7}$$

where $2 \leq d_1 \leq \cdots \leq d_n \leq m - 1$ and all a_{ij}'s are distinct, with $a_{ij} \in \mathbb{C}$.

This is a smooth affine curve and its projective closure $\overline{C}$ in $\mathbb{P}_{\mathbb{C}}^{n+1}$ is singular. Let $\tilde{C}$ be a resolution of $\overline{C}$. The genus of C is then defined to be the genus of $\tilde{C}$. In the following subsections we will in turn describe the projective closure of C, describe a finite map from $\tilde{C}$ to $\mathbb{P}^1$, count the ramification indices of the points of $\tilde{C}$ under this map and finally apply the Riemann-Hurwitz formula to this map to calculate the genus.

5.4.1 Projective Closure of C

We first consider a complete intersection curve C_1 of a special type in $\mathbb{A}_{\mathbb{C}}^3$ defined by,

$$\begin{array}{rcl} y_1^d & = & x^m + a_1 x^{m-1} + \cdots + a_{m-1} x + a_m =: F_1(x) \\ y_2^d & = & x^m + b_1 x^{m-1} + \cdots + b_{m-1} x + b_m =: F_2(x) \end{array} \tag{5.8}$$

with $2 \leq d \leq m - 1$. Let $f_i = y_i^d - F_i$, $i = 1, 2$, and define the ideal I as

$$I = (f_1, f_2).$$

We show in what follows that the ideal I and a Gröbner basis of it contain a certain polynomial. As a consequence of this the projective closure of the curve C_1 can be explicitly defined.

Lemma 5.4 *The ideal I has an element of the form*

$$(y_1^d - y_2^d)^p + f(x, y_1, y_2) \tag{5.9}$$

where $f(x, y_1, y_2)$ has degree less than or equal to pd and if deg $f = pd$, then the leading term of $f(x, y_1, y_2)$ is divisible by x.

Proof: The proof consists of a series of straightforward and tedious calculations which we summarize below. Note that for any ideal I, if $a-b,\ c-d \in I$, then $a^n - b^n,\ ac - bd \in I$ for any integer $n \geq 1$. Thus, $y_1^{dk} - F_1^k$ (with leading monomial x^{mk}) and $(y_1^d - y_2^d)^l - F_3^l$ (with leading monomial $x^{(m-i)l}$) are both in I. Hence the polynomial

$$y_1^{dk}[(y_1^d - y_2^d)^l - F_3^l] + F_3^l[y_1^{dk} - F_1^k]$$

is in I, which simplifies to a polynomial of the form,

$$f_{k,l} = y_1^{dk}(y_1^d - y_2^d)^l - F_1^k F_3^l$$

with leading monomial $x^{km+l(m-i)}$. The degree of $f_{k,l}$ is $km + l(m-i)$ and this number belongs to the subsemigroup of nonnegative integers generated by m and $m-i$. It is well known that if $gcd(a,b) = 1$, then the semigroup generated by ac and bc, for any nonnegative integers a, b and c, contains all the integers which are divisible by c and are greater than $N = c(ab - a - b)$. Hence for every $n > N$, for some N large enough, and divisible by $gcd(m, m-i)$ there is a polynomial in I with leading term x^n.

Fix an integer $p > m$ divisible by $gcd(m, m-i)$ satisfying $pd > N$ and consider the polynomial

$$\phi_p \quad = \quad (y_1^d - y_2^d)^p - F_3^p \in I. \tag{5.10}$$

Its leading monomial is $x^{p(m-i)}$ which can be eliminated by subtracting a suitable constant times the polynomial $f_{m-i,p-m}$. Since $pd > N$, for every integer n divisible by $gcd(m, m-i)$ and in the interval $[pd, p(m-i)]$ there are nonnegative integers k_n and l_n satisfying $k_n m + l_n(m-i) = n$. Dividing both sides of this equation by $m-i$ and observing that $n \leq p(m-i)$ and $m/(m-i) > 1$ we obtain the crucial inequality

$$k_n + l_n \leq p.$$

This inequality now assures us that the degree of the $y_1^{dk_n}(y_1^d - y_2^d)^{l_n}$ part of f_{k_n,l_n} has degree less than pd. Thus if α_1 denotes the leading coefficient of ϕ_p and $k_1 = m-i$, $l_1 = p-m$, then

$$\deg(\phi_p - \alpha_1 f_{k_1,l_1}) = n_2 \quad < \quad p(m-i) = \deg\phi_p$$

and

$$(\phi_p - \alpha_1 f_{k_1,l_1})(0, y_1, y_2) \quad = \quad (y_1^d - y_2^d)^p + \textit{lower degree terms}.$$

Let $m = ac$ and $m - i = bc$ with $(a, b) = 1$. We then have two cases:

Case 1: $c = 1$ If $\deg(\phi_p - \alpha f_{k_1,l_1}) = n_2 > pd$ then we can find nonnegative integers k_2 and l_2 such that $k_2 m + l_2(m - i) = n_2$ and

$$\deg(\phi_p - \alpha_1 f_{k_1,l_1} - \alpha_2 f_{k_2,l_2}) = n_3 < n_2$$

where α_2 is the leading coefficient of $\phi_p - \alpha_1 f_{k_1,l_1}$. Continuing in this manner we eventually obtain a polynomial whose leading form is $(y_1^d - y_2^d)^p + f(x, y_1, y_2)$ where $x|LT(f)$ as claimed.

Case 2: $c > 1$ If $n_2 \leq pd$, then we are done. If $n_2 > pd$ and is divisible by c then we continue as in case 1 above by subtracting suitable polynomials of I and thus reducing the degree. Therefore we might assume without loss of generality that $n_2 > pd$ and is not divisible by c.

Let $c_1 = (ac, bc, n_2)$. Observe that $c_1 | c$ since $(a, b) = 1$. So $c_1 \leq c$. However $c_1 | n_2$ but $c \not| n_2$, so $c \neq c_1$. Therefore $c_1 < c$, which assures the finiteness of the following procedure:

We can write

$$\phi_p - \alpha_1 f_{k_1,l_1} \quad = \quad H(x, y_1, y_2) - F_4(x)$$

where $\deg F_4(x) = n_2$ and $\deg H(x, y_1, y_2) = (k_1 + l_1)d < pd$.

The subsemigroup of $\mathbb{N}$ generated by $m = ac, m - i = bc$ and n_2 contains all the integers which are greater than some N and are divisible by c_1. Fix an integer p divisible by c_1 and is such that $pd > N$. Define the polynomial ϕ_p as in equation (5.10) (with the new value of p). Define polynomials

$$f_{jkl} \quad = \quad H^j y_1^{dk} (y_1^d - y_2^d)^l - F_4^j F_1^k F_3^l \in I$$

with $j, k, l \geq 0$. As before if $\deg f_{j,k,l} \leq p(m - i)$, then $j + k + l < p$.

We are now again at the stage where a suitable constant multiple of the polynomial $f_{j,k,l}$ is subtracted from ϕ_p to remove the leading x-term of ϕ_p and since $j + k + l < p$ the resulting polynomial is of the type which allows further reduction of x-terms without introducing any y_1 or y_2 terms of degree higher than the degree of $(y_1^d - y_2^d)^p$. And how we will continue is going to be determined according to whether $c_1 = 1$ or $c_1 > 1$. Since c_1 is strictly less than c, this process must stop after finitely many steps.

This then proves that there is a polynomial of the form (5.9) in the ideal I. $\square$

Corollary 5.5 *A reduced Gröbner basis for the ideal I with respect to the graded lexicographic order with $y_1 > y_2 > x$ contains an element of the form*

$$(y_1^d - y_2^d)^k (y_1^r - y_2^r)^l - F(x, y_1, y_2) \tag{5.11}$$

where

$$\begin{array}{rl} i) & k > 0,\ l \geq 0,\ r|d \\ ii) & \deg F \leq kd + rl \\ iii) & If deg F = dk + rl,\ then x|LT(F(x, y_1, y_2)). \end{array}$$

Moreover the leading term of any other element in the reduced Gröbner basis is divisible by x.

Proof: The ideal $LT(I)$ of leading terms of I contains a certain y_1^{pd} coming from (5.9). Therefore the Gröbner basis G of I contains an element g whose leading monomial is y_1^k for some $k \leq d$. Homogenizing g with respect to z and setting $z = 0$, $x = 0$ gives a homogeneous form $g(y_1, y_2)$ of degree k. (To see why we also need $x = 0$ for the points at infinity see the proof of Corollary (5.7).) The zero set of I^h and G^h must be the same. Moreover since the curve C_1, (recall equation (5.8)), has points at infinity the system

$$\begin{aligned} (y_1^d - y_2^d)^p &= 0 \qquad (5.12) \\ h(y_1, y_2) &= 0 \qquad (5.13) \end{aligned}$$

must have at least one solution with $(y_1, y_2) \neq (0, 0)$. Let $g(y_1, y_2) = \sum_{i=0}^{k} \alpha_i y_1^{k-i} y_2^i$ where $\alpha_i \in \mathbb{C}$. All the solutions of the first equation (5.12) are of the form $y_2 = \alpha y_1$, where $\alpha^d = 1$. To find a common solution of the system substitute $y_2 = \alpha y_1$ into the second equation (5.13). This gives

$$(\sum_{i=0}^{k} \alpha_i \alpha^i) y_1^k = 0.$$

Since $y_1 \neq 0$ we must have $\sum_{i=0}^{k} \alpha_i \alpha^i = 0$ for all d-th roots α of unity.

Hence $g(y_1, y_2) = (y_1^d - y_2^d)^l g_1(y_1, y_2)$ for some integer l and for some polynomial $g_1(y_1, y_2)$.

If there is an element h in the reduced Gröbner basis whose leading term is y_2^{dk} for some integer k, then h is of the form

$$h(x, y_1, y_2, z) = A \cdot (y_1^d - x^m - \cdots) + B \cdot (y_2^d - x^m - \cdots)$$

for some polynomials A and B. The degree of h must also be dk since $y_1 > y_2 > x$. We now start guessing what terms should A and B contain: B must have a $y_2^{d(k-1)}$ term so that we can have y_2^{dk} as the leading term in h. But this gives a term of the form $y_2^{d(k-1)}x^m$ which should be cancelled by having a term of the form $y_2^{d(k-1)}$ in A. This however will give $y_1^d y_2^{d(k-1)}$ which should be cancelled. To cancel it B must have $y_1^d y_2^{d(k-2)}$. Continuing in this manner we see that B should eventually contain a $y_1^{d(k-1)}$ term but to cancel the $y_1^{d(k-1)}x^m$ term arising from the multiplication we must have a $y_1^{d(k-1)}$ term in A. This gives y_1^{dk} as a term of h and it cannot be cancelled. This however contradicts the assumption about the leading term of h since $y_1 > y_2$. Hence we conclude that the leading term of an element in the Gröbner basis is either of the form y_1^{dk} or is divisible by x.

We now return to g. It is now clear that any point at infinity will be contributed by g alone. After homogenizing g with respect to z and setting $x = 0$ and $z = 0$ we have $g(y_1, y_2) = 0$ giving all the roots for the points at infinity. Since any root of g is in the common solution set of I^h, then it must also satisfy $(y_1^d - y_2^d)^p = 0$ so it must be an r-th root of unity where $r|d$. This then proves that the structure of g is as claimed. $\square$

Conjecture 5.6 *In equation (5.11) of Corollary (5.5) we actually have $l = 0$.*

In our calculations with *Maple V* we always obtained $l = 0$. However here we neither need nor see a way of proving this conjecture...

Corollary 5.7 *Let C_1 be the curve defined by* (5.8). *Its projective closure has only the following points at infinity:*

$$[0 : 1 : \alpha : 0] \text{ where } \alpha^d = 1.$$

Proof: Homogenizing the ideal I with respect to z and setting $z = 0$ gives the description of the points at infinity. Note that f_1 is in I and $f^h(x, y_1, y_2, z) = y_1^d z^{(m-d)} - (x^m + a_1 x^{m-1} z + \cdots + a_m z^m)$. Setting $z = 0$ gives $f^h(x, y_1, y_2, 0) = -x^m$. Hence the x coordinates of all the points at infinity are zero. To find the y_1 and y_2 components of the points at infinity we consider the homogenization with respect to z of a reduced Gröbner basis and set $x = 0$ and $z = 0$. From corollary (5.5) we see that the only surviving element is g and setting $g^h(0, y_1, y_2, 0) = 0$ gives the points at infinity in the claimed form, where g^h denotes the homogenization with respect to z. $\square$

Corollary 5.8 *Let C_2 be the curve in $\mathbb{A}^3$ defined by,*

$$\begin{aligned} y_1^{d_1} &= x^m + a_1 x^{m-1} + \cdots + a_{m-1}x + a_m \\ y_2^{d_2} &= x^m + b_1 x^{m-1} + \cdots + b_{m-1}x + b_m \end{aligned} \tag{5.14}$$

with $d_1 < d_2 < m - 1$. Then the projective closure $\overline{C_2}$ of the curve C_2 in $\mathbb{P}^3$ is the union of C_2 and the point

$$[x : y_1 : y_2 : z] = [0 : 1 : 0 : 0]$$

Proof: With the same approach used in Lemma 5.4, we can show that the ideal generated by the above polynomials has an element of the form,

$$(y_1^{d_1} - y_2^{d_2})^p + f(x, y_1, y_2)$$

where $f(x, y_1, y_2)$ has degree less than or equal to pd_1, and deg $f(0, y_1, y_2) < pd_1$. Homogenizing this with respect to z, we obtain

$$(y_1^{d_1} z^{d_2-d_1} - y_2^{d_2})^p + f_h(x, y_1, y_2, z)$$

Setting $z = 0$ gives $y_2 = 0$, since $x = 0$ follows from homogenizing one of the generating polynomials and setting $z = 0$. This proves the corollary. $\square$

Combining Lemma 5.4 and Corollary 5.8 we generalize these two results to the curve given by (5.7).

Corollary 5.9 *Let C be the curve in $\mathbb{A}^{n+1}$ defined by (5.7), with $d_1 \leq d_2 \leq \ldots \leq d_n$. Assume that for some s the first s d_i's are equal, i.e. $d = d_1 = d_2 = \cdots = d_s < d_{s+1} < \cdots < d_n$. Then the projective closure $\overline{C}$ of the curve C in $\mathbb{P}^{n+1}$ is the union of C and the points of the form,*

$$[x : y_1 : \cdots : y_s : y_{s+1} : \cdots : y_n : z] = [0 : 1 : \alpha_2 : \cdots : \alpha_s : 0 : \cdots : 0]$$

where $\alpha_2^d = \cdots = \alpha_s^d = 1$.

$\square$

5.4.2 A Finite Morphism to $\mathbb{P}^1$

In order to compute the genus of a nonsingular model $\tilde{C}$ of the projective closure $\overline{C}$ of C we first define a finite morphism from $\tilde{C}$ to $\mathbb{P}^1$.

There exists a finite morphism

$$\begin{aligned} \varphi : C &\to \mathbb{C} \\ (x, y_1, \cdots, y_n) &\mapsto x \end{aligned}$$

C is embedded into $\mathbb{P}^{n+1}$ the same way $\mathbb{C}$ embeds into $\mathbb{P}^1$. The morphism φ extends to $\overline{C}$ algebraically by defining

$$\begin{aligned} \varphi : \overline{C} &\to \mathbb{P}^1 \\ [x : y_1 : \cdots : y_n : 1] &\mapsto [x : 1] \\ [0 : y_1 : \cdots : y_n : 0] &\mapsto [1 : 0] \end{aligned}$$

See also the parametrization (5.16) for a justification of this definition. If $\tilde{C}$ is a resolution of $\overline{C}$ then C and $\tilde{C}$ are isomorphic everywhere except at finitely many points which correspond to the points at infinity and φ extends over to $\tilde{C}$ by sending all the points at infinity to $[1 : 0]$ as above.

Thus we have a map

$$\varphi : \tilde{C} \to \mathbb{P}^1$$

which is a morphism of degree $d_1 d_2 \cdots d_n$.

5.4.3 Ramifications of φ

We first examine the $n = 2$ case with $d = d_1 = d_2$. Consider the curve C_1 given by equations (5.8). For the points in the affine plane we can take x as a local parameter. When x is not equal to any of the a_{ij}'s then the ramification of φ at x is 1. When $x = a_{ij}$, then the ramification of φ at x is d. (For the general case of equation (5.7) the ramification at a_{ij} is $d_1 \cdots \hat{d}_i \cdots d_n$, where $\hat{d}_i$ denotes that the term should be omitted.)

To examine the points at infinity choose a local parameter t with $x = 1/t$. Then we have

$$y_1^d = x^m + a_1 x^{m-1} + \cdots + a_{m-1} x + a_m$$

$$
\begin{aligned}
&= (1/t)^m + a_1(1/t)^{m-1} + \cdots + a_{m-1}(1/t) + a_m \\
&= (1 + a_1 t + \cdots + a_{m-1}t^{m-1} + a_m t^m)/t^m.
\end{aligned}
$$

Let

$$
\begin{aligned}
d &= ac \\
m &= bc, \quad (a,b) = 1, \quad c \geq 1.
\end{aligned} \tag{5.15}
$$

Define a new local parameter T such that

$$T^a = t.$$

Then the above parametrization of y_1^d becomes

$$y_1^{ac} = (1 + a_1 T^a + \cdots + a_m T^{abc})/T^{abc}.$$

Similarly we have

$$y_2^{ac} = (1 + b_1 T^a + \cdots + b_m T^{abc})/T^{abc}.$$

Let $H_1(T)$ and $H_2(T)$ be power series such that $y_1^{ac} = H_1^{ac}(T^a)/T^{abc}$ and $y_2^{ac} = H_2^{ac}(T^a)/T^{abc}$. Then the points around infinity are parametrized as

$$P(\alpha_1, \alpha_2, T) = \left[\frac{T^b}{\alpha_1 T^a H_1(T^a)} : 1 : \frac{\alpha_2}{\alpha_1} \frac{H_2(T^a)}{H_1(T^a)} : \frac{T^b}{\alpha_1 H_1(T^a)} \right], \tag{5.16}$$

where α_1 and α_2 are d-th roots of unity. Note that $H_1(0) = H_2(0) = 1$ and thus the points at infinity are of the form $[0 : 1 : \alpha_2/\alpha_1 : 0]$ as claimed in Corollary (5.9). In the T-plane let T_1 and T_2 be two points such that $T_2 = \lambda T_1$ where λ is an a-th root of unity. We have $T_1^a = T_2^a$ but $T_1^b \neq T_2^b$ since $(a,b) = 1$. Hence $P(\alpha_1, \alpha_2, T_1) \neq P(\alpha_1, \alpha_2, T_2)$. As T ranges in the T-plane $P(\alpha_1, \alpha_2, T)$ describes a branch of the curve at infinity. There are then $d^2/a = dc$ branches at infinity. Since there are d points at infinity, around each such point there are then c branches making the total of dc branches. Each branch corresponds to a different point on the resolution so there are dc points on the resolution corresponding to the points at infinity, i.e. the cardinality of the set $\varphi^{-1}([1:0]) \subset \tilde{C}$ is dc. Total ramification index for the preimage of any point under φ, i.e. the degree of φ, is d^2. This gives a ramification index of a for each point in the resolution corresponding to the point at infinity.

In the general case when $d = d_1 = \cdots = d_n$, the total ramification index of φ is d^n, there are $d^{n-1}c$ branches at infinity each having ramification index a. This is the case for the curve define with the equations (5.18).

In the most general case, see equations (5.7), when $d = d_1 = \cdots = d_s < \cdots < d_n$ there are $d^{s-1}c$ branches at infinity each with ramification index

$ad_{s+1}\cdots d_n$. In this case the cardinality of $\varphi^{-1}([a_{ij}:1])$ is $d_1\cdots\hat{d_i}\cdots d_n$ and the ramification index of each such point is d_i-1. The total degree of φ is $d^s d_{s+1}\cdots d_n$.

5.4.4 The Genus Calculation

The Riemann-Hurwitz formula for the map φ takes the form

$$\begin{aligned} g_C &= 1-\deg\varphi+\tfrac{1}{2}\textstyle\sum_{x\in C}(e_x-1) \\ &= 1-\deg\varphi+\tfrac{1}{2}\textstyle\sum_{x\in\varphi^{-1}([*:1])}(e_x-1)+\tfrac{1}{2}\textstyle\sum_{x\in\varphi^{-1}([1:0])}(e_x-1), \end{aligned} \tag{5.17}$$

where e_x denotes the ramification index.

Theorem 5.10 [2] *Let C be the complete intersection curve given by,*

$$\begin{aligned} y_1^d &= (x-a_{11})\cdots(x-a_{1m}) \\ y_2^d &= (x-a_{21})\cdots(x-a_{2m}) \\ \vdots & \quad \vdots \\ y_n^d &= (x-a_{n1})\cdots(x-a_{nm}) \end{aligned} \tag{5.18}$$

where $d+1\le m$, and all a_{ij}'s are distinct. The genus of C is given by the formula

$$g_C = 1-\frac{1}{2}(d-mnd+mn+c)\,d^{n-1} \tag{5.19}$$

where $c=(d,m)$.

Proof: The degree of φ is d^n. The ramification index at finite points x such that $\varphi(x)\neq a_{ij}$ is 1 and for each point $x\in C$ for which $\varphi(x)=a_{ij}$ the ramification index is d. There are d^{n-1} points in $\varphi(x)=a_{ij}$ and the number of a_{ij}'s is mn. This gives $\frac{1}{2}mnd^{n-1}(d-1)$ for the first summation in (5.17).

There are $d^{n-1}c$ points on the resolution of the projective closure of C corresponding to points at infinity. Each such point has ramification index a. This then gives $\frac{1}{2}d^{n-1}c(a-1)$ for the second summation in (5.17). Putting these in and simplifying gives the seeked formula. □

Remark 5.11 *Putting in $d=2$, $c=1$ we recover Stepanov's formula $1+(mn-3)2^{n-2}$, see [42, p37, Lemma 1]. Stepanov arrives at this formula by* *constructing an explicit basis for the differential forms of the curve. He works over a finite field F_q of characteristic $p > 2$.*

We also have an explicit "counting the differentials" method for the genus of the curve C of equation (5.18). See also the equations (5.15) for the conventions in use. For any point in the affine space let x be a local parameter and consider the regular 1-form

$$\omega_{i_1,\dots,i_\sigma}^{(j_1,\dots,j_\sigma)} = \frac{dx}{y_{i_1}^{j_1}\cdots y_{i_\sigma}^{j_\sigma}}$$

where $1 \le \sigma \le n$, $1 \le i_1 < \cdots < i_\sigma \le n$ and $1 \le j_1, ..., j_\sigma \le d-1$. By checking the order of vanishings of x and y_i's it can be shown that the form $\omega_{i_1,\dots,i_\sigma}^{(j_1,\dots,j_\sigma)}$ is regular at any point in the affine space. Let x_∞ be any point at infinity on the projective closure of C. Let ν_∞ denote the order of vanishing of a function at x_∞. Choosing $t = 1/x$ as a local parameter around x_∞ we observe that

$$\begin{aligned} \nu_\infty(x) &= -a \\ \nu_\infty(y_i) &= -b. \end{aligned}$$

Let $\bar{\omega}_{i_1,\dots,i_\sigma}^{(j_1,\dots,j_\sigma)}$ denote the expression for $\omega_{i_1,\dots,i_\sigma}^{(j_1,\dots,j_\sigma)}$ around x_∞. We then have

$$\nu_\infty(\bar{\omega}_{i_1,\dots,i_\sigma}^{(j_1,\dots,j_\sigma)}) = (j_1 + \cdots + j_\sigma)b - a - 1$$

and if $P(x)$ is a polynomial then $P(x)\omega_{i_1,\dots,i_\sigma}^{(j_1,\dots,j_\sigma)}$ is regular at x_∞ if and only if $\deg P(x) \le ((j_1+\cdots+j_\sigma)b - a - 1)/a$. We can then give a basis for the regular differential 1-forms;

$$\begin{aligned} \{x^r \omega_{i_1,\dots,i_\sigma}^{(j_1,\dots,j_\sigma)} \quad | \quad & \sigma = 1, ..., n,\ 1 \le i_1 < \cdots < i_\sigma \le n, \\ & 1 \le j_1, ..., j_\sigma \le d-1, \\ & 0 \le r \le ((j_1 + \cdots + j_\sigma)b - a - 1)/a\ \}. \end{aligned}$$

The cardinality of this set then gives the genus of the curve C. It turns out that the required formula is

$$g(C) = \sum_{\sigma=1}^{n} \quad \sum_{1\le i_1<\cdots<i_\sigma\le n} \quad \sum_{j_1=1}^{d-1} \cdots \sum_{j_\sigma=1}^{d-1} [\![\tfrac{(j_1+\cdots+j_\sigma)b-1}{a}]\!], \tag{5.20}$$

where $[\![\quad]\!]$ denotes the greatest integer function. Note that this formula now works on any algebraically closed field of any characteristic, when $a \neq 0$.

Stepanov has calculated this sum for $d = 2$ and $c = 1$ over a field of characteristic $p > 2$, [42, p372], (in that case $d = a = 2$ and $m = b$ is odd).

We finally give the formula for the most general case.

Corollary 5.12 [2] *Let C be the complete intersection curve given by (5.7), with $d_1 \leq d_2 \leq \ldots \leq d_n$ and with the first s d_i's equal to d. The genus of C is given by the formula*

$$\begin{aligned} g_C &= 1 - \frac{1}{2}\left(d - mnd + ms\right) d^{s-1} d_{s+1} \cdots d_n - \frac{1}{2} d^{s-1} c \\ &\quad - \frac{md^s}{2} \sum_{i=s+1}^{n} d_{s+1} \cdots \hat{d}_i \cdots d_n, \end{aligned} \tag{5.21}$$

where $c = (d, m)$.

□

Remark 5.13 *This corollary can be proved in the same way as Theorem (5.10). The ramification values required for the formula are given at the end of section (5.4.3).*

Remark 5.14 *Note that when we put $s = n$ in the above formula (5.21) we recover the formula (5.19) of Theorem (5.10). However this is only an algebraic phenomena since geometrically the two formulas are derived from different configurations at infinity.*

Chapter 6

Conclusion

We have shown that in affine l-space with $l \geq 4$, minimal number of generators of the tangent cone of a monomial curve $(\mu(I(C)_*))$ can be arbitrarily large, contrary to the case $l = 3$ shown by Robbiano and Valla. Thus, in higher dimensions there is a more complex phenomenon, which is closely related with the structure of the corresponding semigroup. The logical continuation may be to use the determined families of monomial curves having Cohen-Macaulay tangent cone to give a sort of classification of semigroups and thus classification of monomial curves.

We studied the problem by using the ring $k[x_1, x_2, \cdots, x_l]/I(C)_*$, since we tried to use Gröbner theory to find the generators of $I(C)_*$ and to check the regularity of an element. This computational aspect helped us a lot; this result would not have been obtained by considering the semigroup ring for checking the Cohen-Macaulayness of the tangent cone of the monomial curve because it does not tell anything about the number of the generators of $I(C)_*$. Cohen-Macaulayness of the tangent cones of the families of monomial curves made Hilbert series and Hilbert polynomial computations possible. Thus, it is a joyful example of using computational methods in solving geometric problems.

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Vita

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