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Aeschylus said that his tragedies were fragments from the great banquets of Homer; Athenaeus VIII, 39, 347e.

PREFACE

The basic facts in the theory of algebraic surfaces were discovered, in the latter half of the nineteenth century and the beginning of the twentieth, by Max Noether, Picard, Poincaré and in particular by the members of the classical Italian school of algebraic geometry, Castelnuovo, Enriques and Severi. Their results were the starting point for the next stage in the development of algebraic geometry, which was based on the application of topological, analytic and algebraic methods.

During this stage it became clear that the results making up the "classical" theory of algebraic surfaces fall into two fundamentally different classes.

Some of them are special cases of general theorems about algebraic varieties (or schemata) of arbitrary dimensionality. The clearest examples are provided by the theory of Picard varieties or of abelian varieties or by the Riemann-Roch theorem. It is interesting to note that almost all the results in the classical survey of Zariski [16] are exactly of this kind. At the present time there are in existence many excellent expositions of the theories relating to this part of the subject.

The results of the second class deal specifically with algebraic surfaces. Here belong such basic features of the subject as the criterion for whether an algebraic surface is rational or ruled, the solution of the problem of Lurőth, the theory of minimal models, and the great complex of results which are grouped together by the Italian algebraic geometers under the heading of "classification of algebraic surfaces".

It seems that none of these results can be extended to varieties of higher dimension without the most essential changes, and at the present time even the very nature of such changes remains entirely unknown. Some attempts have been made to provide these results with proofs that are rigorous from a modern point of view and are based on present-day techniques, and also to extend the results as far as possible. The first (and basic) publications in this direction are the proof presented by Kodaira [48] for the criterion of rationality and the articles and book of Zariski [19] on the problem of minimal models. The purpose of the present book is to give a connected account of this whole range of questions. Below we give a short description of the contents of the book.

The entire theory is based on the connection between rational mappings of a variety into a projective space and the classes of divisors on the variety (see, for example, [30]). For algebraic curves it is well known that "almost everywhere" (for nonhyperelliptic curves) the mapping that corresponds to the canonical class is birational and defines the so-called canonical model of the curve uniquely up to a projective transformation. Thus the problem of birational classification is reduced to questions of the projective classification of curves in space.

A mapping corresponding to a canonical class of multiplicity two or three plays the same role for all curves other than rational and elliptic ones. For those curves a mapping corresponding to a canonical class of any multiplicity will not be birational. This method is in general inapplicable for their description, which, however, is easily obtained from other considerations.

The question is whether one obtains an analogous situation for algebraic surfaces. One first considers the class of those surfaces for which a canonical class of any multiplicity gives a birational mapping. It turns out that it is always sufficient to take a canonical class with a multiplicity not larger than nine (but a multiplicity of three, which plays the same role in the theory of curves, may be insufficient, as is shown by examples). The surfaces of this kind may be characterized in a simple manner: they are those nonrational surfaces for which the index of selfintersection of the canonical class is positive.

The rest of our task is the description of the remaining surfaces, those for which no canonical class of any multiplicity defines a birational mapping. These surfaces are analogous, from this point of view to the rational and elliptic curves. Their constructive description is also to a great extent analogous to the description of those curves. Namely, the surfaces we are considering fall into the following five groups: (1) rational surfaces; (2) two-dimensional abelian varieties; (3) ruled surfaces, i.e., surfaces made of families of rational curves; (4) surfaces made of families of elliptic curves; and (5) certain surfaces that are similar to abelian varieties in that their canonical class is zero, but which, unlike abelian varieties, have their first Betti number equal to zero.

In order to examine all these algebraic surfaces, we divide them into four groups on the basis of the value of an important invariant, which we denote by κ. The symbol κ stands for the maximal dimension of the image of the surface under rational mappings corresponding to different multiplicities of the canonical class. It is clear that κ is always less than or equal to two. If the linear systems corresponding to all the multiplicities of the canonical class are empty, then we set $\kappa = -1$. Thus κ may take on the four values -1, 0, 1 or 2. The goal of the classification is to give the character of the surfaces with a given value of κ with the aid of the so-called numerical invariants (the index of the selfintersection of a canonical class, plurigenera defined by the formula (11), and the irregularity defined by the formula (12)) and to give a constructive description of them. The results of the classification are given in the table at the end of the introduction.

The book also contains results outside the above mentioned theory, which are, however, related to it.

We shall discuss in detail the following: the theory of birational transformations of surfaces, the theory of minimal models, and Noether's theorem.

The theory of birational transformations of surfaces is based on the concept of the σ-process. This is a birational transformation

$$f:\ V \to V'$$

of nonsingular surfaces V and V', which is biregular everywhere except at a point $P \in V$ and a curve $C \subset V'$, where, moreover, f^{-1} is regular and $f^{-1}(C) = P$. The following are basic results:

(1) if ϕ is a birational mapping

$$\phi:\ V \to V'$$

of nonsingular surfaces such that ϕ^{-1} maps V' regularly onto V, then ϕ is the product of a finite number of σ-processes;

(2) any birational transformation of a nonsingular surface onto a nonsingular surface is the product of a finite number of σ-processes and a finite number of transformations inverse to σ-processes.

The theory of minimal models studies those surfaces V (called minimal) which are such that any regular birational transformation $f:\ V \to V'$ is biregular. Every surface is birationally equivalent to a minimal one, from which it is obtained, consequently, by a finite number of σ-processes. The basic theorem says that in the class of surfaces birationally equivalent to each other, there is only one minimal one if the surfaces are not ruled.

The minimal models of ruled (in particular, rational) surfaces are all described.

Finally, Noether's theorem relates to the structure of the group of all birational transformations of a projective plane (or, what is the same, of the group of the automorphisms of the field of rational functions $k(x, y)$ of two variables). It shows that this group is generated by the so-called quadratic transformations:

$$x' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}, \qquad y' = \frac{a_3x + b_3y + c_3}{a_4x + b_4y + c_4}$$

The classical results presented in this book may almost all be found in the survey of Enriques [59]. The present work is very closely connected with Enriques' book. The proofs of a large part of the theorems are based on ideas of Enriques. At the same time, it would hardly be possible to carry out the details of Enriques' proofs, for example, following the customs of the time and his school, he frequently limited himself to the consideration of a "general" case, not choosing the most unpleasant cases that might be examined. On the other hand, for certain questions we can supplement the classical results with new ones. This is true, for example, of certain results in Chapters V, VII, and IX. Finally, there are a few divergences from assertions of Enriques.

We do not aim for the greatest possible generality in the conditions imposed on the base field. All results are true if this field coincides with the field of complex numbers. The majority of arguments, however, retain their validity if the base field is algebraically closed and has characteristic 0, and some arguments remain valid for any algebraically closed field. These considerations are discussed in more detail in each chapter.

The present book is based on reports on seminars in the theory of algebraic surfaces held in 1961–1962 and 1962–1963 under the leadership of I. R. Šafarevič. The texts of the reports were then worked over, and certain parts were rewritten. The individual chapters were written by the following authors: Chapters I, II and III by A. B. Žižčenko; Chapters IV and VII by I. R. Šafarevič; Chapter V, §§1 and 2, by Ju. I. Manin §§3–6 by Ju. R. Vaĭnberg and Ju. I. Manin, §7 by A. N. Tjurin; Chapter VI by B. G. Moĭšezon; Chapters VIII and X by B. G. Averbuh; Chapter IX by G. N. Tjurin.*

* Translator's note: In a more recent article, *On special types of Kummer and Enriques surfaces*, Izv. Akad. Nauk SSSR 29 (1965), 1095–1118, Averbuh fills out some gaps in the classification which he began in Chapters VIII and X. This article has been translated as the Appendix to the present volume.

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TABLE OF CONTENTS

INTRODUCTION

The present book uses many facts from the theory of algebraic varieties, mostly from the theory of algebraic surfaces and curves. We shall list the basic results, knowledge of which is indispensable for the reading of this book, with references to the literature available to the reader. Moreover, we shall introduce below certain definitions and the proofs of some results which, although included in periodicals, may not be considered general knowledge.

All preliminary definitions and results (the definition of an algebraic variety, the field of functions over it, etc.) will be found by the reader in the books of Hodge and Pedoe [55] and Lang [30].

We assume the knowledge of the well-known basic definitions and facts of the theory of rational and birational transformations: the definition of a rational and of a birational transformation, the concepts of exceptional and fundamental subvarieties under birational transformations, the basic properties of fundamental and exceptional subvarieties, the concept and basic properties of a regular (rational) mapping, the concept of a normal variety, of normalization (Hodge and Pedoe [55], Lang [30]).

An algebraic correspondence T between algebraic varieties V and V', in particular a rational or birational mapping, is defined by the graph $U \subset V \times V'$. We shall say that the pair (x, x'), $x \in V$, $x' \in V'$ belongs to the correspondence if $x \times x' \in U$; also, the point $x' \in V'$ will correspond to $x \in V$, if $x \times x' \in U$. If $W \subset V$ is some subvariety of V and T is an algebraic correspondence between V and V', then by $T(W)$ we shall mean the set of points in V' corresponding to the points of W; thus, $x' \in T(W)$ if there exists a point $x \in W$ such that $x \times x' \in U$. As is known, $T(W)$ will be an algebraic set [55]. We shall call $T(W)$ a complete image (or total transform). A rational mapping $T\colon V \longrightarrow V'$ induces a homomorphism of the function fields $k(V') \longrightarrow k(V)$; this homomorphism will be denoted by T^*.

The concept of divisor of a function, the definition of the linear and algebraic equivalence of cycles, the construction of the group of classes of linear equivalence and of the group of classes of algebraic equivalence for cycles, the concept of the space of functions associated with a divisor, may all be found by the reader in the books [8] and [11] and in the article [12].

The basic facts about these groups (in particular, the finiteness of the rank of the Néron-Severi group of the classes of algebraic equivalence for divisors) are given in the book of Hodge and Pedoe [55] and in [8].

The theory of the intersection of algebraic cycles on algebraic varieties is set forth in Serre's article [51], and for algebraic surfaces, in Zariski's book [23].

The possibility of using intersections to introduce multiplication in the group of the classes of linear equivalence, thanks to which this group acquires the structure of a model ring (Chow ring of an algebraic variety), is proved in the article of Samuel [43] and in the notes of Chevalley's seminar [46].

The definition of linear systems of divisors on an algebraic variety and their connection with rational mappings is included in Lang's book [30]. The theorem of Bertini on this subject is given in §3, Chapter I, of the present work.

This work uses many facts from the theory of algebraic curves. The Riemann-Roch theorem, the formula for the genus of covering of a curve, the formula for the arithmetic genus of a curve on a surface, and the Riemann-Roch theorem for a curve with singularities, may be found in the books of Chevalley [58] and Serre [49].

The Riemann-Roch theorem for a surface and the formula for the arithmetic genus of a surface is found in the book of Serre [49], the article of Zariski [21], and the article of Borel and Serre [9].

The basic properties of differential forms on algebraic varieties and their behavior under regular mappings is set forth in Lang's book [30].

Finally, the basic properties of abelian varieties, Albanese varieties, etc., are included in Lang's book [31].

The concept of the local ring of a point, of a subvariety, and theorems about the behavior of local rings under birational transformations may be found in the books of Hodge and Pedoe [55] and of Lang [30].

The basic method used in this work is the method of coherent sheaves. The description of this method, its connection with the classical method of linear systems, and also the basic theorems about coherent algebraic sheaves is found in the articles of Zariski [21] and Serre [50].

We now introduce two definitions which are not well known.

Let there be given a rational mapping $T\colon V \longrightarrow V'$ and let C be some division on V with a generic point x. Since the fundamental points of T have dimension not exceeding $n-2$, the mapping T is regular at the point x, and consequently the point $y = T(x)$ is defined when $k(y) \subset k(x)$, where clearly $k(y) \subset k(V')$. We consider a subvariety C' on V' with a generic point y.

Two cases are possible:

1. $\dim C' = \dim C$, i.e., $tr^{k(y)/k} = tr^{k(x)/k}$.

We denote by n_C the number $[k(x)\colon\ k(y)]$; then we call the divisor $n_C C'$ the image (proper, algebraic) of the divisor C.

2. $\dim C' < \dim C$, i.e., $tr^{k(y)/k} < tr^{k(x)/k}$. In this case we shall say that the image of C is null (it is convenient to say in this case that $n_C = 0$) or is a point ($T(x)$, since such a point is defined). When $\dim V = \dim V'$, a mapping thus defined gives a homomorphism of the group of divisors $D(V)$ into the group $D(V')$, which we shall indicate by $T\colon\ D(V) \longrightarrow D(V')$.

The image of the divisor $T(C)$ (sometimes written $T[C]$ to distinguish it from the total transform) under the mapping will be called an algebraic (or proper) transform.

If the mapping T is birational, then, as easily follows from the properties of birational mappings, n_C will be either 0 or 1.

We consider in this case the inverse mapping $T^{-1}\colon\ V' \longrightarrow V$, and let D' be a divisor on V'.

The image $(T^{-1})(D')$ is defined. If $(T^{-1})(D') = D \neq 0$, $D \subset V$, then, as is not difficult to verify, $T(D) = T((T^{-1})D') = D'$.

When T is a regular birational mapping without fundamental points on V, the mapping T is clearly an epimorphism:

$$T\colon\ D(V) \xrightarrow{\text{onto}} D(V').$$

Now let $T\colon\ V \longrightarrow V'$ be a regular mapping of V onto V' (possibly birational). In this case we shall define a mapping $T^*\colon\ D(V') \longrightarrow D(V)$ in the following manner.

Let C' be a divisor on V', $P' \in C'$. As is known from [55], the divisor C' is then, in a neighborhood of the point P', a local P'-component of a divisor $(g_{P'})$ of some function $g_{P'} \in k(V)$ (V' is nonsingular). The equation $g_{P'} = 0$ will be called the local equation of the divisor C' at the point P'. Now let P be some point on V corresponding to the point $P' \in V'$ (it is possible that P does not correspond to a single point, but perhaps to some divisor of V).

The function $g_{P'}$ may be regarded as a certain function on V (since the inclusion $k(V') \subseteq k(V)$ exists). It is not difficult to see that, since the point P' belongs to a zero divisor of the function $g_{P'}$ on V', the point P belongs to a zero divisor of the function $g_{P'}$ on V, where, moreover, if the divisor D on V corresponds to the point P', then that divisor D will be included in a zero divisor of the function $g_{P'}$ with multiplicity equal to the multiplicity of the point P', as a point of the divisor C' (see Hodge and Pedoe [55]).

Considering an affine covering $\{U'\}$ of a variety V' and a system of local equations $\{g_{P'} = 0\}$ for a divisor C' in this covering, we obtain upon passing to the variety V a system of local equations $\{g_{P'} = 0\}$ corresponding to some covering $\{U\}$ of the variety V. It is possible to show that the system $\{g_{P'} = 0\}$ in the covering $\{U\}$ will be consistent; i.e., that we obtain some divisor C on V.

The divisor on V thus determined is denoted by $T^*(C')$ and is called the pre-image of the divisor $C' \subset V'$.

It is well known that when $T\colon V \longrightarrow V'$ is regular, the mappings T and T^* take linearly equivalent divisors into linearly equivalent ones, and algebraically equivalent divisors into algebraically equivalent ones, and also they induce homomorphisms of Chow rings (see the articles of Borel and Serre [9] and Samuel [43]).

Besides these general remarks, in certain chapters reference will be made to several special facts from the theory of algebraic and analytic varieties (for example, in Chapter IX). Exact indications to the appropriate literature will be given in these places.

We now introduce basic symbols and certain basic formulas to which we shall frequently refer in the text.

(1) $D \sim C$, $D \overset{V}{\sim} C$ –linear equivalence of the cycles D and C (on V);

(2) $D \approx C$, $D \overset{V}{\approx} C$ –algebraic equivalence of the cycles D and C, and also the homology of the cycles D and C (on V);

(3) $D \geq 0$ –an effective divisor;

(4) (f), $(f)_V$ –a divisor of the function f on the variety V;

(5) $\mathfrak{L}(D)$ –space of functions such that $(f) + D \geq 0$ on V;

(6) $|D|$ –a linear system of divisors (linearly) equivalent to D;

(7) O_P –the local ring of the point $P \in V$;

(8) $C \cdot D$ –the intersection cycle of the two cycles D and C;

(9) $(C \cdot D)$ –index of intersection of the curves C and D on a surface;

(10) K, K_V –the canonical class of the surface V (sometimes also an arbitrary representative of that class);

(11) $P_n = l(nK)$ –n-genus, $p_g = l(K)$ –geometric genus;

(12) q, $q(V)$ –the dimension of the space of one-dimensional forms of the first kind on a surface, irregularity of the surface V;

(13) $p_a(V)$ –the arithmetic genus of the surface V, according to the definition $p_a(V) = 1 - q + p_g$;

(14) $S(V)$ –the Néron-Severi group of the surface V (the factor group of the group of divisors of V over the subgroup of divisors algebraically equivalent to 0);

(15) $A(V)$ –the Albanese variety of the variety V, $\dim A(V) = q(V)$, and $\alpha_V\colon V \longrightarrow A(V)$ the canonical mapping, everywhere regular on V;

(16) $\rho(V)$ –Picard number of the surface V, the rank of the group $S(V)$ (this number is finite [8], [16]);

(17) χ, $\chi(V)$ –Euler characteristic of the surface;

(18) $F(D)$ –coherent sheaf on the surface corresponding to the divisor D (sometimes also $0[D]$, $\Omega(D)$) (Zariski [21]).

The numbers P_n, q, and p_a are birational invariants.

To see this, let T: $V \longrightarrow V'$ be a birational mapping of V onto V'. Each element $\omega \in \mathfrak{L}(nK_{V'})$ can be considered as a differential form $\phi(dx \wedge dy)^n$, regular on V'.

The inverse image of this form of V, a form $\overline{\phi}(d\bar{x} \wedge d\bar{y})^n$, where $\overline{\phi}$, $\bar{x}$, $\bar{y} \in k(V)$, will be a regular form on $V - S$ (S being the set of fundamental points of T on V). Since $\dim S \leq n - 2$, this form will be regular everywhere on V, since the set of poles of the form is a divisor on V. Thus $P_n(V) \geq P_n(V')$, and conversely, i.e., $P_n(V) = P_n(V')$. One shows in exactly the same way that $q(V) = q(V')$.

BASIC FORMULAS

If D is a divisor on the surface D, and $p_a(D)$ is its arithmetic genus, then [49]

$$p_a(D) = \frac{(D\cdot(D+K))}{2} + 1. \tag{1}$$

The Riemann-Roch formula on the surface V for the divisor D is

$$l(D) = \frac{(D(D-K))}{2} + p_a(V) - l(K-D) + \Delta_D, \tag{2}$$

where Δ_D is the superabundance of D and is equal to $\dim H^1(V, F(D)) \geq 0$.

The Riemann-Roch inequality is

$$l(D) + l(K-D) \geqslant \frac{(D(D-K))}{2} + p_a(V). \tag{3}$$

The formula for the arithmetic genus of a surface [9] is

$$p_a(V) = \frac{(K^2)+\chi}{12}. \tag{4}$$

The formula for the arithmetic genus of an irreducible curve C is

$$p_a(C) = g + \delta, \qquad \delta = \sum_p \delta_p, \tag{5}$$

where g is the genus of the nonsingular model of C (the geometric genus of C); $\Sigma\delta_p$ is extended over all singular points of the curve C, $\delta_p > 0$ and δ_p is not smaller than the number of points of the nonsingular model of C that map into P under the canonical projection (Serre [49]).

Table of Types of Algebraic Surfaces

κ	Invariants	Classes divided into	Types of surfaces	Chapters, §§
-1	$P_{12}=0$	$q=0,\ P_2=0$	Rational	Ch. III; Ch. V, §4.
		$(q>1,\ p=0) \Leftarrow \Rightarrow (K^2)<0$ $q=1,\ P_{12}=0$	Ruled	Ch. IV; Ch. V, §7.
0	$12K=0$	$P_2=1,\ p=0,\ q=0$	Enriques surface, $2K=0$.	Ch. VIII, §1; Ch. X.*
		$P_2=1,\ p=1,\ q=0$	Regular surfaces with $K=0$.	Ch. IX.
		$P_4=1,\ p=1,\ q=2$	Two-dimensional abelian varieties.	Ch. VIII, §4.*
		$P_{12}=1,\ q=1$	Surfaces with a bundle of elliptic curves. The basis of the bundle is a projective line. All fibers, except for a finite number of multiple fibers, are isomorphic. $2K=0$, or $3K=0$, or $4K=0$, or $6K=0$.	Ch. IV, §§7, 8; Ch. VII, §9.
1	$(K^2)=0$ $12K\neq 0$		Surface with a bundle of elliptic curves, with the exception of those that belong to the above type.	Ch. VII.
2	$(K^2)>0$		A linear system $\lvert 9K\rvert$ gives a birational mapping into a projective space of dimension P_9-1.	Ch. VI.

Minimal models of surfaces are understood in this table. The first column contains the value of the invariant κ. The second has the characteristic of the class of surfaces in terms of the other invariants: P_n, q, (K^2), and K. The third column, where filled in, contains a division into narrower classes of surfaces, while the fourth has the constructive character of the same surfaces. Finally, the last column shows the chapters and sections of the book in which the surfaces are described.

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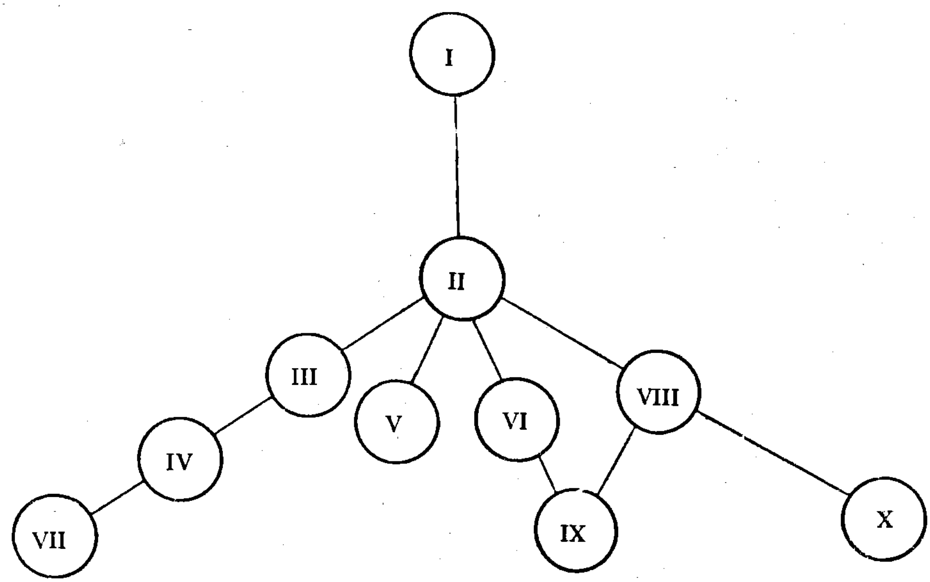
* Translator's note: See also the Appendix.

Schema of Dependence of the Chapters*



* Translator's note: The Appendix depends on Chapters VIII–X.

CHAPTER I

BIRATIONAL TRANSFORMATIONS

In this chapter the description of one important type of birational transformation—the σ-process, is given. The application of a finite sequence of σ-processes makes it possible to annihilate the fundamental points of a birational transformation. It is proved that the application of a finite number of σ-processes leads to the resolution of the singularity of a curve lying on a surface. A proof of Zariski's theorem on multiple linear systems is presented.

§1. σ-processes

In the sequel an essential example of a birational transformation will be the so-called σ-process (other names: "local quadratic transformation," "blowing up a point," "local modification," "local dilatation").

By a σ-process at a nonsingular point Q of the algebraic surface (variety) V we will understand a birational transformation σ: $V \longrightarrow V^*$ satisfying the following conditions:

(1) σ is regular everywhere on V except at the point Q;

(2) σ^{-1} is regular everywhere on V^*;

(3) to the point Q corresponds some line L (respectively, space P^{d-1} if $\dim V = d$) on V^*;

(4) all the points of the line L (space P^{d-1}) are nonsingular on V^*.

Thus a transformation σ is a biregular mapping of $V - Q$ onto $V^* - L$, where $\sigma^{-1}(L) = Q$. We will show that it is possible to introduce a σ-process at any nonsingular point of a surface V. Thus, let Q be a nonsingular point on a surface V and let $\{U\}$ be a covering of the surface with affine sets. From this covering we will construct a new collection of affine sets $\{U'\}$ and will show that this will be a covering of some surface V^* such that a certain mapping σ: $V \longrightarrow V^*$ will satisfy all the conditions of a σ-process at the point $Q \in V$.

If an affine set $U \in \{U\}$ does not contain the point Q, then we include it in our collection $\{U'\}$. If $Q \in U$, then it is possible to choose local parameters x, y at the point Q (which is nonsingular) such that x and y are regular functions on all of U.

We consider the direct product $U \times P^1$ of a set U and a projective line P^1. We designate by (t_0, t_1) the projective coordinates of P^1 and consider the subvariety $U' \subset U \times P^1$ defined in the following manner:

$$U' = \{(u;\ t_0,\ t_1) \subset U \times P^1,\ t_0 y(u) - t_1 x(u) = 0\}.$$

It is clear that the projective line $Q \times P^1 = L_u$ lies on U'. It is easy to see that the mapping (projection)

$$\sigma_U\colon\ (u;\ t_0,\ t_1) \longrightarrow u$$

will be an isomorphism outside of L_u; the line L_u is itself projected into the point Q.

The subvariety U' is the union of two affine coverings of sets U_0' and U_1'; U_0' consists of the points of U' such that $t_0 \neq 0$, while U_1' of the points of U' such that $t_1 \neq 0$. We note that at points $Q' = (Q;\ t_0,\ t_1) \in U_0' \cap L_u$ the functions x and $y/x - c^*$ (where $c^* = t_1/t_0$) will serve as local coordinates, while at points $Q' = (Q;\ t_0,\ t_1) \in U_1' \cap L_u$ the functions y and $x/y - b^*$ will serve (here $b^* = t_0/t_1$). In the first case the local equation of the line L_u at the point Q' has the form $x = 0$, while in the second case it is $y = 0$. The sets U_0' and U_1' will be called the preimages of the set $U \in \{U\}$. It is possible to show that the points of the line L_u are in one-to-one correspondence with the set P of tangential directions at the point Q. To see this, let α be some branch of a curve on U with center at the point Q. This branch has a parametric representation

$$\alpha:\begin{cases} x = \alpha_1\tau^{r_1} + \ldots = \varphi(\tau), \\ y = \beta_1\tau^{s_1} + \ldots = \psi(\tau), \end{cases}$$

where the tangent to this branch is defined as the ratio of the numbers r_1 and $s_1 (r_1 > s_1,\ r_1 < s_1)$, and when $r_1 = s_1$ as the ratio $\alpha_1 \colon \beta_1$.

To this branch will correspond some branch α^*: $(\phi(\tau),\ \psi(\tau),\ \phi(\tau)\colon \psi(\tau))$ on U', the center of which will be either the point $(0,\ 0,\ 1\colon 0)$ (if $s_1 > r_1$), or the point $(0,\ 0,\ 0\colon 1)$ (if $r_1 > s_1$), or the point $(0,\ 0,\ u_1\colon v_1)$ (if $r_1 = s_1$). Thus the center of the branch α^* is uniquely determined by only the tangential direction to the branch α at the point Q; it is also easy to see that to each point of L_u there corresponds some tangential direction. This correspondence will be denoted by $\mu_u\colon\ L_u \longrightarrow P$.

We wish to show that the collection of affine sets $\{U'\}$ forms a covering of some algebraic surface V^*.

For this it is sufficient to show that the elements of the covering $\{U'\}$ can be patched together in such a way so that as a result we obtain some algebraic surface V^*. We first note that it is possible to choose the original covering $\{U\}$ of the surface V such that the point Q will be included within only one element of this covering; for the sake of definiteness, let us say $Q \in \widetilde{U}$. If now U' and

W' are two elements of the covering $\{U'\}$, then three cases are possible. First of all, U' and W' can be simply two elements of the covering $\{U\}$ that do not contain the point Q. Then we patch them together in the same way that they were patched together as elements of the covering $\{U\}$. If $W' \in \{U'\}$ is an element W of the covering $\{U\}$, while U' is one of the preimages of the set $\widetilde{U}$, for example $\widetilde{U}'_0$, then we may patch U' and W' together in the following way. We write $\widetilde{U}_0 = \sigma_{\widetilde{U}}(U'_0)$. The mapping $\sigma_{\widetilde{U}}$ will clearly be a biregular mapping of $\widetilde{U}'_0 - \widetilde{U}'_0 \cap L_u$ onto $\widetilde{U}_0 - Q$. Inasmuch as $\widetilde{U}_0 \cap W \not\ni Q$, we identify the point $\beta \in U_0 \cap W$ with the point $\sigma_{\widetilde{U}}^{-1}(\beta) \in \widetilde{U}'_0$. Finally, we patch together the sets $\widetilde{U}'_0$ and $\widetilde{U}'_1$ in correspondence with the natural way they are patched together as affine subsets of the variety $\widetilde{U}' \in \widetilde{U} \times P^1$. It is clear that such an identification will be consistent, i.e. as a result we obtain an algebraic surface V^*.

There is thus constructed an algebraic surface V^* satisfying all the conditions of a σ-process (the birational equivalence of V^* and V is clear in view of the fact that $V^* - L$ is biregularly equivalent to $V - Q$).

The following fact was proved in passing: the points of the line L into which Q was blown up are in one-to-one correspondence with the set of tangential directions at the point Q. In exactly the same way we may construct a σ-process for blowing up varieties of any dimension.

If the variety V is imbedded in a projective space P^N, then the variety V^* obtained as a result of the σ-process will also be imbedded in some projective space $P^{N'}$.

This fact may be proved directly. However, there exists another description of a σ-process for imbedded varieties, for which this fact is trivial. Let us assume that the variety V is imbedded in P^N, and let $Q = (1, 0, \cdots, 0)$. Then every point $R \in V$, $R \neq Q$, is projected from Q onto the surface $x_0 = 0$; let S be the corresponding point. If $R = (x_0, x_1, \cdots, x_N)$ then $S = (x_1, \cdots, x_N)$. The pairs of points (R, S) fill out the algebraic variety V^*; the correspondence σ: $R \longrightarrow (R, S)$, σ: $V \longrightarrow V^*$ is the desired one. It is not difficult to establish the (local) isomorphism of these two descriptions of the σ-process.

Thus, we have a birational mapping σ: $V \longrightarrow V^*$ such that the inverse mapping σ^{-1}: $V^* \longrightarrow V$ is regular. In this case there are defined two homomorphisms of the groups of divisors $D(V)$ into the group $D(V^*)$:

$$\sigma:\ D(V) \longrightarrow D(V^*),$$

$$\sigma_*:\ D(V) \longrightarrow D(V^*),$$

where $\sigma(C)$ is the proper (algebraic) transform of the divisor $C \subset V$, and $\sigma_*(C)$ is the total transform of the divisor $C \subset V$. We shall write these homomorphisms similarly, for this will be important in the future. Let C be an irreducible curve on V. In a neighborhood U_P of an arbitrary point $P \in V$ the divisor C is given by the equation $\gamma_P = 0$, where γ_P is a rational (meromorphic) function on V. In view of our condition (that C is an irreducible curve), the function γ_P will be holomorphic in U_P, only on the curve C and having on it a zero of the first order (all in the neighborhood U_P). The equation $\gamma_P = 0$ will be the local equation of the curve C in the neighborhood U_P. If $P_3 \in U_{P_1} \cap U_{P_2}$, then the function $\gamma_{P_1}^{-1}\gamma_{P_2}$ is a unit at P_3. When $P \neq Q$ there exists a neighborhood U_P such that the mapping σ is biregular on U_P. The curve C^* given by the equations $\sigma^*\gamma_P = 0$ in the neighborhoods $U_{P^*} = \sigma(U_P)$ of the points $P^* = \sigma(P)$, $P \neq Q$, will clearly be a curve on $V^* - L$ that is biregularly equivalent to the curve $C - Q$ on $V - Q$.

Now let $\gamma_Q = 0$ be the local equation of a curve in a neighborhood U_Q. Let Q be a point of multiplicity m of the curve C. This means that the decomposition of the functions γ_Q according to the powers of the local parameters (x, y) at the point Q begins with the m th power:

$$\gamma_Q = \gamma_Q^m + \gamma_Q^{m+1} + \cdots,$$

where

$$\gamma_Q^m = a_0 x^m + a_1 x^{m-1} y + \cdots + a_m y^m$$

and not all the a_i are equal to 0.

Now let $P^* = (t_0^*, t_1^*)$ be some point on the line $L \in V^*$. We will assume that $t_0^* \neq 0$; then local coordinates at the point P^* will be the function $x_1 = x$ and $y_1 = t_1/t_0 - c^* = y/x - c^*$, where $c^* = t_1^*/t_0^*$. The local equation of the line L in the neighborhood U_{P^*} will be $x_1 = 0$. The function $\sigma^*\gamma_Q$ in this neighborhood can be written in the form $\sigma^*\gamma_Q = x_1^m[a_0 + a_1(y_1 + c^*) + \cdots + a_m(y_1 + c^*)^m + \cdots]$, from which it follows that the function $\sigma^*\gamma_Q$ has a zero of multiplicity m on L in the neighborhood U_{P^*}.

We consider the function $\sigma_1^*\gamma_Q = \sigma^*\gamma_Q/x_1^m$ in the neighborhood U_{P^*}. As we have just explained, this will be a holomorphic function in this neighborhood.

We consider the local equation $\sigma_1^*\gamma_Q = 0$ in the neighborhood U_{P^*}. It is not difficult to verify that the set of local equations $\{\sigma^*\gamma_P = 0$ if $P \neq Q$, and, consequently, $\sigma(P) \notin L$, and $\sigma_1^*\gamma_Q = 0$, if $P^* \in L\}$ will be a consistent set of equations (i.e. in the intersection of two arbitrary neighborhoods the ratio of corresponding functions will be a unit), and, consequently, will determine some curve C^* on V^*. Here $C^* - L \cap C^*$ will be biregularly equivalent to $C - Q$, and, in view of the fact that it does not contain L as a component, it will be irreducible. If X is a divisor on V, $X = \Sigma k_i C_i$, then, setting $X^* = \Sigma k_i C_i^*$, we obtain a mapping of the group of divisors on V into the group of divisors on $V^*(\bar{\sigma}\colon\ D(V) \to D(V^*))$, which is clearly a monomorphism. It is obvious that the mapping thus constructed is also the mapping σ.

In connection with the above, we note also that if g is a function meromorphic on V and $(g)_V = X = \Sigma k_i C_i$, then, considering the same function on V^* (or, which is the same, considering the function $\sigma^* g$), we obtain $(g)_{V^*} = (\sigma^* g)_{V^*} = X^* + \Sigma k_i m_i L$, where m_i is the multiplicity of the point Q on the curve C_i. The mapping σ_* is obtained from the mapping $\sigma = \overline{\sigma}$ in the following way.

We associate with the curve $C \subset V$, $Q \notin C$ the curve $C^* = \sigma(C) \subset V^*$. If Q is a point of multiplicity m of the curve C, we associate with this curve the curve $C^* + mL \subset V^*$. With the divisor $X = \Sigma k_i C_i$, we associate the divisor $\sigma_*(X) = \Sigma k_i C_i^* + \Sigma k_i m_i L$, where m_i is the multiplicity of the point Q on the curve C_i. In the future we will designate the number $\Sigma k_i m_i$ by $X(Q)$.

Lemma 1. *If* $X \overset{V}{\sim} Y$, *then* $\sigma_*(X) \overset{V^*}{\sim} \sigma_*(Y)$, *and, conversely, if* $\sigma_*(X) \overset{V^*}{\sim} \sigma_*(Y)$, *then* $X \overset{V}{\sim} Y$. *Moreover,* $(X \cdot Y)_V = (\sigma_*(X) \cdot \sigma_*(Y))_{V^*}$.

Proof. Let $X \overset{V}{\sim} 0$, i.e. $X = (g)_V$. We consider $(g)_{V^*}$. Since

$$(g)_{V^*} = X^* + X(Q), \quad L = \sigma_*(X),$$

the first (both the direct and converse parts) statement of the lemma is clear.

Now let X and Y be two divisors on V. If X and Y do not pass through the point Q (and do not have common components), then the index of the intersection $(X \cdot Y)_V$ is determined and the equality $(X \cdot Y)_V = (\sigma_*(X) \cdot \sigma_*(Y))_{V^*}$ is clear, for the mapping σ is biregular at each point of the intersection of the divisors X and Y.

In the general case we replace the divisors X and Y by divisors X' and Y' linearly equivalent to them that do not pass through Q and which do not have common components. This lemma is of course a simple corollary of the properties of the homomorphism σ_* listed in the Introduction. From this, in part, it easily follows that $(L^2) = -1$.

§2. Fundamental points and the σ-process

Let us assume that T: $V \to V'$ is a birational mapping of a nonsingular surface V onto a surface V', and let $P \in V$ be a fundamental point of the mapping T. We apply a σ-process to the surface V at the point P; we obtain a new surface $V_1 = \sigma_1(V)$ and a birational mapping T_1: $V_1 \to V'$; where $T_1 = T\sigma_1^{-1}$.

The mapping T_1 can have fundamental points on the line $L_1 = \sigma_1(P)$; let $P^{(1)}$ be one of these points. We apply to V_1 at $P^{(1)}$ a σ-process, obtaining a surface $V_2 = \sigma_2(V_1)$ and a birational mapping T_2: $V_2 \to V'$, where $T_2 = T_1\sigma_2^{-1}$; denoting a fundamental point of T_2 on L_2 (if such a point exists) by $P^{(2)}$, we applying to V_2 a σ-process at the point $P^{(2)}$, and so on. As will be shown in this section, this process breaks off after a finite number of steps, i.e., after a certain finite number p of steps we shall obtain a surface V_p such that there will be no fundamental points of the transformation T_p on the line $L_p \subset V_p$. The next basic theorem follows easily.

Theorem 1. *Let T: $V \longrightarrow V'$ be a birational mapping of the nonsingular surface V onto V'. Applying to V a finite sequence of σ-processes $\sigma = \sigma_n \cdot \sigma_{n-1} \cdots \sigma_1$, we obtain a surface $\bar{V} = \sigma(V)$ such that the birational mapping $\bar{T} = T\sigma^{-1}$: $\bar{V} \longrightarrow V'$ does not have fundamental points on $\bar{V}$.*

Proof. Let $P \in V$ be a fundamental point of the transformation T. If (x, y) are local parameters at the point P, then the transformation T can, in a neighborhood U_P of this point, be written in the form

$$\bar{z}_1 = \frac{f_1(x, y)}{g(x, y)},$$
$$\cdots\cdots\cdots\cdots$$
$$\bar{z}_s = \frac{f_s(x, y)}{g(x, y)},$$

where $g(0, 0) = 0$ and $f_i(0, 0) = 0$ (since the point P is fundamental). We consider

$$\bar{z}_i = \frac{f_i(x, y)}{g(x, y)}$$

(for simplicity we will omit the index i in the future). Let

$$f(x, y) = f^{\rho}(x, y) + f^{\rho+1}(x, y) + \cdots, \tag{1}$$

$$g(x, y) = g^{\delta}(x, y) + g^{\delta+1}(x, y) + \cdots, \tag{1a}$$

where $f^{\lambda}(x, y)$ and $g^{\mu}(x, y)$ are homogeneous polynomials of x and y of degrees λ and μ respectively.

We denote by F the curve in the neighborhood U_P given by the equation $f(x, y) = 0$, and by G the curve $g(x, y) = 0$. The point $P = (0, 0)$ will be a point of multiplicity ρ of the curve F and a point of multiplicity δ of the curve G. We express the condition that the point $P = (0, 0)$ is fundamental in a geometrical form. The curves F and G can be represented in the form of a sum of locally irreducible components ("branches"):

$$F = \sum_{j=1}^{v} k_j F_j, \quad G = \sum_{j=1}^{w} m_j G_j.$$

It is possible to show that some of the components of the curves F and G

coincide; we isolate all these components:

$$F_j = G_j, \quad j = 1, \cdots, l, \quad l \leq v, \ w,$$

$$F = k_1 F_1 + \cdots + k_l F_l + \sum_{j=l+1}^{v} k_j F_j, \tag{2}$$

$$G = m_1 F_1 + \cdots + m_l F_l + \sum_{j=l+1}^{w} m_j G_j. \tag{2a}$$

Moreover, we shall asume that

$$k_1 \geq m_1, \cdots, k_r \geq m_r, \quad k_{r+1} \leq m_{r+1}, \quad k_l \leq m_l.$$

We call the curves

$$\overline{F} = (k_1 - m_1) F_1 + \cdots + (k_r - m_r) F_r + \sum_{j=l+1}^{v} k_j F_j,$$

$$\overline{G} = (m_{r+1} - k_{r+1}) F_{r+1} + \cdots + (m_l - k_l) F_l + \sum_{j=l+1}^{w} m_j G_j,$$

the reduced pair of curves associated with the curves F and G. The curves $\overline{F}$ and $\overline{G}$ have no common components; it is clear that the point P will be fundamental only if both $\overline{F}$ and $\overline{G}$ pass through it.

The curves $\overline{F}$ and $\overline{G}$ are determined by the numerator and denominator of $\overline{z}$ after all possible simplifications have been made. Of course it would be possible to assume from the beginning that f and g are relatively prime in the neighborhood U_P; in the future, however, it is going to be convenient for us not to simplify and to consider the "complete" curves F and G, rather than the reduced $\overline{F}$ and $\overline{G}$.

The equation $f^{\rho}(x, y) = 0$ defines a curve whose components are tangent to F at the point $P = (0, 0)$; the equation $g^{\delta}(x, y) = 0$ is that of the curve whose components are tangent to G. Here distinct branches of F and G can have coinciding tangents, and the point $(0, 0)$ can be singular on some branches. A branch that has the point $(0, 0)$ as a regular point is said to be linear at that point. We shall assume that the curves G and F do not have the line $x = 0$ as a tangent. Therefore

$$f^{\rho}(x, y) = (y - a_1 x)^{\alpha_1} \cdots (y - a_r x)^{\alpha_r}, \quad \Sigma \alpha_j = \rho, \tag{3}$$

$$g^{\delta}(x, y) = (y - b_1 x)^{\beta_1} \cdots (y - b_t x)^{\beta_t}, \quad \Sigma \beta_j = \delta. \tag{3a}$$

We apply a σ-process to V at the point P; as a result we obtain a surface V_1 with a line $L_1 = \sigma_1(P)$. Local coordinates on V at the point $P^{(1)} = (t_1, t_0) \in L_1$

will be either the functions $(x_1, y_1) = (x, (y/x) - c)$ (at the point $P_c^{(1)}(c, 1) \in L_1$), or $(x_1, y_1) = (x/y, y)$ (at the point $P_\infty^{(1)} = (1, 0) \in L_1$). The birational mapping $T_1 = T\sigma_1^{-1}$ in the neighborhood of the point $P_\infty^{(1)}$ is written in the form

$$\bar{z} = \frac{y_1^{\rho-\delta}[(1 - a_1x_1)^{\alpha_1} \cdots (1 - a_rx_1)^{\alpha_r} + \cdots]}{(1 - b_1x_1)^{\beta_1} \cdots (1 - b_tx_1)^{\beta_t} + \cdots},$$

from which it follows immediately that the point $P_\infty^{(1)}$ is not a fundamental point of the transformation T_1.

At any other point $P_c^{(1)}$ the birational transfomation T_1 is written in the form

$$\bar{z} = \frac{x_1^\rho[(y_1 - a_1 + c)^{\alpha_1} \cdots (y_1 - a_r + c)^{\alpha_r} + x_1\tilde{f}^{\rho+1}(x_1, y_1) + \cdots]}{x_1^\delta[(y_1 - b_1 + c)^{\beta_1} \cdots (y_1 - b_t + c)^{\beta_1} + x_1\tilde{g}^{\delta+1}(x_1, y_1) + \cdots]}.$$

The equation

$$g_1(x_1, y_1) = (y_1 - b_1 + c)^{\beta_1} \cdots (y_1 - b_t + c)^{\beta_t} + x_1\tilde{g}^{\delta+1}(x_1, y_1) + \cdots = 0 \quad (4)$$

is the local equation of the curve G^1, the algebraic image of the curve G under the σ-process; the equation

$$f_1(x_1, y_1) = (y_1 - a_1 + c)^{\alpha_1} \cdots (y_1 - a_r + c)^{\alpha_r} + x_1\tilde{f}^{\rho+1}(x_1, y_1) + \cdots = 0 \quad (4a)$$

is the local equation of the curve F^1, the image of the curve F under the σ-process.

To each tangential direction to the curve at the point $P = (0, 0)$ on V, there corresponds a point of intersection of the image of this curve under the σ-process with a projective line $L_1 = \sigma_1(P)$, where the multiplicity of this point of intersection on the image of the curve is equal to the multiplicity of the corresponding tangent (this is immediately clear from formulas (3) and (4)). Therefore, if the curves F and G had distinct tangents at the point P, then, when $\rho = \delta$, F^1 and G^1 will not intersect on L_1 and, consequently, T_1 will not have fundamental points on L_1. The general case is more complicated, but reduces in the end to this case. Along with the curves F^1 and G^1, which we will denote by $\sigma_{c_1}^1(F)$ and $\sigma_{c_1}^1(G)$, where the index "c_1" indicates that the curves are considered in the neighborhood of the point $P_{c_1}^1 \in L_1$, we shall consider the curves

$$\sigma_{*c_1}^1(F) = \sigma_{c_1}^1(F) + \rho L_1,$$

$$\sigma_{*c_1}^1(G) = \sigma_{c_1}^1(G) + \delta L_1,$$

defined by the numerators and denominators of $\bar{z}$. We shall call these curves the total transforms of the curves G and F.

Applying a σ-process at the point $P^1_{c_1} \in L_1 \subset \sigma_1(V)$, we obtain at some point $P^2_{c_2} \in L_2 \subset \sigma_2\sigma_1(V)$ the curves $F^{(2)} = \sigma^2_{c_2}(F^{(1)}) = \sigma^2_{c_2}(\sigma^1_{c_1}F)$ and $G^{(2)} = \sigma^2_{c_2}(G^{(1)}) = \sigma^2_{c_2}(\sigma^1_{c_1}G)$ and also the complete images

$$\sigma^2_{*c_2}(F) = F^{(2)} + \rho\sigma^2_{c_2}(L_1) + \rho_1 L_2 = F^{(2)} + \rho L^{(2)}_1 + \rho_1 L_2,$$

$$\sigma^2_{*c_2}(G) = G^{(2)} + \delta\sigma^2_{c_2}(L_1) + \delta_1 L_2 = G^{(2)} + \delta L^{(2)}_1 + \delta_1 L_2,$$

where the number ρ_1, as it is not difficult to verify, is equal to the multiplicity of the point $P^1_{c_1}$ on the curve $\sigma^1_{*c_1}(F)$; δ_1 is defined analogously. It is clear that the curve $\sigma_2(L_1)$ will pass through only one point of the line L_2 and will determine at it a linear branch; in general

$$\sigma^n_{*c_n}(F) = F^{(n)} + \Sigma\rho_{i-1}L^{(n)}_i,$$

where $F^{(n)}$ is the algebraic local image of the curve F under the sequence of σ-processes $\sigma_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_1$ and $L^{(n)}_i$ is the algebraic local image of the line under the further σ-processes $\sigma_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_{i+1}$ beyond the i th step; all the $L^{(n)}_i$ will determine linear branches at an arbitrary point $P^n_{c_n}$ on L_n.

Thus, at each step some multiplicity of linear branch is added to the curve F, where, since a linear curve is taken into a linear curve under a σ-process, as a result of the sequence of σ-processes, we obtain an algebraic image of the curve F under the sequence of σ-processes along with some collection of linear branches. We shall first show that the algebraic image of the curve F at each point $P^n_{c_n} \in L_n$ will consist only of linear branches, i.e., a finite sequence of σ-processes permits one to "linearize" branches. For this we shall show that each point of the intersection of L_n with $F^{(n)}$ will be a simple point on each branch $F^{(n)}$ starting with some n. To see this, we compare (2a), (3a) and (4a).

The point P is a point of multiplicity ρ of the curve F; the point $P^1_{a_j} \in L_1$ will be a point of multiplicity α_j of the curve F^1, as follows immediately from these formulas. Thus, if $r > 1$, then each point of the intersection of F^1 with L_1 will have a multiplicity less than ρ, and we obtain the desired statement arguing by induction. Thus the only difficult case is when $r = 1$, i.e., $f^\rho(x, y) = (y - ax)^\rho$; then the curve F has at the point P a unique tangent of multiplicity ρ.

Without loss of generality we can assume that $f^\rho(x, y) = y^\rho$ and $f(x, y) = y^\rho + f^{\rho+1}(x, y) + \cdots$. The curve F^1 will then have a unique point of intersection

$P_0^1 = (0, 0)$ with the line L_1 of the same multiplicity ρ:

$$f_1(x_1, y_1) = y_1^\rho + x_1\widetilde{f}(x_1, y_1) + \cdots.$$

Among the terms of $\widetilde{f}(x, y) - y^\rho$ we choose one of least degree in y: $\gamma x^\nu y^\mu$, $\nu + \mu \geq \rho$. It is clear that $\mu < \rho$ and $\nu + \mu > \rho$; for otherwise $f(x, y) = y^\rho[1 + \cdots]$ and the curve F is equivalent to the curve $y^\rho = 0$, i.e., the point P is a simple point on $y = 0$, i.e., on an irreducible component of F, which is what we want to show.

Thus, there exists a term $\gamma x^\nu y^\mu$, $\mu < \rho$, $\nu + \mu > \rho$. Then $\gamma x_1^{\nu+\mu-\rho} y_1^\mu$ will occur in the decomposition of $f_1(x_1, y_1)$. Applying a σ-process at the point $(0, 0)$ on $L_1 \subset V_1$, we find that $\gamma x_2^{\nu+2\mu-2\rho} y_2^\mu$ will be in the decomposition of f_2, and so on. It is clear that after a finite number of steps we obtain a term of degree less than ρ, while by the same token we reduce the multiplicity of the singularity.

Our first statement has thus been proved.

Now let H_q: $\{h_q(x_q, y_q) = 0\}$ and E_q: $\{e_q(x_q, y_q) = 0\}$ be two distinct (locally) irreducible curves passing through the point $P_0^q(x_q = 0, y_q = 0) \in L_q$ on V_q, where the point P_0^q is a simple point of H_q and E_q.

We apply a σ-process to V_q at the point P_0^q:

$$\sigma_q:\ V_q \longrightarrow V_{q+1},\ L_{q+1} = \sigma_{q+1}(P_0^q).$$

The curves H_{q+1} and E_{q+1}—the images of the curves H_q and E_q under the σ-process—will each intersect L_{q+1} in one (simple) point; if the tangents to H_q and E_q at the point P_0^q were distinct, then the points $H_{q+1} \cdot L_{q+1}$ and $E_{q+1} \cdot L_{q+1}$ will be distinct; while if the tangents coincide, $H_{q+1} \cdot L_{q+1} = E_{q+1} \cdot L_{q+1}$. It turns out that after several applications of σ-processes, these curves will separate, i.e., the points $E_{q+u} \cdot L_{q+u}$ and $H_{q+u} \cdot L_{q+u}$ will become distinct. Thus, since the point P_0^q is nonsingular on E_q and H_q, one can choose local coordinates such that

$$h_q(x_q, y_q) = y_q - \sum_{i=1}^{\infty} d_i^q x_q^i, \tag{5}$$

$$e_q(x_q, y_q) = y_q - \sum_{i=1}^{\infty} l_i^q x_q^i, \tag{5a}$$

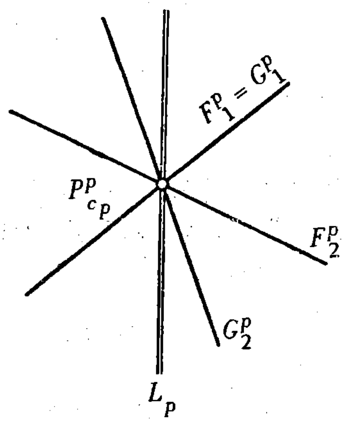
and, since the curves E_q and H_q are distinct, there exists a least i such that $d_i^q \neq l_i^q$.

It is easy to see that after i σ-processes these curves will have separated, i.e. $E_{q+i} \cdot L_{q+i} \neq H_{q+i} \cdot L_{q+i}$. Theorem 1 follows from these two statements.

To see this, by the first statement, we obtain after the application of a finite sequence of σ-processes that, at each point

$$P^n_{c_n} \in L_n \subset \sigma_n \sigma_{n-1} \cdots \sigma_1 V,$$

the curves $\sigma^n_{*c_n}(F)$ and $\sigma^n_{*c_n}(G)$ will have only linear branches.



Let

$$\sigma^n_{*c_n}(F) = \Sigma k^n_j F_j,$$

$$\sigma^n_{*c_n}(G) = \Sigma m^n_j G_j,$$

where F_j and G_j are linear branches at the point $P^n_{c_n}$. By means of further applications of σ-processes we obtain that all the different branches will have distinct tangents; we note that a linear branch blown up under some later process does not touch the images of preceding branches. Thus we obtain that

$$\sigma^p_{*c_p}(F) = k^p_1 F^p_1 + \cdots + k^p_{l_p} F^p_{l_p} + \sum_{j=l_p+1}^{v_p} k^p_j F^p_j + \rho_p L_p, \tag{6}$$

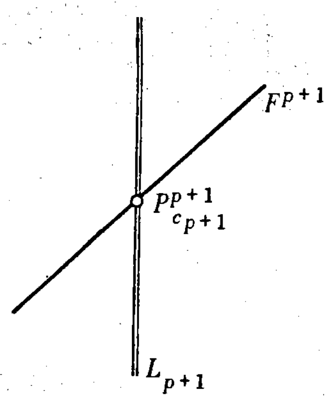
$$\sigma^p_{*c_p}(G) = m^p_1 F^p_1 + \cdots + m^p_{l_p} F^p_{l_p} + \sum_{j=l_p+1}^{w_p} m^p_j G^p_j + \delta_p L_p, \tag{6a}$$

where all the branches are linear and have distinct tangents. If we now apply a σ-process to the point $P^p_{c_p} \in L_p \subset V_p$, then we see that through each point $P^{p+1}_{c_{p+1}} \in L_{p+1}$ not more than two linear branches of the curves $\sigma^{p+1}_{*c_{p+1}}(F)$ and $\sigma^{p+1}_{*c_{p+1}}(G)$ will pass, namely the branches

$$\sigma^{p+1}_{*c_{p+1}}(F) = kF^{(p+1)} + \rho_{p+1} L_{p+1},$$

$$\sigma^{p+1}_{*c_{p+1}}(G) = mF^{(p+1)} + \delta_{p+1} L_{p+1},$$

where $k, \rho_{p+1}, m, \delta_{p+1} \geq 0$.



Two cases are possible:

1. $k \geq m$, $\rho_{p+1} \geq \delta_{p+1}$. In this case the corresponding reduced curves will be such that $\overline{\sigma^{p+1}_{*c_{p+1}}(F)}$ passes through the point $P^{p+1}_{*c_{p+1}}$ while $\overline{\sigma^{p+1}_{*c_{p+1}}(G)}$ does not, or neither curve passes through the point; i.e., this

point is not a fundamental point T_{p+1}. The case $k \leq m$, $\rho_{p+1} \leq \delta_{p+1}$ is completely analogous.

2. $k > m$, $\rho < \delta$ (or symmetrically $k < m$, $\rho > \delta$). In this case the corresponding reduced curves will both pass through the point $P^{p+1}_{*c_{p+1}}$, i.e. this point is a fundamental point for T_{p+1}.

If we apply a σ-process to the point $P^{p+1}_{*c_{p+1}}$, then the line L_{p+2} will correspond to this point on the surface V_{p+2}; two points interest us on L_{p+2}; R_1 – the center of the transformed branch F^{p+1}, and R_2 – the point of intersection of the transformed line L_{p+1} with L_{p+2}.

Let F^{p+2} be the image of the branch F^{p+1} and L'_{p+1} the image of L_{p+1} under σ_{p+2}: $V_{p+1} \to V_{p+2}$. It is clear that T_{p+2} can have fundamental points on L_{p+2} only at the points R_1 and R_2.

At the point R_1

$$\begin{aligned} \sigma^{p+2}_{*R_1}(F) &= kF^{p+2} + (k + \rho_{p+1})L_{p+2}, \\ \sigma^{p+2}_{*R_1}(G) &= mF^{p+2} + (m + \delta_{p+1})L_{p+2}; \end{aligned} \tag{7}$$

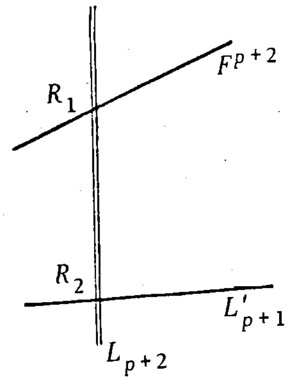
and at the point R_2

$$\begin{aligned} \sigma^{p+2}_{*R_2}(F) &= \rho_{p+2}L_{p+1} + (k + \rho_{p+1})L_{p+2}, \\ \sigma^{p+2}_{*R_2}(G) &= \delta_{p+2}L_{p+1} + (m + \delta_{p+1})L_{p+2}. \end{aligned} \tag{7a}$$

Since $k > m$, $\rho < \delta$, then either

(a) $k + \rho \geq m + \delta$, or

(b) $k + \rho < m + \delta$.



In case (a) the point R_1 will not be a fundamental point for the transformation T_{p+2}; in case (b) the point R_2 will not be a fundamental point for T_{p+2}. In the case of equality, there will be no fundamental points of T_{p+2} on L_{p+2}. Thus, T_{p+2} will have no more than one fundamental point on L_{p+2}.

In general, if the decomposition of the curves $\sigma^{p+2}_*(F)$ and $\sigma^{p+2}_*(G)$ at an arbitrary point of the line L_{p+2} can be written in the form

$$\sigma^{p+2}_*(F) = k_1D + \rho_{p+2}L_{p+2},$$

$$\sigma^{p+2}_*(G) = m_1D + \delta_{p+2}L_{p+2},$$

then we see that

$$|k_1 - m_1| + |\rho_{p+2} - \delta_{p+2}| < |k - m| + |\rho_{p+1} - \delta_{p+1}|.$$

Thus, using induction on the sum

$$|k_i - m_i| + |\rho_{p+i} - \delta_{p+i}|,$$

we obtain that at some step μ this number has become 0, from which follows immediately the nonexistence of fundamental points for $T_{p+\mu}$ on $L_{p+\mu}$.

The theorem is proved.

We note that from the statements about the possibility of the "linearization" of branches and about the "separation" of branches immediately follows

Theorem 2. *Any curve E of a surface V can be transformed into a nonsingular curve by means of the application of a finite sequence of σ-processes to the surface V.*

Clearly these σ-processes must be applied at points of V that are singular points of the curve E.

§3. Theorems of Bertini

First Theorem. *Let V be a d-dimensional algebraic variety in P^n. We consider on V the linear system $\Pi(\lambda_0, \cdots, \lambda_s)$ without fixed components that is given by the system of hypersurfaces*

$$\Phi_\lambda = \lambda_0 f_0 + \cdots + \lambda_s f_s = 0. \tag{1}$$

A rational mapping of V onto a variety $W \subset P_s$ corresponds to this linear system (cf. the Introduction). If $\dim W \geq 2$, then the generic member of the linear system is an irreducible subvariety $V' \subset V$ of multiplicity 1.

Proof. Let $(1, \xi_1, \cdots, \xi_n)$ be the nonhomogeneous coordinates of a generic point of V and let us assume that $f_0(1, \xi_1, \cdots, \xi_n) \neq 0$. Then

$$(f_1(\xi)/f_0(\xi), \cdots, f_s(\xi)/f_0(\xi))$$

will be a generic point of W.

Suppose that $\dim W = 2$; then we can assume that $\eta_1 = f_1(\xi)/f_0(\xi)$ and $\eta_2 = f_2(\xi)/f_0(\xi)$ are algebraically independent over the field k.

We will first show that if λ is an independent parameter, then the element of our linear system cut out by the hypersurface

$$f_1 + cf_2 = \lambda f_0,$$

where c is some element of the field k, has the form mS, where S is an irreducible subvariety of V.

For this it is sufficient to show that for some $c \in k$ the field $k(\eta_1 + c\eta_2)$ will be algebraically closed in the field $k(V) = k(1, \xi_1, \cdots, \xi_n)$ (i.e. the field

$k(V)$ is a regular extension of the field $k(\eta_1 + c\eta_1))$.

Thus, in this case the point $(1, \xi_1, \cdots, \xi_n)$ determines some (absolutely) irreducible variety V^* over the field $k(\eta_1 + c\eta_2)$, whose dimension is equal to $d-1$, where $d = \dim V$ [55]. Since the point (ξ) lies on V, $V^* \subseteq V$.

On the other hand, V^* will lie on the hypersurface of Π defined by the equation

$$f + cf_2 = tf_0,$$

where $t = \eta_1 + c\eta_2$ is some transcendental element over k. It is also easy to see that each point of the intersection $\Pi \cdot V$ belongs to V^*, for one obtains from the generic point (ξ) the variety V as a specialization over $k(\eta_1 + c\eta_2)$.

Thus, to prove that $V \cdot \Pi = mV^*$, it remains for us to show that it is possible to choose a $c \in k$, such that the field $k(\eta_1 + c\eta_2)$ will be algebraically closed in $k(\xi)$. This statement is a well-known lemma of Zariski (cf. Hodge and Pedoe [55], pp. 93–95 of the Russian translation).

We have thus shown that the common element of our linear system $\Pi(\lambda_0, \cdots, \lambda_s)$ cut out on V by the hypersurface $\Phi_\lambda = 0$ where $(\lambda_0, \cdots, \lambda_s)$ are independent parameters, has the form $\Pi(\lambda_0, \cdots, \lambda_s) = mS$, where S is a (irreducible) subvariety of V. We now show that $m = 1$, i.e. that the hypersurface (1) cuts out on V the subvariety S with a multiplicity of one.

For this it is sufficient to show that the hypersurface (1) is not tangent to V at a generic point of the subvariety S.

Let us consider a generic point (ζ) of the subvariety S that is contained in the intersection of the subvariety V with the hypersurface (1); this will be some point over the field $k(\lambda_0, \cdots, \lambda_s)$ satisfying the equation

$$\Phi_\lambda(\zeta) = \lambda_0 f_0(\zeta) + \cdots + \lambda_s f_s(\zeta) = 0. \tag{2}$$

Extending the differentiation $\partial\lambda_i$ effective in the field $k(\lambda_0, \cdots, \lambda_s)$ to some differentiation D_i in the field $k(\lambda_0, \cdots, \lambda_s)(\zeta)$ and applying it to (2), we obtain:

$$f_i(\zeta) + \sum_k \left\{\lambda_0 \frac{\partial f_0(\zeta)}{\partial \zeta_k} + \cdots + \lambda_s \frac{\partial f_s(\zeta)}{\partial \zeta_k}\right\} D_i\zeta_k = 0. \tag{3}$$

If f is an arbitrary element of an ideal determining the variety V, then

$$f(\zeta_0, \cdots, \zeta_n) = 0, \tag{4}$$

since $(\zeta) \in S \subset V$.

Applying to (4) the differentiation D_i, we obtain

$$\frac{\partial f}{\partial \zeta_0} D_i\zeta_0 + \cdots + \frac{\partial f}{\partial \zeta_n} D_i\zeta_n = 0, \quad i = 1, \cdots, s. \tag{5}$$

This means that the point $(D_i\zeta_0, \cdots, D_i\zeta_n)$ is contained in the subspace tangent to V at the point (ζ).

Now let the hypersurface (1) be tangent to V at the point (ζ). Then, as easily follows from a calculation of dimensions, the subspace tangent to V is contained in the tangent hyperplane

$$\sum_{j=0}^{n} z_j \frac{\partial \Phi(\zeta)}{\partial \zeta_j} = 0 \tag{6}$$

to the hypersurface (1) (one can assume the point (ζ) to be nonsingular on V). Then the point $(D_i\zeta_0, \cdots, D_i\zeta_n)$ satisfies equation (6) and

$$\sum_k \left[\lambda_0 \frac{\partial f_0(\zeta)}{\partial \zeta_k} + \cdots + \lambda_s \frac{\partial f_s(\zeta)}{\partial \zeta_k}\right] D_i\zeta_k = \sum_{j=0}^{n} \frac{\partial \Phi(\zeta)}{\partial \zeta_j} D_i\zeta_j = 0. \tag{7}$$

Comparing (3) and (7), we obtain the equations

$$f_r(\zeta) = 0, \quad r = 0, \cdots, s.$$

This means that the subvariety S with generic point (ζ) is a fixed component of our linear system, which is a contradiction. The theorem is proved.

We now assume that there corresponds to the linear system Φ_λ a mapping π of the variety V onto a curve C with generic point $(\eta) = (f_1(\xi)/f_0(\xi), \cdots, f_s(\xi)/f_0(\xi))$. We write $k(\xi) = K$, $K(\eta) = K_1$. The field K/K_1 is the field of functions on the subvariety $V_1 = \pi^{-1}(\eta) \subset V$. The subvariety V_1 can be reducible. We then consider the field $\bar{K}_1 = \overline{k(\eta)}$, the algebraic closure of the field K_1 in the field K. If $\dim V = 2$, then $\dim V_1 = 1$. We denote by $\bar{C}$ a nonsingular model of the field $\overline{k(\eta)} = \bar{K}_1$; let $\overline{(\eta)}$ be a generic point of $\bar{C}$.

The inclusion $\bar{K}_1 \subset K$ determines a rational mapping $\bar{\pi}$: $V \longrightarrow \bar{C}$. Since the field $\bar{K}_1 = k(\bar{\eta})$ is closed in K, the curve $\bar{V}_1 = \bar{\pi}^{-1}(\bar{\eta})$ will have the form mS, where S is an irreducible curve. Exactly as in the proof of the preceding theorem, we can show that $m = 1$.

Moreover, since $k(\eta) \subset k(\bar{\eta})$, there exists a covering r: $\bar{C} \longrightarrow C$.

Thus, one may make a commutative diagram of the rational mappings

$$\begin{array}{ccc} V & \xrightarrow{\bar{\pi}} & \bar{C} \\ & {}_{\pi}\searrow \quad \swarrow_{r} & \\ & C & \end{array} \tag{T_1}$$

where r is a covering, and a generic fiber $\bar{\pi}^{-1}(\bar{\eta})$ of the mapping $\bar{\pi}$ is an irreducible curve.

When a mapping of a surface V onto a curve C corresponds to a linear system, we will say that this system is composed of a pencil. Each linear system that consists of a pencil consists of an irreducible pencil in the sense that there exists for it a diagram (T_1) with the properties given above.

Let Φ be an irreducible pencil (not necessarily linear) of curves on V. There corresponds to it a rational mapping π: $V \longrightarrow C$ such that C is a curve and a generic curve of the pencil $E_x = \pi^{-1}(x)$, where x is a generic point of C. The curve C is usually called a base of the pencil. Clearly, the field of functions on V, $k(V)$, is equal to $K'(E_x)$, where $K' = k(C)$. We will assume that the curve E_x is rational (has an arithmetic genus of 0) over the field K'. Then, as is known [58], the field $K'(E_x)$ will be a field of functions on a curve of second order $\phi(u, v) = 0$ with coefficients in the field K'.

If the original field k was algebraically closed, then the field K' will be quasi-algebraically closed and the curve $\phi(u, v) = 0$ will have at least one rational point over K' [52]. But this means [58], that the field $k(V) = K'(E_x)$ will be a pure transcendental extension of the field K':

$$k(V) = K'(t) = k(C)(t),$$

from which it follows that the surface V is birationally equivalent to the direct product $C \times P^1$ of the curve C on the line P^1. Such a surface is said to be ruled. The result obtained belongs to Noether.

Theorem of Noether. *If there exists on the surface V a pencil, a generic curve of which is nonsingular and rational, then the surface V is ruled. Moreover, if C is a base of the pencil, then V is birationally equivalent to $P^1 \times C$. In particular, if C is a line, i.e. if the pencil is linear, then V is birationally equivalent to $P^1 \times P'^1$, i.e. V is rational.*

Second Theorem of Bertini. *A generic element of a linear system Π_λ on an algebraic variety V cannot have singular points that are not base points of the system Π_λ or singular points of V.*

Let a linear system Π_λ be given by the system of hypersurfaces (1).

We reduce the general case to the case of the pencil

$$\lambda f_0 + \mu f = 0. \tag{8}$$

Thus, adjoining $(\lambda_0, \cdots, \lambda_s)$ to the original field k and considering the system given by the pencil (8) (where $\mu f = \lambda_1 f_1 + \cdots + \lambda_s f_s$), we see that if the point P is a singular point of a generic element of the system and is not a singular point of V, then $f_0(P) = 0$, assuming that the second theorem of Bertini is true for pencils. Analogously $f_1(P) = \cdots = f_r(P) = 0$, i.e. the point P is a base point of the system (1).

Thus we consider the pencil $\lambda f_0 + \mu f = 0$. It is clear that the singular points of the subvarieties that are elements of this pencil satisfy the equation

$$fDf_0 - f_0 Df = 0, \tag{9}$$

where D is any differentiation in the field $k(\xi)$ over k, and (ξ) is a generic point of V.

These equations determine some algebraic subset W on V, $W = \sum W_i$, where the W_i are irreducible subvarieties on V.

Let (x_i) be a generic point of W_i. We can assume that $f_0(x_i) \neq 0$; we consider $\nu = \lambda/\mu = -f_1(x_i)/f_0(x_i)$.

Applying the differentiation D to

$$\nu f_0(\xi) + f_1(\xi) = 0,$$

we obtain

$$f_0(\xi) D(\nu) + \nu D(f_0(\xi)) + D(f_1(\xi)) = 0. \tag{10}$$

Since (x_i) satisfies equations (9) and (10),

$$\nu D(f_0(x_i)) + D(f_1(x_i)) = 0 \tag{11}$$

and from (10) and (11) it follows that $D(\nu) = 0$ for any differentiation in $k(\xi)$ over k (in particular, for any differentiation $\bar{D}$ in $k(x_i)$ over k). Thus ν is algebraic over k, and this means that there exist only a finite set of elements of the pencil (8) having singular points that are not base points of the pencil and not singular points of V. The theorem is proved. The proof given belongs to Akizuki [2].

§4. Zariski's theorem on multiple linear systems

Theorem. *Let T be a rational mapping of a nonsingular surface V onto a surface V'. This mapping can be given by a linear system L of curves on the surface V that do not have fixed components. Then for a sufficiently large number h, the complete linear system $|hL|$ does not have base points, i.e. the mapping corresponding to this system is regular.*

Proof. Since a mapping on the surface corresponds to the system L, a generic member of this system is an irreducible curve (cf. the proof of the first theorem of Bertini). Let C be some irreducible curve of the system L. We will show that $(C^2) \geq 1$. Now $\dim L \geq 2$, for a mapping on the surface corresponds to this system. Thus, for any point $P \in C$, there exists a curve $C' \in L$ passing through P and distinct from C. Since the curves C and C' cannot have common components (C is irreducible), $(C \cdot C') \geq 1$, i.e. $(C^2) \geq 1$.

We consider the exact sequence of sheaves

$$0 \longrightarrow F((h-1)D') \longrightarrow F(hD') \longrightarrow F_C(hD' \cdot C) \longrightarrow 0, \tag{T$_1$}$$

where D' is some divisor (not necessarily effective) linearly equivalent to the curve C and not passing through the singular points $Q_1, \cdots, Q_s$ of the curve C.

The sheaf $F_C(hD' \cdot C)$ is such that

$$F_C(hD' \cdot C)_x = \begin{cases} \text{the set of rational functions } \phi \text{ on } C \text{ such that} \\ (\phi) + C \cdot hD' > 0 \text{ if } x \text{ is a simple point of the} \\ \text{curve } C\text{; the set of rational functions on } C \text{ regular} \\ \text{in } x, \text{ if } x \in \{Q_1, \cdots, Q_s\}. \end{cases}$$

From the sequence (T_1) we obtain the exact cohomology sequence

$$0 \to H^0(V, F((h-1)D')) \to H^0(V, F(hD')) \to H^0(C, F_C(hD' \cdot C)) \to$$
$$\to H^1(V, F((h-1)D')) \to H^1(V, F(hD')) \to H^1(C, F_C(hD' \cdot C)) \to \cdots. \qquad (T_2)$$

Since $hD' \cdot C = hC^2 \geq h$ and the divisor $hD' \cdot C$ does not contain singular points of the curve C, by the Riemann-Roch theorem for a curve with singularities, $H^1(C, \mathfrak{L}_C(hD' \cdot C)) = 0$ if $hC^2 > 2\pi - 2$ (1), where π is the arithmetic genus of the curve C. From the sequence (T_2) we see that for numbers h satisfying condition (1) the numbers $r_h = \dim H^1(V, \mathfrak{L}(hD'))$ form a nonincreasing sequence, and consequently there exists an integer m_0 such that $r_m = \text{const}$ for all $h \geq m_0$. From this it follows that $H^1(V, F(hD')) \approx H^1(V, F((h-1)D'))$ for $h \geq m_0 + 1$, and we obtain the following exact sequence:

$$0 \to H^0(V, F((h-1)D')) \to H^0(V, F(hD')) \xrightarrow{r_C} H^0(C, F_C(hD' \cdot C)) \to 0. \qquad (T_3)$$

In this sequence

$$H^0(V, F((h-1)D')) = \mathcal{L}_V((h-1)D'),$$
$$H^0(V, F(hD')) = \mathcal{L}_V(hD'),$$
$$H^0(C, F(hD' \cdot C)) = \mathcal{L}_C(hD' \cdot C).$$

Our goal is to show that the system $|hD'|$, i.e. the system of divisors $D_\phi = (\phi) + hD'$, where $\phi \in \mathfrak{L}_V(hD')$, does not have base points, i.e. points generic for all D_ϕ. For this purpose it is sufficient to show that the system of divisors $|hD' \cdot C|_C$, i.e. the system of divisors $\mathfrak{E}_{\bar{\phi}} = (\bar{\phi}) + hD' \cdot C$, where $\bar{\phi} \in \mathfrak{L}_C(hD' \cdot C)$, does not have base points on the curve C. Now if some point $\mathfrak{p} \in V$ is a base point of the system $|hD'|$, then $\mathfrak{p} \in C$, for $hC \in |hD'|$, and, consequently, passes through $\mathfrak{p}$. From the fact that the mapping r_C is an epimorphism it follows that each divisor of the system $|hD' \cdot C|_C$ cuts out on the curve C some divisor of the system $|hD'|$. From this we quickly obtain that the existence of a base point $\mathfrak{p}$ of the system $|hD'|$ implies the existence of a base point of the system

$|hD' \cdot C|_C$. We now show that for large h the system $|hD' \cdot C|_C$ does not have base points. It is clear that if C is a nonsingular curve, then for $hD' \cdot C = hC^2 > 2\pi - 2$ (in this case $\pi = g(C)$, the geometric genus of the curve C), the system of divisors $|hD' \cdot C|_C$ does not have base points and everything is proven. There remains to be considered the case when C is a curve with singular points $Q_1, \cdots, Q_s$. We write $S = \bigcup_i Q_i$. Let $\widetilde{C}$ be a normalization of the curve C, and let μ: $\widetilde{C} \longrightarrow C$ be a regular mapping of $\widetilde{C}$ onto C such that μ: $\widetilde{C} - S' \longrightarrow C - S$, where $S' = \mu^{-1}(S)$, is a biholomorphic map. We denote by O' the sheaf of local rings on C, by $\widetilde{O}$ the sheaf of local rings on $\widetilde{C}$ and by O, the direct image of the sheaf $\widetilde{O}$: $O = \mu(\widetilde{O})$. As follows from the properties of a normalization, the sheaf O is such that

$$O_x = \begin{cases} O'_x, & \text{if } x \notin S; \\ \bigcap \widetilde{O}_{P_i}, & \text{if } x = Q,\ Q \in S; \\ P_i \in \mu^{-1}(Q) & \text{in this case } O_Q \text{ is the complete closure of the} \\ & \text{ring } O'_Q \text{ in the field of functions } K = k(C). \end{cases}$$

Clearly $O \supset O'$.

We consider the annihilator $\mathfrak{S}$ of the sheaf of modules O/O', i.e. the sheaf whose fiber at the point $x \in C$ is the ideal $\mathfrak{S}_x \subset O_x$ consisting of all elements $f \in O_x$ such that $f \cdot g \in O'_x$ for all $g \in O_x$. Since O and O' are coherent algebraic sheaves, $\mathfrak{S}$ will have these properties.

If x is a simple point of the curve C, then $O_x = O'_x$ and $\mathfrak{S}_x = O'_x$; if $x = Q$, $Q \in S$, then $O'_Q \neq O_Q$ and $\mathfrak{S}_Q \subset O'_Q$, for O_Q is a ring with identity. Moreover, in this case $\mathfrak{S}_Q$ cannot contain 1, and consequently

$$O_Q \supset O'_Q \supset k + \mathfrak{S}_Q,$$

i.e., in particular, $\dim O'_Q/\mathfrak{S}_Q \geq 1$.

The condition for an element $f \in O_Q$ to belong to the ideal $\mathfrak{S}_Q$ can be expressed in the following form, which will be convenient for us in the future.

Since the ring O'_Q is the intersection of a finite number of regular local rings $\widetilde{O}_P$, $P \in \mu^{-1}(Q)$, the intersection of the ideal $\mathfrak{S}_Q$ with any ring $\widetilde{O}_P$ will again be an ideal $\mathfrak{S}_Q^P$. If we denote by t_P a local uniformizing at the point P of the curve $\widetilde{C}$, then it is clear that any ideal of the ring $\widetilde{O}_P$ has the form $t_P^k \widetilde{O}_P$; in particular, $\mathfrak{S}_Q^P = t_P^{n_P} \widetilde{O}_P$.

We consider the divisor $\mathfrak{S}_Q = \Sigma_{P \in \mu^{-1}(Q)} n_P P$ on the curve $\widetilde{C}$. It is clear that a function $f \in O'_Q$ belongs to $\mathfrak{S}_Q$ if and only if $f \equiv O \bmod \mathfrak{S}_Q$, i.e. if $f = t_P^{n_P} u$, where $u \in \widetilde{O}_P$ for each $P \in \mu^{-1}(Q)$. In other words, a necessary and sufficient condition for a function $f \in O'_Q$ to belong to the ideal $\mathfrak{S}_Q$ is for it to vanish to order less than n_P at each point $P \in \mu^{-1}(Q)$. The degree $n_Q = \Sigma_{P \in \mu^{-1}(Q)} n_P$ of the divisor $\mathfrak{z}_Q$ is equal to the dimension of $O_Q/\mathfrak{z}_Q$. The divisor $\mathfrak{z} = \Sigma_{Q \in S} \mathfrak{z}_Q$ on the curve $\widetilde{C}$ is usually called the conductor of the curve C. The degree of the divisor $\mathfrak{z}$ is denoted by $n_{\mathfrak{z}}$, $n_{\mathfrak{z}} = \Sigma_{Q \in S} n_Q$. It follows from the above that $\mathfrak{z} \approx F_{\widetilde{C}}(-\mathfrak{z})$.

We now consider the exact sequence of (coherent algebraic) sheaves

$$0 \longrightarrow \mathfrak{z} \longrightarrow O' \longrightarrow M_C \longrightarrow 0,$$

where M_C is the factor sheaf $O_C/\mathfrak{z}$. Since $\mathfrak{z}_Q = O'_Q$ at points $Q \not\subset C - S$, the sheaf M_C is concentrated on S. Moreover, in view of what has been said, at each point $Q \in S$ the group $M_Q = O'_Q/\mathfrak{z}_Q$ is a finite k-module of dimension ≥ 1. Let D (in our case $D = hD' \cdot C$) be some divisor on the curve C, $D \cap S \neq \emptyset$. To it corresponds a divisor $\widetilde{D} = \mu^{-1}(D)$ on $\widetilde{C}$ that clearly has the same degree as D. Moreover, since the sheaf $\mathfrak{z}$ is isomorphic with the sheaf $F_{\widetilde{C}}(-\mathfrak{z})$, we obtain the exact sequence of sheaves:

$$0 \longrightarrow F_{\widetilde{C}}(\widetilde{D} - \mathfrak{z}) \longrightarrow F_C(D) \longrightarrow M_C \longrightarrow 0$$

(it is essential here that $D \cap S = \emptyset$). From this we obtain the exact cohomology sequence:

$$\begin{aligned} 0 \longrightarrow H^0(\widetilde{C}, F_{\widetilde{C}}(\widetilde{D} - \mathfrak{z})) \longrightarrow H^0(C, F_C(D)) \longrightarrow H^0(C, M_C) \longrightarrow \\ \longrightarrow H^1(\widetilde{C}, F_{\widetilde{C}}(\widetilde{D} - \mathfrak{z})) \longrightarrow \cdots. \end{aligned} \qquad (T_4)$$

If the degree of the divisor D is sufficiently large (if $hD' \cdot C - n_{\mathfrak{z}} = hC^2 - n_{\mathfrak{z}} > 2\pi - 2$, where π is the arithmetic genus of C = the genus of $\widetilde{C}$), then $H^1(\widetilde{C}, F_{\widetilde{C}} \times (\widetilde{D} - \mathfrak{z})) = 0$ and the linear system of divisors $|\widetilde{D} - \mathfrak{z}|$ on $\widetilde{C}$ does not have fixed points. From the second proposition and from the exact sequence (T_4) it follows that no simple point of the curve C can be a base point of the system $|hD' \cdot C|_C$; from the first statement and from the properties of M_C it follows that also no point $Q \in S$ can be a base point of our system. The theorem is proved.

Zariski's theorem is proved in [22]. The present proof comes from an idea of Kodaira [25].

CHAPTER II

MINIMAL MODELS

In this chapter a theorem about the decomposition of a birational transformation into a sequence of σ-processes is proved and minimal models of surfaces are studied. It is shown that each birational class of surfaces has a relatively minimal model. It is also shown that all classes of surfaces except for classes of ruled surfaces have (absolute) minimal models.

Let V be a nonsingular projective algebraic surface. Then it is possible to introduce a partial order in the class $\{V\}$ of the birational nonsingular surfaces equivalent to it. We shall say that a nonsingular surface $V' \in \{V\}$ dominates the surface $V'' \in \{V\}$ if there exists a regular birational mapping T: $V' \longrightarrow V''$. We will denote this relation $V' > V''$.

If the mapping T is biregular, we shall identify the surfaces V' and V''; thus, we shall consider surfaces up to biregular equivalence.

A minimal model of the class $\{V\}$ will be a nonsingular surface $\overline{V} \in \{V\}$ such that $\overline{V} < V'$ for any $V' \in \{V\}$, $V' \neq \overline{V}$. If a minimal model exists it is unique.

In this chapter we explain which classes $\{V\}$ of surfaces have minimal models.

The following is a basic result.

All classes of surfaces, except for ruled surfaces, have minimal models.

We shall call a surface $V' \in \{V\}$ a relatively minimal model (r.m.m.) if there does not exist a surface $V'' \in \{V\}$ such that $V' > V''$.

We shall show that in each class $\{V\}$ there exists at least one relatively minimal model.

Thus, let V_1 be an arbitrary surface of a class $\{V\}$. Then either there does not exist a surface $V_2 \in \{V\}$, $V_2 < V_1$, and the surface V_1 is itself a relatively minimal model, or such a surface V_2 does exist. The surface V_2 either is a relatively minimal model or there exists a surface $V_3 \in \{V\}$ such that $V_2 > V_3$. Repeating this argument we either obtain a relatively minimal model V_n, $n < +\infty$ or we obtain an infinite sequence $V_1 > V_2 > V_3 > \cdots$ of surfaces of the class $\{V\}$. We shall show that the existence of such a sequence is impossible.

Thus, since each regular mapping T_i: $V_i \longrightarrow V_{i+1}$ is not biregular, there exists on V_i an exceptional curve C_i (an irreducible curve such that $T_{i*}(C_i) = Q_i$, where Q_i is a point of the surface V_{i+1} (cf. the Introduction, [55])). The regular mapping T_i induces an epimorphism (mapping) of the groups of divisors

$$T_{i*}\colon\ D(V_i) \longrightarrow D(V_{i+1}),$$

and also, since algebraic equivalence is preserved by a regular mapping, the epimorphism

$$T_{i*}\colon\ S(V_i) \longrightarrow S(V_{i+1}).$$

Since the homomorphism T_{i*} has a nonzero kernel (since it includes the class of algebraic equivalence of the curve C_i), while the rank of the group $S(V_1)$ is finite, there cannot exist an infinite chain of surfaces $V_1 > V_2 > V_3 > \cdots$.

Thus there exists in each class $\{V\}$ at least one relatively minimal model. Moreover, for any surface $\bar{V}' \in \{V\}$, there clearly exists a relatively minimal model $V' \in \{V\}$ such that $V' < \bar{V}'$. It is also clear that if a minimal model does not exist in the class $\{V\}$, then there exists in this class at least two relatively minimal models for which neither $V_1 > V_2$ nor $V_2 > V_1$. We consider the birational mapping $T\colon\ V_1 \longrightarrow V_2$. Since T is nonregular, there exists a point $Q_1 \in V_1$, fundamental for T, i.e., such that $T(Q_1)$ is some curve E on the surface V_2.

We now consider the mapping T^{-1}. If this mapping, which necessarily has fundamental points on V_2, did not have fundamental points on E, then there would be a surface $V_3 \in \{V\}$ and a regular mapping $T'\colon\ V_2 \longrightarrow V_3$ such that $T'(E)$ would be a point on V_3 (cf. Lemma 2, §1). This would mean that V_2 is not a relatively minimal model, contrary to the assumption. Thus, from the nonexistence of a minimal model in some class $\{V\}$ it follows that there exists in the class a relatively minimal model V with a curve E lying on it which has the following properties:

(1) the curve E is the total preimage of some point $Q \in V'$ under the birational mapping $T\colon\ V \longrightarrow V'$, $E = T^{-1}(Q)$;

(2) the mapping T has at least one fundamental point on the curve E.

Such a curve E is said to be an exceptional curve of the second kind; these curves will be studied in detail in the sequel. From the preceding arguments it is clear that each relatively minimal model of a class $\{V\}$ without a minimal model carries an exceptional curve of the second kind.

We shall show later that if an exceptional curve of the second kind lies on a relatively minimal model, then an irreducible exceptional curve of the second kind lies on it also.

The basic proposition of this chapter will follow from the following theorem (cf. Theorem A, §4):

If an irreducible exceptional curve of the second kind E lies on a relatively minimal model V, then this surface is ruled if $(E^2) = 0$, and is rational if $(E^2) > 0$.

It is also possible to show that if a class $\{V\}$ does not have a minimal model, then any surface $V' \in \{V\}$ carries an exceptional curve of the second kind; this statement, however, will not be needed.

§1. Exceptional curves of the first kind

An algebraic curve E on a nonsingular surface V is said to be an exceptional curve if there exists a birational transformation $T\colon V \longrightarrow V'$ of the surface V onto a nonsingular surface V' such that E is the total preimage under T of some point $Q' \in V'$, i.e. $E = T^{-1}(Q')$.

If the transformation T is a regular mapping at every point of the curve E, then this curve is said to be an exceptional curve of the first kind; in the contrary case, it is called an exceptional curve of the second kind.

In this section we shall study exceptional curves of the first kind. We shall show that in some sense the exceptional curves of the first kind are exhausted by the lines obtained from the surface as a result of applying σ-processes to it.

We first prove a lemma about regular transformations.

Lemma 1. *Let $T\colon V' \longrightarrow V$ be a birational transformation of a surface V' onto V that is regular at some nonsingular point $Q' \in V'$, let $Q = T(Q')$ be a nonsingular point of the surface V, and let $O_{Q'} > O_Q$ (i.e., Q is a fundamental point of the transformation T^{-1}). We consider further the surface $V^* = \sigma(V)$ obtained from the surface V as a result of the application of a σ-process at the point Q. And we consider the birational transformation $T'\colon V' \longrightarrow V^*$, $T' = \sigma T$, then T' will be regular at the point Q'.*

$$\begin{array}{ccc} V' & \xrightarrow{T} & V \\ & {\scriptstyle T'}\searrow \quad \swarrow{\scriptstyle \sigma} & \\ & V^* & \end{array}$$

Proof. Let Q^* be some point on V^* corresponding to $Q' \in V'$ under the transformation T'. From the definition of T' and σ, and also from the regularity of T at the point Q, it easily follows that $Q^* \in L = \sigma(Q)$. Let $(\bar{x}, \bar{y})$ be local parameters at the point Q^*. The functions $\bar{x}, \bar{y}$ can be chosen (cf. the description of a σ-process) such that $\bar{x} = x$, $\bar{y} = y/x$, where (x, y) are local parameters at the point Q. It is clear that $x, y \in \mathfrak{M}'$ where $\mathfrak{M}'$ is the maximal ideal of the local ring $O_{Q'}$. To prove our assertion it is sufficient to show that $\bar{x}, \bar{y} \in O_{Q'}$. Let us assume the contrary, i.e., that the point Q' is a fundamental point of the transformation T'. Then, as follows easily from definition, only the line $L = \sigma(Q)$ can correspond on V^* to the point Q'. The local equation of L in a neighborhood of the point Q^* is $\bar{x} = 0$. If one denotes by $\nu_L(\phi)$ the order of the function $\phi \in \Sigma$ on the line L, then $\nu_L(x) = 1$, from which one obtains that $x \notin \mathfrak{M}'^2$ (the point Q'

on V' corresponds to the curve L [55]). Since the point Q is a fundamental point of the transformation T^{-1}, there exists on the surface V' a curve (irreducible) C', such that $T(C') = Q$. It is clear that x and y vanish on C', and, since $x \notin \mathfrak{M}'^2$, $x = 0$ can serve as a local equation of the curve C' in a neighborhood of the point Q'. From this it follows that x divides y in the local ring $O_{Q'}$, i.e. $y/x \in O_{Q'}$, which contradicts our assumption. The lemma is proved.

Lemma 2. *Let V be a nonsingular surface and let E be an exceptional curve of the first kind on V. Then there exists a nonsingular surface G and a birational mapping T: $V \longrightarrow G$ which is everywhere regular on V and biregular on $V - E$, where, moreover $E = T^{-1}(P)$ for P a point on G.*

Proof. By the premise of the lemma there exists a nonsingular surface V' and a birational mapping T': $V \longrightarrow V'$ that is regular at all points $Q \in E \subset V$, where $E = (T')^{-1}(P')$, for P' a point of V'. The transformation $(T')^{-1}$ can have fundamental points on V' distinct from P'. Applying a finite sequence of σ-processes (at points $\neq P'$), we obtain a new nonsingular surface $V'' = T_0(V')$ such that the birational mapping $T'' = T_0 T'$: $V \longrightarrow V''$ is regular on E, $(T'')^{-1}$ has a fundamental point only at $P'' = T_0(P')$, and $E = (T'')^{-1}(P'')$, i.e. $(T'')^{-1}$ is regular on $V'' - P''$ (Theorem 1, §2, Chapter I).

Let L be the linear system on V (without fixed components) corresponding to the transformation T''. We consider the complete linear system $|hL|$ for a sufficiently large h and consider the corresponding transformation T and the surface $G = T(V)$. We shall show that the normalization G_N of the surface G ($G_N = N^{-1}(G)$) and the transformation $T_N = N^{-1}T$ corresponding to it satisfy the requirements of the lemma. First of all, the transformation T is everywhere regular on V (by a theorem of Zariski).

The transformation T may be considered to be composed of two transformations: the transformation $T_{|L|}$ of the surface V corresponding to the complete linear system $|L|$, and the transformation T_h of the surface $V_{|L|} = T_{|L|}(V)$ corresponding to some complete system of sections of the surface $V_{|L|}$ with the hypersurfaces of degree h. Since the surface $V_{|L|}$ is obtained from the surface V by the transformation $T_{|L|}$ corresponding to the system $|L| \supseteq L$, $V_{|L|}$ is birationally equivalent to V (since V'' was birationally equivalent to V; this immediately follows from the properties of mappings corresponding to linear systems, cf. the Introduction). Thus, the mapping T: $V \longrightarrow G$ is everywhere regular on V and is birational.

Let D be some curve on V, $D \neq E$, and let ξ be a generic point of the curve D. Then, since $\operatorname{tr}[T''(\xi)/k] = 1$ (since T'' does not contract the curve D into a point) and since $k(T(\xi)) \supset k(T''(\xi))$, for the transformation T corresponds to the linear system $|hL|$, we have $\operatorname{tr}[k(T(\xi))/k] = 1$, i.e., the curve $D \neq E$ is taken into a curve by the transformation T, and is not contracted into a point.

The linear system L consists, by definition, of effective divisors D_λ, such that

$$D_\lambda = (F_\lambda) + D, \; F_\lambda = \lambda_0 F_0 + \cdots + \lambda_d F_d,$$

where $F_0, \cdots, F_d$ are linearly independent rational functions on V and D is a divisor on V such that $(F_j) + D \geq 0$, $j = 0, \cdots, d$. The mapping of the surface V corresponding to the system L is given in the following manner in the neighborhood U_i:

$$V \ni P \xrightarrow{T''} (F_0 R_D^i(P), \ldots, F_d R_D^i(P)),$$

where R_D^i is a local equation of D in the neighborhood U_i.

It is clear that in order for the curve E to be taken into a point under a birational transformation S corresponding to a linear system M (without fixed components), it is necessary and sufficient that $(E \cdot \bar{M}) = 0$, where $\bar{M}$ is a generic curve of the system M. From this we obtain immediately that since $(E \cdot \bar{L}) = 0$, $(E \cdot h\bar{L}) = 0$, the curve E is transformed into a point by T. Thus the mapping $T\colon V \longrightarrow G$ regularly transforms the curve E into some point $P \in G$, and does not contract any other curve $D \neq E$ into a point.

Along with the surface G we consider its derived normal model G_{N} and the commutative triangle (T_1) of birational mappings, where T and N are regular mappings, and $T_{\mathrm{N}} = N^{-1}T$.

$$\begin{array}{ccc} V & \xrightarrow{T} & G \\ & {}_{T_{\mathrm{N}}}\nwarrow \quad \nearrow_{N} & \\ & G_{\mathrm{N}} & \end{array} \qquad (\mathrm{T}_1)$$

In the future we shall need two basic facts from the theory of normalization of surfaces:

(1) for any point $Z \in G$, $N^{-1}(Z)$ is a finite set of points;

(2) if the mapping T_{N}^{-1} is nonregular at a point $R \in G_{\mathrm{N}}$, then a curve on V corresponds to the point R under the mapping T_{N}^{-1} (analogously, if T_{N} is nonregular at a point $Q \in V$, then a curve on G_{N} corresponds to the point Q).

The proof of these facts is given, for example, in the book of Hodge and Pedoe ([55], Theorem III).

From this and from the properties of the mapping T it immediately follows that: (a) the mapping T_{N} is regular everywhere on V; (b) the mapping T_{N}^{-1} is regular on $G_{\mathrm{N}} - N^{-1}(P)$, where $P = T(E)$; (c) $T_{\mathrm{N}}(E) = N^{-1}(P) = R$ (as an image of an irreducible curve).

Thus $V - E$ is biregularly equivalent to $G_{\mathrm{N}} - N^{-1}(P)$. It remains to show that $R = N^{-1}(P)$ is a nonsingular point of the surface G_{N}. For this we consider the triangle of birational mappings (T_2), where we denote by $\bar{T}$ the mapping $T''T_N^{-1}$.

$$\begin{array}{ccc} V & \xrightarrow{T''} & V'' \\ & {\scriptstyle T_N}\searrow & \uparrow {\scriptstyle \bar{T}} \\ & & G_N \end{array} \qquad (T_2)$$

We shall show that the mapping $\bar{T}$ is biregular at the point $R \in G_N$, from which will immediately follow that the point R is simple. From the definition of $\bar{T}$ it follows that $\bar{T}(R) = P''$, i.e. $\bar{T}$ is regular at the point R. If $(\bar{T})^{-1}$ were nonregular at the point P'', then there would exist a curve $E_1 \subset G_N$, $E_1 \neq \bar{T}^{-1}(P'')$, $E_1 \ni R$. We consider the curve $D = T_N^{-1}(E)$, $D \subset E$. It is clear that $D \neq E$; moreover, since $\bar{T} = T''T_N^{-1}$, and T_N is regular, $T''(D) = P''$, which contradicts the assumption (that E is a total preimage of the point P'' under the transformation T''). The lemma is proved.

These lemmas make it possible to describe the exceptional curves of the first kind. Let V be a nonsingular surface, and let E be an exceptional curve of the first kind on it. This means that there exists a birational mapping T of the surface V onto some nonsingular surface V' such that T is regular at all the points of E, $T(E) = Q'$.

Then, as follows from Lemma 2, there exists a birational mapping $\bar{T}$ of the surface V onto a nonsingular surface $\bar{V}$ which is regular everywhere on V and biregular on $V - E$, such that $E = \bar{T}^{-1}(\bar{Q})$ where $\bar{Q} \in \bar{V}$. Applying a σ-process to the surface $\bar{V}$ at the point $\bar{Q}$, we obtain a nonsingular surface V^* with a line L on it such that $L = \sigma(\bar{Q})$. We consider the (birational) mapping $T^* = \sigma\bar{T}$ of the surface V onto V^*. From Lemma 1 it follows that T^* is regular everywhere on E and, consequently, on all of V, and is biregular on $V - E$

$$\begin{array}{ccc} V & \xrightarrow{\bar{T}} & \bar{V} \\ & {\scriptstyle T^*}\searrow \swarrow {\scriptstyle \sigma} & \\ & V^* & \end{array}$$

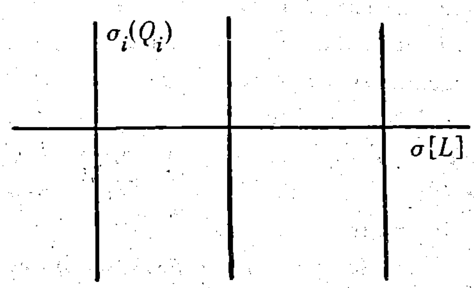
$$\left(V - E \overset{T^*}{\approx} V^* - L\right).$$

Here $(T^*)^{-1}(L) = E$. It is clear that the curve E will consist of the (algebraic) image of the curve L under the transformation $(T^*)^{-1}$ and of exceptional curves of the transformation T^* that correspond to fundamental points of the transformation $(T^*)^{-1}$ on the line L. If E is an irreducible curve, the transformation $(T^*)^{-1}$ does not have fundamental points on L, i.e. T^* is a biregular mapping of V onto V^*. Thus, the following theorem has been proved:

Theorem 1. *Any irreducible exceptional curve of the first kind is obtained (up to a biregular transformation of the surface) as a result of the application of* *a σ-process at some point of the nonsingular surface. We obtain from this, in particular, that if E is an irreducible exceptional curve of the first kind, then $(E^2) = -1$, $p_a(E) = 0$.*

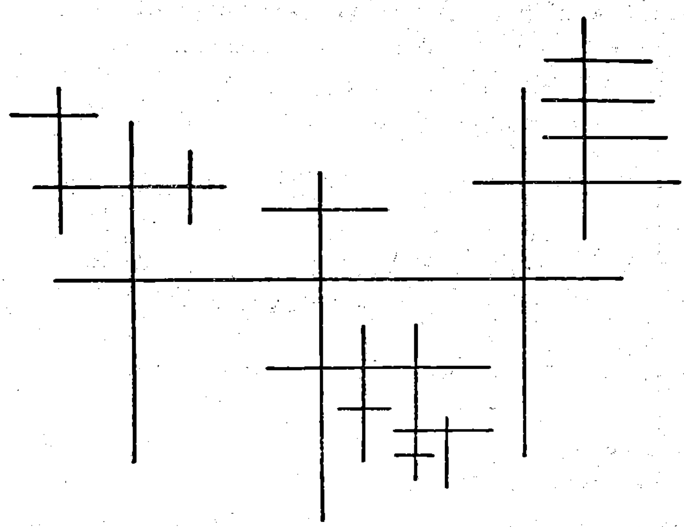
In the rest of the section we use the notation of the preceding proofs.

Now let E be a reducible exceptional curve of the first kind. We denote by $E_0 = (T^*)^{-1}[L]$ the algebraic inverse image of the line L under the mapping T^*. Since $E_0 \neq E$, there exist on the line L points $Q_1, \cdots, Q_r$ that are fundamental points of the transformation $(T^*)^{-1}$. If we write $E_i = (T^*)^{-1}(Q_i)$, then, clearly each curve E_i is again an exceptional curve of the first kind. Applying a σ-process at each point Q_i of the surface V^*, we obtain a surface V_1^*, on which there lies the line $\widetilde{L}_1$ depicted in the figure,



$$\widetilde{L}_1 = \sigma[L] + \Sigma\sigma_i(Q_i).$$

As follows from Lemma 1, there exists a regular mapping $T_1^* = \sigma_r\sigma_{r-1} \cdots \sigma_1 T^*$ of the surface V^* onto V_1^*, such that $(T_1^*)^{-1}(\widetilde{L}_1) = E$. Repeating this process, we find that any exceptional curve of the first kind E has the form



of a tree consisting of irreducible rational nonsingular curves L_i. The curve E has only double singular points formed by the intersections of the L_i; the curves L_i and L_j have no more than one point of intersection. This tree is connected;

of the sheaf $F_L(L \cdot \bar{E}_m)$ will be 1 and a function ψ that has a unique zero on L (cf. the preceding argument about the system $|\bar{E}_{m+1}|$). We denote the inverse images of these functions in $H^0(V, F(\bar{E}_{m-1}))$ by χ_1 and χ_2; this means that in U_i

$$\chi_1 e^i_m|_L = 1, \quad \chi_2 e^i_{m-1}|_L = \psi^i.$$

(Clearly, in some U_j, $j \neq i$, it is possible that

$$\chi_1 e^i_{m-1}|_L = \psi^i, \quad \chi_2 e^i_{m-1}|_L = 1.)$$

We consider the (global) functions $\xi_1 = \chi_1/f_q$, $\xi_2 = \chi_2/f_q$. Since $f_q e^i_{m+1}|_L = 1$ in any neighborhood U_i, and since $e^i_{m+1} = e^i_{m-1} \cdot l^i$, where l^i is a local equation of L in U_i, the functions ξ_1 and ξ_2 are holomorphisms in a neighborhood $U(L)$ of the curve L and vanish on L.

Our goal is to show that the functions ξ_1 and ξ_2, considered on $\bar{V}$, are local parameters at a point $P \in \bar{V}$. In other words, we will show that there exists a biregular mapping $U(\xi_1, \xi_2)$ of a neighborhood O onto $U(P) = \Phi(U(L)) \subset \bar{V}$.

First of all, there exists a biregular mapping of $U(L) - L$ into $U - O$:

$$x: \; z \to (\xi_1(z), \; \xi_2(z)).$$

For, if the point $q \in L$, $q \in U_i \cap U(L)$, then either $\xi_2/\xi_1 = \chi_1/\chi_2$ or $\xi_1/\xi_2 = \chi_1/\chi_2$ can serve as a local parameter on L; let the first case be true. Then, clearly, $(\xi_1, \xi_2/\xi_1)$ is a system of local coordinates on V in the neighborhood $U_i \cap U(L)$. But in this case $(\xi_1, \xi_2/\xi_1)$ are local parameters at any point $z \in U_i \cap U(L) - L$. For this it is only necessary to repeat the arguments given during the discussion of the σ-process (the neighborhood $U_i \cap U(L)$ is itself obtained, as may easily be seen, from the neighborhood $U(\xi_1, \xi_2)$ with the aid of the σ-process $\xi_1 t_2 - \xi_2 t_1 = 0$).

We now consider the mapping Φ in the neighborhood $U(L)$. In $U(L) \cap U_i$ the function $f_q \cdot e^i_{m+1} \neq 0$, and therefore Φ is written in nonhomogeneous coordinates in the form

$$\Phi^i(t) = \left(\frac{f_0(z)}{f_q(z)}, \ldots, \frac{f_{q-3}(z)}{f_q(z)}, \xi_1^i(z), \xi_2^i(z), 1\right),$$

where all the $f_k(z)/f_q(z)$ are holomorphic functions of $z \in U(L) \cap U_i$ and, consequently, are holomorphic functions of ξ_1 and ξ_2 if $\xi_1, \xi_2 \in U(\xi_1, \xi_2)$. But then all the functions $f_k(z)/f_q(z)$ are holomorphic functions of ξ_1 and ξ_2 in all of $U(\xi_1, \xi_2)$, since they cannot have isolated poles.

Thus the mapping $\Phi \circ x^{-1}$ is a biholomorphic mapping of $U(\xi_1, \xi_2)$ into $\Phi(U(L)) = U(P) \subset \bar{V}$, which is what we needed to show.

§3. Exceptional curves of the second kind

We now turn to exceptional curves of the second kind. Let $T\colon V \longrightarrow V'$ be a birational transformation of a nonsingular surface V onto a nonsingular V', let $Q' \in V'$, and let $E = T^{-1}(Q')$ be the total preimage of the point Q', i.e., an exceptional curve. We consider a so-called "dominating" surface G, i.e., a nonsingular surface birationally equivalent to V and V', such that there exist (birational) mappings $f\colon G \longrightarrow V$ and $f'\colon G \longrightarrow V'$ that are everywhere regular on G. Such a surface can be constructed using the method of Lemma 2, §1.

$$\begin{array}{ccc} & G & \\ {\scriptstyle f}\swarrow & {\scriptstyle T} & \searrow{\scriptstyle f'} \\ V & \longrightarrow & V' \end{array}$$

The triangle of mappings is commutative. It is clear that $E = T^{-1}(Q') = f((f')^{-1}(Q'))$, from which it follows immediately that an exceptional curve of the second kind is connected, for $\bar{E} = (f')^{-1}(Q')$ is an exceptional curve of the first kind (and thus connected), and f is a regular mapping.

Now let D be an irreducible component of the curve E. The algebraic inverse image $f^{-1}[D]$ of the curve D is an irreducible curve on G, and, moreover, $f^{-1}[D] \subset (f')^{-1}(Q') = \bar{E}$. But the curve $\bar{E}$ is an exceptional curve of the first kind, and thus all its irreducible components are rational curves. Consequently the curve D is also rational.

We introduce a σ-process at the point $Q' \in V'$; we obtain a surface V_1' with the line L on it. We consider the triangle of mappings $T_1 = \sigma_1 T$.

$$\begin{array}{ccc} V & \xrightarrow{T} & V' \\ & {\scriptstyle T_1}\searrow \quad \swarrow{\scriptstyle \sigma_1} & \\ & V_1' & \end{array}$$

It is clear that $E = T_1^{-1}(L)$. Let us assume that there exists a point $Q_1' \in L$ such that $E = T_1^{-1}(Q_1')$. Then we introduce a σ-process at the point $Q_1' \in V_1'$, obtaining a surface V_2', and so on. Thus we obtain a sequence of points $Q' < Q_1' < Q_2' < \cdots$ lying respectively on the surfaces $V', V_1', V_2', \cdots$ and such that the curve E is the total preimage of each of these points under the corresponding mapping. But then, as follows from §3, Chapter I, this sequence breaks off at a finite step, i.e., there exists a surface $\bar{V}'$ and a point $\bar{Q}'$ on it (called the "maximal contraction"), and also a birational transformation $\bar{T}\colon V \longrightarrow \bar{V}'$ such that $E = \bar{T}^{-1}(\bar{Q}')$, where the following conditions hold: if we introduce a σ-process at the point $\bar{Q}' \in \bar{V}'$, then no point $\bar{Q}^* \in \bar{L}^* = \sigma(\bar{Q}') \subset \bar{V}^*$ can have the curve E as a total preimage. It is clear that $E = (\bar{T}^*)^{-1}(\bar{L}^*)$; here $\bar{V}^* = \sigma(\bar{V}')$, $\bar{T}^* = \sigma\bar{T}$.

$$\begin{array}{ccc} V & \xrightarrow{\bar{T}} & \bar{V}' \\ & {\scriptstyle \bar{T}^*}\searrow & \downarrow{\scriptstyle \sigma} \\ & & \bar{V}^* \end{array}$$

Now let the curve E be reducible and have $q > 1$ irreducible components. Since the curve $\bar{L}^*$ is irreducible, the transformation $(\bar{T}^*)^{-1}$ must have (at least one) fundamental point P' on $\bar{L}^*$. Let $E_1 = (\bar{T}^*)^{-1}(P')$. In view of the choice of the surface $\bar{V}'$ and the point $\bar{Q}'$, E_1 will be an exceptional curve strictly included in the curve E, i.e. consisting of $s < q$ components. Two cases may arise: either E_1 is an exceptional curve of the second kind, or it is one of the first kind. In the second case there exists a birational everywhere regular mapping T_0: $V \to \bar{V}_0$, under which $E_1 = T_0^{-1}(Q_0)$. We consider the curve $E_0 = T_0(E)$. Then $E_0 = (\bar{T}_0)^{-1}(\bar{Q}')$, and, since the transformation $\bar{T}$

$$\begin{array}{ccc} V & \xrightarrow{T_0} & V_0 \\ {\scriptstyle \bar{T}}\searrow & & \swarrow{\scriptstyle \bar{T}_0} \\ & \bar{V}' & \end{array}$$

was not regular on E, the transformation $\bar{T}_0$ will not be regular everywhere on E_0, i.e., E_0 will be an exceptional curve of the second kind, where the number of components of the curve E_0 will be $< q$ (it will be equal to $q - s$, where s is the number of components of E_1). Thus, in both the first and second cases we obtain an exceptional curve of the second kind (lying, perhaps, on another surface of the same birational class as the original surface V) with a number of components less than q. We summarize all the results obtained.

Theorem 1. *An exceptional curve of the second kind is connected and all its irreducible components are rational curves. Moreover, if an exceptional curve of the second kind E lies on a surface V, then there exists a (nonsingular) surface of the same class containing an irreducible (and, consequently, rational) exceptional curve of the second kind. In addition, it follows immediately from the proof of the last statement, that if V is a relatively minimal model, then there already exists on it an irreducible exceptional curve of the second kind (in fact, the above described curve E_1 will necessarily be an exceptional curve of the second kind with a smaller number of components than E).*

Lemma 1. *Let E be an irreducible exceptional curve of the second kind on the surface V, and let T: $V \to V'$ be a birational transformation such that $E = T^{-1}(P')$, $P' \in V'$. Let, further, the point $P \in E$ be a fundamental point of the transformation T (such a point always exists, since E is an exceptional curve of the second kind) and let σ: $V \to \bar{V}$ be a σ-process at the point $P \in E$. Then $\sigma[E]$ is also an exceptional curve (perhaps of the first kind).*

Proof. We consider a commutative triangle of mappings. Let $L = \sigma(P)$. Then

$$\begin{array}{ccc} V & \xrightarrow{T} & V' \\ \sigma\downarrow & \nearrow \bar{T} & \\ \bar{V} & & \end{array}$$

$$\bar{T}^{-1}(P') \subseteq \sigma[E] + L.$$

We will show that $\bar{T}^{-1}(P') = \sigma[E]$. In view of the fact that P is a fundamental point of the transformation T, there exists an (irreducible) curve E' on V' passing through P', and such that $T^{-1}[E'] = P$. We consider some point $Q' \in E'$, $Q' \neq P'$, at which the mapping T^{-1} is regular. But then the mapping $\bar{T}^{-1}$ will also be regular at this point (Lemma 1, §1); we denote the corresponding point (clearly lying on the curve L) by Q. Then $O_{Q'} > O_P$, $O_{Q'} \geq O_Q$. If we now assume that $\bar{T}[L] = P'$, then it is possible to suppose without loss of generality that $\bar{T}$ is regular at the point Q, and consequently $O_Q > O_{P'}$. From this it would follow that $O_{Q'} > O_{P'}$, which is clearly impossible. The lemma is proved.

If the curve $\sigma[E]$ is again an exceptional curve of the second kind, we again apply a σ-process at the point $P' = \sigma[E]$, a fundamental point of the transformation $\bar{T}$, and so on. It is clear that after a finite number of steps we obtain a curve $\bar{E}$ which is an exceptional curve of the first kind. The following theorem is thus proved.

Theorem 2. *Let E be an exceptional curve of the second kind on V. Then there exists a surface G obtained from V by an antiregular birational transformation g: $G \to V$ that is a product of a finite number of σ-processes, such that $g^{-1}[E]$ is an (irreducible) exceptional curve of the first kind, i.e., a rational curve without singularities. From our arguments it follows in particular that all the singular points of an irreducible exceptional curve of the second kind E are necessarily fundamental points of the corresponding transformation; in the process of transforming the exceptional curve of the second kind into an exceptional curve of the first kind by applying σ-processes to the surface V at the singular points of E, we resolve these singularities (i.e., we desingularize the curve E).*

From these properties of irreducible exceptional curves of the second kind we now obtain, using the Lemma of §1, Chapter I, certain numerical characteristics of such curves.

Let E be an irreducible exceptional curve of the second kind on V. We write $E^2 = k$, $g(E) = 0$, $\pi(E) = p$. If E is a nonsingular curve, then $p = 0$. For the transformation of E into an exceptional curve of the first kind, one needs $n > 0$ σ-processes with centers at the points $Q_0 \in E$, $Q_1 \in E_1 = \sigma_1[E], \cdots$ $\cdots, Q_{n-1} \in E_{n-1} = \sigma_{n-1}[E_{n-2}]$; $E' = \sigma_n[E_{n-1}]$. We have $(E^2) = (E'^2) + n$ (Lemma 1, §1), and since $(E'^2) = -1$; $(E^2) = -1 + n \geq 0$, $-(KE) = 1 + n \geq 2$. If the curve E has singular points $P_1, \cdots, P_r$ of multiplicities $s_1, \cdots, s_r$, then in the process of transforming it into an exceptional curve of the first kind we must resolve all the singularities with the aid of σ-processes.

Using Lemma 1, §1, it is easy to find the numerical characteristics of such a curve:

$$(E^2) = -1 + n + \sum_{i=1}^{r} s_i^2, \tag{1}$$

$$\pi(E) = \sum_{i=1}^{r} \frac{1}{2} s_i(s_i - 1), \tag{2}$$

$$-(KE) = 1 + n + \sum_{i=1}^{r} s_i, \quad \text{where } n + r > 0, \quad n, r \geq 0. \tag{3}$$

We obtain the following proposition about irreducible exceptional curves of the second kind.

Theorem 3. *An irreducible exceptional curve of second kind E is a rational curve, $(E^2) \geq 0$; if $(E^2) = 0$, then E is a nonsingular curve and $\pi(E) = 0$.*

§4. Basic theorem

We now turn to the proof of a basic theorem (cf. the introduction to Chapter II), namely we will show that if some class B of nonsingular surfaces does not have an (absolute) minimal model, then this class consists of ruled surfaces. First of all, the absence of an absolute minimal model means that an irreducible exceptional curve of the second kind lies on any relatively minimal model of this class (Theorem 1, §3), where $(E^2) \geq 0$ (Theorem 3, §3). We will now prove the following theorem, from which the basic one will follow.

Theorem A. *If an irreducible exceptional curve of the second kind lies on a relatively minimal surface V (i.e. a surface without an exceptional curve of the first kind), then the surface V is ruled if $(E^2) = 0$ and is rational if $(E^2) > 0$.*

When $(E^2) = 0$ the curve E is nonsingular (Theorem 3, §3). This case is not difficult to examine. The proof of the corresponding theorem presupposes the following lemma, which will be used frequently in the future.

Lemma. *If an irreducible exceptional curve of the second kind C whose index of selfintersection $(C^2) \geq 0$ lies on a nonsingular algebraic surface V, and if $(C \cdot K) < 0$ (where K is a canonical divisor on V), then all the plurigenera P_n of the surface V are equal to 0.*

Proof. We assume, to the contrary, that some $P_n = l(nK) > 0$. Then there exists an effective divisor D (or 0) such that $nK \sim D$ (~ 0). But then $(D \cdot C) \geq 0$. In fact, if one represents D in the form of the sum of irreducible curves $\Sigma n_i C_i$,

$n_i > 0$, then $(D \cdot C) = \Sigma n_i(C_i \cdot C)$, where $(C \cdot C_i) \geq 0$ if $C_i \neq C$ and $(C_i \cdot C) \geq 0$ if $C_i = C$, since $(C^2) \geq 0$. From this we have $(D \cdot C) \geq 0$, which contradicts the assumption of the lemma $((D \cdot C) = n(K \cdot C) < 0)$. Thus, all the $P_n = 0$.

Corollary. *If an exceptional curve of the second kind lies on the surface V, then all the plurigenera are equal to zero (cf. formula 3, §3, and Theorem 1).*

From this, on the basis of results of Chapter IV, it already follows that the surface V is ruled.

Theorem. *Let V be a nonsingular algebraic surface which contains a nonsingular irreducible rational curve E such that $(E^2) = 0$. Then V is birationally equivalent to a ruled surface.*

Proof. Since $\pi(E) = 0$ and $E^2 = 0$, from the formula $\pi(E) = ((E^2) + (E \cdot K))/2 + 1$ we immediately obtain that $(E \cdot K) = -2$; hence, by the preceding lemma, $l(n \cdot K) = 0$, $n > 0$. We apply the Riemann-Roch theorem to the curve nE:

$$l(n \cdot E) \geqslant \frac{1}{2}((n \cdot E)^2) - \frac{1}{2}(K \cdot (nE)) + 1 + p_a - l(K - nE);$$

since $((nE)^2) = n^2(E^2) = 0$, $(K \cdot nE) = -2n$ and $l(K - nE) \leq l(K)(E > 0)$, we obtain the inequality

$$l(nE) \geq n + p_a. \tag{1}$$

From the exact sequence of sheaves

$$0 \longrightarrow F((n-1)E) \longrightarrow F(nE) \longrightarrow F_E(E \cdot nE) \longrightarrow 0$$

we obtain the exact cohomology sequence:

$$0 \longrightarrow H^0(V, F((n-1)E)) \longrightarrow H^0(V, F(nE)) \longrightarrow H^0(E, F_E(E \cdot nE)) \longrightarrow \cdots. \tag{2}$$

Since $(E \cdot nE) = n(E \cdot E) = 0$, and E is a nonsingular rational curve, $\dim H^0(E, F_E(E \cdot nE)) = l_E(0) = 1$, and from the exact sequence (2) we have

$$l((n-1)E) \leq l(nE) \leq l((n-1)E) + 1. \tag{3}$$

Comparing inequalities (1) and (3), we obtain that, starting with some n, $l(nE) = n + C$, where C is some constant, and, consequently, for sufficiently large n

$$l(nE) = 1 + l((n-1)E). \tag{4}$$

Since the curve E is irreducible, (4) means that for sufficiently large n the linear system $|nE|$ does not have fixed components. At the same time a generic curve of this system is reducible. In fact, let D be a generic curve of the system $|nE|$ (n sufficiently large) and let P be a point on D. Then, as follows from the inequality of (1) there exists a curve $D' \in |nE|$, $D' \neq D$, passing through the point P. By assumption $(D' \cdot D) = (nE \cdot nE') = 0$, from which it follows that the curves

D and D' must have a common component (passing through the point P). Consequently, by the theorem of Bertini, the system $|nE|$ is composed of some irreducible pencil of curves; we denote this pencil by L. We will show that the curve E is itself a curve of this pencil. In fact, the number n can be chosen from the beginning large enough so that the system $|(n-1)E|$ does not have fixed components. This means that the system $|(n-1)E|$ contains a curve D such that D does not have E as a component. Therefore the system $|nE|$ contains a curve $D' = D + E$ that has E as a simple component. In view of this the curve E must be a curve of the irreducible pencil of which the system $|nE|$ consists. Thus, we have found on the surface V a pencil L, a member of which is the irreducible nonsingular rational curve E. From this, by the theorem of Noether (§2, Chapter I), the surface V is ruled. The proof of the second part of Theorem A is more difficult.

The proof will follow from a theorem of Castelnuovo which asserts that if a nonsingular surface V is such that $P_2 = 0$, and $p_a = 1$, then the surface V is rational (Chapter III).

The genus P_2 of our surface V containing the irreducible exceptional curve of the second kind E with $(E^2) > 0$, is equal to 0 by the corollary to Lemma 1.

We will show that $p_a(V) = 1$. First of all, since the geometric genus $P_1 = 0$, it is sufficient to show that the irregularity q of the surface V is equal to 0, for $p_a = p - q + 1$. The irregularity of the surface is the dimension of its Albanese variety A. We consider the canonical mapping $\alpha_V\colon\ V \longrightarrow A$, and let $V' = \alpha_V(V)$. Since V is a nonsingular surface, the mapping α is regular on V. The image $\alpha(E)$ of the exceptional curve of the second kind will be a point, for in the contrary case there would exist on A a rational curve $\alpha(E)$ (since E is rational), which is impossible. From this, in particular, it follows that any effective curve $U \sim mE$, where $m > 0$, is taken by the mapping α_V into a point $P = \alpha(E) \in A$.

In fact, $\alpha(U)$ will be some effective curve on A, where $\alpha(U) \approx \alpha(mE) \approx 0$. This means that, finally, $\alpha(U) = Q \in A$. On the other hand, since $(U \cdot E) = m(E^2) > 0$, i.e. since the curves U and E intersect, $Q = P$. From this we immediately obtain that $\alpha_V(V) = P$, for, if m is sufficiently large, the linear system $|mE|$ has a positive dimension (since $(E^2) > 0$, this follows immediately from the Riemann-Roch theorem), and, consequently, through each point of the surface V there passes an effective curve $U \sim mE$. Since $\alpha(V)$ generates A, $\dim A = 0$, and our assertion is proved.

CHAPTER III

CRITERIA OF RATIONALITY

§1. Adjoint systems

We consider a linear system $|D|$ of curves on a surface V corresponding to a linear space of functions $\mathfrak{L}(D)$. This system consists of (effective) curves C such that $D \sim C > 0$. A system adjoint to $|D|$ is the system $|D + K|$ where K is a canonical divisor on V. The system $|D + K|$ will also be called the first system adjoint to the system $|D|$; along with it we will consider second, third, etc. adjoint systems: the nth adjoint system is the system $|D + nK|$. We now give some conditions that assure the emptiness of the nth adjoint system of a given system $|D|$ for a large n.

We denote by K_a an anticanonical divisor on V i.e. the divisor $-K$. The linear system $|K_a| = |-K|$ is called an anticanonical system on V. We will say that a linear system $\mathfrak{L}$ exists, if there can be found a divisor $D \geq 0$, $D \in \mathfrak{L}$.

Lemma 1. *If an anticanonical divisor on V is positive, then for any divisor D on V, the nth adjoint system $|D + nK|$ does not exist for a sufficiently large n.*

The existence of an anticanonical system means that there exists a divisor $Z \sim -K$, $Z > 0$.

Let H be the hyperplane section of the surface V. Then $(H \cdot K) = -(H \cdot Z) = r < 0$. We write $(H \cdot D) = s$. Then $((D + nK) \cdot H) = s + nr < 0$ if $n > -s/r$. Hence, clearly, the divisor $D + nK$ cannot be linearly equivalent to any effective divisor, i.e., the system $|D + nK|$ does not exist.

Lemma 2. *Let V be a surface on which there exist no exceptional curves of the first kind, and let there exist on V an effective curve $\mathfrak{E}$ such that $(\mathfrak{E} \cdot K) < 0$. Then for any divisor D on V, the nth adjoint system $|D + nK|$ does not exist for a sufficiently large n.*

Let $\mathfrak{E} = \Sigma n_i C_i$, where the C_i are irreducible curves and $n_i > 0$. Then, since $(\mathfrak{E} \cdot K) < 0$, one can find a curve C_i such that $(C_i \cdot K) = -r < 0$.

From the formula for the arithmetic genus

$$\pi(C_i) = \frac{(C_i^2) + (C_i \cdot K)}{2} + 1$$

it follows that, since $\pi(C_i) \geq 0$, there exist two possibilities:

(a) $\pi(C_i) = 0$, $(C_i^2) = (C_i \cdot K) = -1$,

(b) $\pi(C_i) \geq 0$, $(C_i^2) \geq 0$.

Case (a) is impossible, because we have assumed that there are no exceptional curves of the first kind on the surface V.

We consider the index of intersection $((D + nK) \cdot C_i)$. Since $(K \cdot C_i) = -r<0$, $((D + nK) \cdot C_i) < 0$ if n is sufficiently large. Thus there cannot exist an effective curve $C \sim D + nK$, for $(C \cdot C_i) \geq 0$ (in view of the fact that $(C_i^2) \geq 0$).

Corollary 1. *Let* $(K^2) < 0$. *Then for any divisor* D *there exists an* n *such that the system* $|D + nK|$ *does not exist.*

Proof by contradiction. Assume that the system $|D + nK|$ exists for all n. This system cannot consist of only 0 for large n, for if that were true, then $D + nK \sim 0$, i.e. $(D^2) = n^2(K^2)$ for large n, which is impossible for $n^2 > (D^2)/(K^2)$. Thus for any sufficiently large number n there exists an effective curve $\mathfrak{S} \sim D + nK$. It is clear that $(\mathfrak{S} \cdot K) = ((D + nK) \cdot K) < 0$, if n is sufficiently large. Therefore, in view of Lemma 2, the system $|D + nK|$ does not exist.

Lemma 3. *Let a curve* $D > 0$ *on the surface* V *be such that the adjoint system* $|D + K|$ *does not exist. Then* $\pi(D) \leq 1 - p_a(V)$.

In fact, the nonexistence of the system $|K + D|$ means that $l(K + D) = \dim \mathfrak{L}(K + D) = 0$. Applying the Riemann-Roch theorem to the divisor $-D > 0$, we obtain:

$$0 = l(-D) \geq \frac{1}{2}(D^2) + \frac{1}{2}(D \cdot K) + p_a(V) - l(K + D). \qquad (1)$$

From (1) it follows that:

$$\pi(D) = \frac{(D^2) + (D \cdot K)}{2} + 1 \leq 1 - p_a(V),$$

since $l(K + D) = 0$.

Corollary 2. *If* C *is an irreducible curve on the surface* V *with* $p_a(V) = 1$ *and the system* $|K + C|$ *does not exist, then* $\pi(C) = 0$.

In this case $\pi(C) \leq 0$. Since the curve C is irreducible $\pi(C) \geq 0$, hence $\pi(C) = 0$.

§2. A theorem of Castelnuovo

The theorem of Castelnuovo we prove is the following: every nonsingular surface with $p_a(V) = 1$, $P_2(V) = 0$ is birationally equivalent to a rational surface. Since p_a and P_2 are birational invariants, we can assume from the beginning that our surface V is a relatively minimal surface, i.e., there does not exist an exceptional curve of the first kind on it, i.e., an irreducible curve X such that

$$(X^2) = -1, \quad \pi(X) = 0.$$

By Noether's lemma, to prove this theorem it is sufficient to find on the surface V a linear pencil L of curves, a generic member of which will be an (irreducible) curve with an arithmetic genus of 0, i.e., an irreducible rational nonsingular curve.

Let the surface V contain an irreducible curve C such that $\pi(C) = 0$, $(C^2) \geq 0$. Then, using the Riemann-Roch theorem, we obtain that $l(C) \geq (C^2) - \pi(C) + 2 = (C^2) + 2 \geq 2$, since $p_a(V) = 1$, $l(K - C) < l(K) = P_1 = 0$. Thus, one can find on the surface V a curve C such that: (1) $\pi(C) = 0$, (2) $l(C) \geq 2$.

The second condition guarantees the existence of a linear pencil of curves on V of which the curve C is a member. There exists an $f \in \mathfrak{L}(C)$, $f \neq 0$, since $l(C) = \dim \mathfrak{L}(C) > 1$. We consider the mapping of the surface V onto a projective line P^1 given by the function f. With this we define a linear pencil of curves L on V of which the curve C will be a member. A generic member of this pencil is the curve $C_\lambda = f^{-1}(\lambda) = (f - \lambda)_0$, where λ is a generic point of the line P^1, is clearly an irreducible curve (since the curve C is irreducible), and $\pi(C_\lambda) = 0$, since $C_\lambda \sim C$, and, consequently $\pi(C_\lambda) = \pi(C)$. Therefore, by the theorem of Noether, the surface V is rational.

Thus, to prove the theorem it is sufficient for us to find on the surface V an irreducible curve C satisfying the conditions $\pi(C) = 0$, $(C^2) \geq 0$.

We have to consider three cases: (1) $(K^2) = 0$, (2) $(K^2) < 0$, (3) $(K^2) > 0$.

(1) $(K^2) = 0$.

By the Riemann-Roch theorem, in this case $l(-K) \geq 1$. If $D \in |-K|$, then $D \neq 0$, since otherwise

$$P_2 = l(2K) = l(0) = 1.$$

Thus, an anticanonical system exists. Let E be a sufficiently high multiplicity of a hyperplane section of the surface V. Then $l(E + K) > 1$, and by Lemma 1 about adjoint systems: $l(E + rK) = 0$ for a sufficiently large $r > 0$. Thus, one can find a number $n \geq 1$ such that

$$l(E + nK) \geq 1,$$
$$l(E + (n+1)K) = 0.$$

Let $D' \in |E + nK|$, $D' = \Sigma a_i C_i$, where the C_i are irreducible curves and $a_i > 0$. The index of intersection $(K \cdot D') = (K \cdot (E + nK)) = (K \cdot E) + (nK^2) = -(E \cdot D) < 0$, from which it follows that at least for one curve C_i; the index of intersection $(K \cdot C_i) < 0$.

Since $C_i \leq D'$,

$$l(K + C_i) \leq l(K + D') = l(E + (n+1)K) = 0,$$

from which we have, by Lemma 3 about adjoint systems, that $\pi(C_i) = 0$. Thus we have found on the surface an irreducible curve C_i such that

$$(K \cdot C_i) < 0, \quad \pi(C_i) = 0.$$

By the formula for the virtual genus,

$$\pi(C_i) = \frac{(C_i^2) + (K \cdot C_i)}{2} + 1,$$

we obtain that $(C_i^2) \geq -1$. Since the case $(C_i^2) = -1$ is impossible (we assumed from the beginning that there exist no exceptional curves of the first kind on V), we have $(C_i^2) \geq 0$, which is what we wanted to show.

(2) $(K^2) < 0$.

Let E be some multiplicity of a hyperplane section of the surface V, where $l(E + K) > 2$. By the corollary of Lemma 2 about adjoint systems, the system $|E + sK|$ does not exist for sufficiently large s. One can find, consequently, a number $n > 0$ such that

$$l(E + nK) \geq 2,$$
$$l(E + (n+1)K) \leq 1.$$

From the first inequality it follows that there exists on the surface V a linear pencil L of curves. A generic element D of this pencil is represented in the form $D = A + \Sigma C_i$, where A is the fixed component of the pencil, and the C_i are irreducible curves with $(C_i^2) \geq 0$, and where moreover there exists at least one curve C_i. By the Riemann-Roch theorem

$$\pi(C_i) \leq l(K + C_i) \leq l(K + D) \leq 1.$$

If $\pi(C_i) = 0$, then the theorem is proved (since $(C_i^2) \geq 0$).

Let us assume that $\pi(C_i) = 1$, from which it follows that $l(K + C_i) = 1$, and let $0 \leq D \in |K + C_i|$. The divisor $D \neq 0$, for otherwise $K \sim -C_i$ and $(K^2) = (C_i^2) \geq 0$, which contradicts our main assumption $((K^2) < 0)$. Thus we have found a divisor $D > 0$, $D \sim K + C_i$. Since $\pi(C_i) = 1$, $(K \cdot C_i) = -(C_i^2) \leq 0$ and $(D \cdot K) = ((K + C_i) \cdot K) = (K^2) + (K \cdot C_i) < 0$. This means that at least one of the irreducible components of D, a curve G, has a negative index of intersection with K: $(G \cdot K) < 0$. At the same time $(G \cdot C_i) = 0$ since

$$(D \cdot C_i) = ((K + C_i) \cdot C_i) = 0, \quad (C_i^2) \geq 0.$$

Thus we have found an irreducible curve G such that $(G \cdot K) < 0$ and $(G^2) < 0$

(since $(G \cdot D) = (G \cdot (K + C_i)) < 0$). This is possible only when $(G \cdot K) = -1$, $(G^2) = -1$ and $\pi(G) = 0$, i.e., when an exceptional curve of the first kind exists on V. Therefore the case $\pi(C_i) = 1$ is impossible, and the theorem is proved.

(3) $(K^2) > 0$.

By the Riemann-Roch theorem $l(-K) \geq 2$, i.e., there exists a pencil L of curves D, $D \sim -K$, $D = A + \Sigma C_i$ where A is the fixed component of the pencil L, and $(C_i^2) \geq 0$.

If D is a reducible curve, then $D - C_i > 0$ and $\pi(C_i) \leq l(K + C_i) \leq l(C_i - D) = 0$, which is what we need to show. Let D be an irreducible curve, i.e., $D = C_i$, $(D^2) \geq 0$. Then, since $D \sim -K$, $\pi(D) = 1$, and by Lemma 1 about adjoint systems, for any hyperplane section E corresponding to some imbedding of V in a projective space, there can be found a number n_E such that

$$l(E + n_E K) \geq 1,$$

$$l(E + (n_E + 1)K) = 0.$$

Let $G \in |E + n_E K|$. We assume first that $G \neq 0$. Then $(G \cdot K) \leq 0$, since $K \sim -D$ and $(D^2) = (K^2) > 0$. Therefore one can find an irreducible component C of the curve G such that $(C \cdot K) \leq 0$.

Since $\pi(C) \leq l(K + C) \leq l(K + G) = 0$, $(C^2) + (K \cdot C) = -2$, and either $(C \cdot K) < 0$ and $(C^2) \geq 0$, i.e., C is the desired curve, or $(K \cdot C) = 0$, $(C^2) = -2$.

In the second case, by the Riemann-Roch theorem

$$l(D - C) = l(-K - C) \geq \frac{(D^2)}{2} - l(2K + C),$$

and, since $l(2K + C) \leq l(K + C) = 0$,

$$l(D - C) \geq 1.$$

Thus there exists a curve $H \in |D - C|$; since $C \not\sim D$, $((C^2) = -2,\ (D^2) > 0)$, then $H > 0$. Because $(H \cdot K) = -(C \cdot K) - (K^2) < 0$, one can find an irreducible component H_0 of the curve H, such that $(H_0 \cdot K) < 0$. Since the case $(H_0^2) = -1$ is impossible, and $\pi(H_0) \leq l(K + H_0) \leq l(H + K) = l(-C) = 0$, $(H_0^2) = -1$, then the curve H_0 is the desired curve.

It remains to consider the case, when, for any hyperplane section E of the surface V, a divisor $G \in |E + n_E K|$ is null. We will show that this assumption leads to a contradiction. In fact, this means that any hyperplane section $E \sim n_E K$, and since each divisor D of the surface V can be represented in the form of a difference $D = E_1 - E_2$, where E_1 and E_2 are hyperplane sections of V corresponding to two (distinct) imbeddings of V in a projective space [24], the group of the classes of divisors of V is isomorphic to the group of integers Z, and a divisor (curve) D is a generator of this group. Since, moreover, $h^{2,0} = h^{0,2} = p = 0$, this group is isomorphic to the group $H^2(V, Z)$ and the curve D, as a generator of this group, has an index of selfintersection of ± 1. On the other hand, by the formula of Noether-Enriques

$$\frac{(K^2)+\chi}{12} = 1 + p_g - q, \text{ i.e. } (K^2) = 9,$$

since $p_g = q = 0$ and $\chi = 2 - 4q + b_2 = 3$. This contradiction completes the proof of the theorem. It is clear that all of the proof given by Kodaira [48] will remain valid when the base field is any algebraically closed field of characteristic 0, with the exception of the last argument. There exists a proof by Zariski of the theorem of Castelnuovo for the case of an algebraically closed base field of characteristic $p > 0$ [18–20].

The converse theorem, that if a surface V is rational, then $p_a(V) = 1$ and $P_2(V) = 0$, is obvious. In fact, on the projective space P^2, a canonical divisor $K = -3E$, where E is a line on P^2, and therefore $P_n(P^2) = 0$, $n = 1, 2, \cdots$. In exactly the same way $q(P^2) = 0$, and hence $p_a(P^2) = p(P^2) - q(P^2) + 1 = 1$. From this it follows that, since the numbers P_2 and p_a are birational invariants, $p_a(V) = 1$, $P_2(V) = 0$ for any rational surface V. Thus, the theorem of Castelnuovo gives criteria for the rationality of an algebraic surface.

The theorem of Castelnuovo provides a solution of the problem of Lüroth for poles of algebraic functions of second degree of transcendency over a base field. The problem is the following: let K be a field isomorphic to the field of rational functions of two independent variables, and let K' be a subfield of the field K of finite index:

$$K' < K, \quad [K: K'] < \infty.$$

Will the field K' be isomorphic to a field of rational functions of two independent variables?

The theorem of Castelnuovo permits a positive answer to this question. Thus, the field K is a field of algebraic functions on a rational surface V; the field K', a field of algebraic functions on some surface V'; to the inclusion $K' \subset K$ corresponds a rational mapping $T\colon V \longrightarrow V'$. Since $P_2(V) \geq P_2(V')$, $q(V) \geq q(V')$ (cf. the Introduction) and V is a rational surface, $P_2(V') = q(V') = 0$, from which it follows, by the theorem of Castelnuovo, that the surface V' is rational, and, consequently, the field K' is isomorphic to a field of rational functions of two variables.

A variety V' such that there exists a rational mapping T of a rational variety V onto V' is said to be unirational. The result obtained may be formulated in the following way: every unirational surface is rational. It is unknown whether the analogous statement for varieties of dimension greater than two is true. The examples constructed in [63] of unirational varieties do not decide the question, for the proofs that these varieties are nonrational are unconvincing (cf. [64]).

We note also that there are examples of surfaces V with $p_a(V) = 1$, $p_g(V)=0$, $P_2(V) \neq 0$ (and thus not rational).

These surfaces were constructed by Enriques (cf. Chapter X). On the other hand, Severi raised the question of whether there were any rational surfaces characterized by the conditions $p_g(V) = 0$, $H_1(V, \mathbf{Z}) = 0$ (we recall that the condition $H_1(V, \mathbf{Z}) = 0$ divides into two conditions: $q = 0$ which, when $p = 0$, causes $p_a = 1$, and $\operatorname{Tor} H_1(V, \mathbf{Z}) = 0$). The answer to this question is unknown.

We note that the problem of Lüroth for fields of second degree of transcendency has been decided negatively when the base field k is not algebraically closed. Corresponding examples hold for the case when k is the real field [62] or a finite field [61].

CHAPTER IV

RULED SURFACES

Ruled surfaces will be studied in this chapter, and it will be proved, in particular, that they are characterized by the condition $P_{12} = 0$. If they are nonrational, then $q > 0$. It will be shown that if $q > 1$, then the condition $p_g = 0$ is sufficient for the surface to be ruled. For $q = 1$ this condition is insufficient. All the surfaces with $p_g = 0$, $q = 1$ will be found, and it will be verified that those among them for which $P_{12} = 0$ are ruled. Thus, we give in this chapter a classification of all the surfaces with $p_g = 0$ and $q > 0$, not just a classification of the ruled nonrational surfaces. We will assume that the base field k is the field of complex numbers. Nevertheless, almost all the arguments remain valid when k is an algebraically closed field of characteristic 0. We will make special note of the places where the assumption $k = \mathbf{C}$ is essential.

§1. Elementary properties

Definition. *The surface V is said to be ruled if it is birationally equivalent to the direct product of an algebraic curve with a projective line.*

Theorem 1. *If the surface V is ruled, i.e., if V is birationally equivalent to $B \times P^1$, where B is an algebraic curve, then $P_n(V) = 0$, $n \geq 1$ and the irregularity q of the surface V coincides with the (geometric) genus of the curve B.*

Proof. In view of the birational invariance of the numbers P_n and q we can assume that

$$V = B \times P^1, \tag{1}$$

where B is a nonsingular curve. In general, let $X = B \times C$ where B and C are nonsingular curves and let α and β be differential forms of degree 1 on B and C respectively, and let (α) and (β) be their divisors. If π_1 and π_2 are projections of V onto B and C, then the form

$$\omega = \pi_1^*(\alpha) \wedge \pi_2^*(\beta)$$

is a two-dimensional differential form on V. It is clear that

$$(\pi_1^*(\alpha)) = \pi_1^{-1}((\alpha)) = (\alpha) \times C,$$
$$(\pi_2^*(\beta)) = \pi_2^{-1}((\beta)) = B \times (\beta),$$

$$K(B\times C) = K(B)\times C + B\times K(C), \tag{2}$$

$$(\omega) = (\alpha)\times C + B\times (\beta). \tag{3}$$

From this it follows that, if $V = B \times P^1$, then

$$K \cdot (b \times P^1) = -2((B \times a)\cdot(b \times P^1)) = -2 < 0, \quad b \in B, \ a \in P^1$$

and consequently, $(nK \cdot (b \times P^1)) < 0$ for any $n > 0$. If we had $P_n > 0$ for at least one $n > 0$, there would exist a divisor D, $D > 0$, $D \sim nK$. Then necessarily $(D \cdot (b \times P^1)) < 0$. But this is impossible: if $D = \Sigma n_i c_i + \Sigma m_j (b_j \times P^1)$, then $(D \cdot (b \times P^1)) = \Sigma n_i (C_i \cdot (b \times P^1)) \geq 0$.

For the proof of the assertion about irregularity, we use the fact that, for any varieties B and C, the Albanese variety possesses the property

$$A(B \times C) = A(B) \times A(C).$$

In particular, from (1) we obtain

$$A(V) = A(B) \times A(P^1) = A(B),$$

and since

$$q = \dim A(V), \quad g = \dim A(B),$$

where g is the genus of the curve B, this proves our assertion.

The basic problem of this chapter consists in proving the converse assertion in its stricter form: a surface is ruled if $P_{12} = 0$. Here we shall begin from the theorem of Noether (Chapter I, §3). We reformulate that theorem geometrically: a rational mapping

$$\pi: V \longrightarrow B$$

corresponds to the imbedding $K_1 = k(B) \longrightarrow K = k(V)$, where if ξ is a generic point of the curve B, then the field K/K_1 is a field of functions on the curve $\pi^{-1}(\xi)$. Thus the theorem of Noether can be given the following formulation:

Theorem 2. *If there exists a rational mapping*

$$\pi: V \longrightarrow B$$

of a surface V onto a curve B, such that the inverse image $\pi^{-1}(\xi)$ of a generic point ξ of the curve B is an irreducible curve of genus 0, then V is a ruled surface.

We shall use this theorem for the particular case in which π is a regular mapping. Then it determines a fibering of V into nonintersecting fibers $F_b = \pi^{-1}(b)$, $b \in B$. We shall frequently call π a fibering and F_b its fibers. The manner of the construction of such a fibering is based on the consideration of the Albanese mapping.

§2. The Albanese mapping for $p_g = 0,\ q > 0$

Theorem 3. *If* $\alpha_V : V \longrightarrow A(V)$ *is the Albanese mapping of a surface* V *for which* $p_g = 0$, $q > 0$, *then*: 1) $\alpha_V(V)$ *is an algebraic curve*; 2) $\alpha_V(V)$ *is nonsingular*; *and* 3) *the genus of* $\alpha_V(V)$ *is equal to* q.

Proof of 1). Since the variety $\alpha_V(V)$ generates all of $A(V)$, and $\dim A(V) = q > 0$, it is also true that $\dim \alpha_V(V) > 0$, i.e., $\dim \alpha_V(V) = 1$ or 2.

We shall show that $\dim \alpha_V(V) \neq 2$. Let us assume that $\dim \alpha_V(V) = 2$, and let a be a nonsingular point of the surface $\alpha_V(V)$. Since the mapping α_V is defined up to a translation of the variety $A(V)$, we can assume that $a = 0$. Let σ be a bivector, in the tangent space to $A(V)$ at the point 0, corresponding to the plane tangent to $\alpha_V(V)$ at this point; and let s be a two-dimensional element of the Grassmann algebra such that $(s, \sigma) \neq 0$. We denote by ω the invariant differential form corresponding to s. It is clear that ω is a differential form of the first kind on $A(V)$. From the fact that $(s, \sigma) \neq 0$, it follows that ω is not identically equal to zero at the point 0. From this it follows that the differential form of the first kind $\alpha_V^*(\omega)$ on V does not vanish identically, and this contradicts the assumption $p_g = 0$.

Proof of 2). Let $\alpha_V(V) = B$. Let B_N be a normalization of B and let $N: B_N \longrightarrow B$ be the canonical mapping. Since N is a birational equivalence, there exists a rational mapping $\nu : B \longrightarrow B_N$, $\nu = N^{-1}$. Let $\phi = \nu \cdot \alpha_V$, $\phi : V \longrightarrow B_N$. We have a commutative diagram

$$\begin{array}{ccc} A(V) & \xrightarrow{\alpha(\varphi)} & A(B_N) \\ \psi_2 \uparrow & & \uparrow \psi_1 \\ B & \xleftarrow{N} & B_N . \end{array} \tag{4}$$

Here ψ_2 is the imbedding of B in $A(V)$. Since the (geometric) genus of the curve B, and thus also that of B_N, is distinct from 0 (cf. [31], Chapter II), the canonical mapping ψ_1 of the curve B_N into $A(B_N)$ is also an imbedding, which we can assume to be the identity mapping. The mapping $\alpha(\phi)$ is regular, and consequently the mapping $N' : B \longrightarrow B_N$, $N' = (\alpha(\phi)|_B) \cdot \psi_2$ is also regular. From the diagram (4) it follows that $NN' = 1$, $N'N = 1$, i.e., N is a biregular equivalence of B and B_N, so that $B = B_N$ does not have singular points.

Proof of 3). Since $B = B_N$, (4) reduces to the diagram

$$\begin{array}{ccc} A(V) & \xrightarrow{\alpha(\varphi)} & A(B) \\ & {}_{\psi_2}\nwarrow \quad \nearrow_{\psi_1} & \\ & B & \end{array}$$

According to the universal mapping property of an Albanese variety there exists a mapping $\eta: A(B) \longrightarrow A(V)$, giving the commutative diagram

$$\begin{array}{ccc} A(V) & \overset{\eta}{\leftarrow} & A(B) \\ {\scriptstyle\psi_2}\nwarrow & & \nearrow{\scriptstyle\psi_1} \\ & B & \end{array}$$

From this it follows that $\psi_2 = \eta\psi_1$ and $\psi_1 = \alpha(\phi)\psi_2$, i.e.,

$$\psi_1 = \alpha(\phi)\eta\psi_1, \quad \psi_2 = \eta\alpha(\phi)\psi_2.$$

Since $\operatorname{Im}\psi_1$ and $\operatorname{Im}\psi_2$ generate $A(B)$ and $A(V)$, it follows that $\alpha(\phi)\eta = 1$, $\eta\alpha(\phi) = 1$, i.e., $A(V)$ is isomorphic to $A(B)$. This proves 3) and Theorem 3.

We denote α_V by π and obtain a regular mapping (or fibering)

$$\pi: V \longrightarrow B$$

onto a nonsingular curve B of genus q.

Theorem 4. *A generic fiber of the fibering π is irreducible.*

Proof. We saw that the assertion of Theorem 4 is equivalent to the fact that the field $k(B)$ is algebraically closed in the field $k(V)$. If this is not so, let K' be the algebraic closure of $k(B)$ in $k(V)$ and let B' be a nonsingular model of the field K'. The inclusion $k(B') = K' \subset k(V)$ determines a rational mapping $V \longrightarrow B'$ and thus an epimorphism $A(V) \longrightarrow A(B')$.

From this it follows that the genus g of the curve B' is not greater than q. But B' is a covering of B (since $k(B) \subset k(B')$) and consequently $g \geq q$. We see that $g = q$. According to the formula of Hurwitz for the genus of a covering, the equality $g = q$ is possible only when $g = q = 1$. Thus, $A(V) = B$, $A(B') = B'$, and we have the mappings

$$A(V) \overset{\varphi}{\rightarrow} B', \quad B' \overset{\psi}{\rightarrow} B, \ \psi\varphi = \alpha_V.$$

According to the universality property of the Albanese variety, there exists a mapping $\chi: B \longrightarrow B'$, giving a commutative diagram

$$\begin{array}{ccc} A(V) & \overset{\alpha_V}{\rightarrow} & B \\ {\scriptstyle\varphi}\searrow & & \swarrow{\scriptstyle\chi} \\ & B' & \end{array}$$

From this, as in the proof of Theorem 3, it is easy to obtain that χ and ψ are isomorphisms, i.e. that $B' = B$ and $K' = k(B)$. The theorem is proved.

§3. The case $q > 1$

In this section we will always assume that for the surface V

$$gp_g = 0, \quad q > 1. \tag{5}$$

Lemma 1. *For a surface satisfying the conditions* (5),

$$(K^2) < 0.$$

Proof. According to formula (4) of the Introduction

$$\frac{(K^2)+\chi}{12} = 1 - q + p = 1 - q,$$

so that

$$(K^2) = -\chi + 12(1-q).$$

Since $\chi = 2 - 4q + b_2$, we have

$$(K^2) = -2 + 4q - b_2 + 12(1-q),$$
$$(K^2) = 8(1-q) + 2 - b_2.$$

Since $q \geq 2$, $b_2 \geq 1$, it follows that $(K^2) \leq -7$.

Corollary. *If E is a hyperplane section, then $l(E+mK) = 0$ for sufficiently large m. If $l(E+nK) > 0$ and $l(E+mK) = 0$, for $m > n$, then*

$$E + nK \sim D > 0, \quad D \neq 0 \quad \text{and} \quad p_a(C) \leq q \tag{6}$$

for any cycle C for which $0 < C < D$.

All the assertions except $D \neq 0$ follow from Lemma 3 (Chapter III, § 1). The assertion $D \neq 0$ follows from the fact that otherwise we would have $E \sim -nK$ while $(E^2) > 0$, $(K^2) < 0$.

Lemma 2. *If C is an irreducible curve on V, π is the fibering introduced in the preceding section, and F_b is one of its fibers, then either $C = F_b$ or*

$$p_a(C) \geq (C \cdot F)(q-1) + 1. \tag{7}$$

Proof. If $C \neq F_b$, then the mapping $f = \pi/C$ defines C as a covering of B. The degree of this covering is equal to $C \cdot F$. In fact by definition this degree is equal to the degree of the divisor $f^{-1}(b)$. Considering π on V locally as a function and using the equation

$$\deg(\pi/C)_0 = ((\pi_0) \cdot C)$$

($(\pi)_0$ and $(\pi/C)_0$ are zero divisors of the functions π and π/C on V and C respectively), we obtain that $\deg f^{-1}(b) = (F_b \cdot C)$. Applying now the formula of Hurwitz for the genus of a covering, we obtain the inequality $\gamma \geq (F \cdot C)(q-1) + 1$ for the geometric genus γ of the curve C, and since $p_a(C) \geq \gamma$, (7) follows from this.

Lemma 3. *If the genus g of a generic fiber F of a fibering π is not 0, then $(F \cdot K) \geq 0$.*

By assumption

$$\frac{(F \cdot (F+K))}{2} + 1 \geqslant 1,$$

and since $(F^2) = 0$, we have $(F \cdot K) \geq 0$.

Corollary. *Under the conditions given for the divisor D in (6), $(D \cdot F) \geq 3$.*

Since $D \sim E + nK$, $(D \cdot F) = (E \cdot F) + n(K \cdot F) \geq (E \cdot F)$. It is clear that $(E \cdot F)$ is the degree of the curve F. If this curve had degree 1 or 2, it would be rational, which would contradict the assumption that $g > 0$.

Theorem 5. *An algebraic surface V with the invariants $p_g = 0$ and $q > 1$ is ruled.*

In the proof we may assume that V is a relatively minimal model and, consequently, does not contain exceptional curves of the first kind.

We consider the mapping $\pi: V \longrightarrow B$ constructed in the preceding section. It satisfies all the assumptions of Theorem 2 except that we still do not know that the genus g of a generic fiber is equal to 0. If we can prove this, Theorem 5 will follow from Theorem 2.

We assume that $g \geq 1$. Let the divisor D whose existence is proven in the corollary of Lemma 1 have the form $D = \Sigma_1^m n_i C_i$, where the C_i are distinct irreducible curves. According to the corollary of Lemma 3, $(D \cdot F) \geq 3$.

We consider separately three cases, which together include all the possibilities:

1) for some C_i, for instance for $i = 1$, $(C_i \cdot F) \geq 2$,

2) for all C_i, $(C_i \cdot F) \leq 1$ but $m \geq 2$,

3) $D = nC$, $(C \cdot F) = 1$, $n \geq 3$.

In case 1) we can calculate $p_a(C_1)$ in different ways on the basis of Lemma 2 and formula (6). We obtain the contradiction

$$p_a(C_1) \leqslant q, \quad p_a(C_1) \geqslant 2(q-1) + 1 = 2q - 1.$$

In case 2), for at least one of the C_i, for instance for $i = 1$, $(C_i \cdot F) = 1$, since $(D \cdot F) > 0$. By assumption there exists another curve C_2. We apply formula (6) to $C = C_1 + C_2$:

$$p_a(C_1 + C_2) \leq q.$$

Since $p_a(C) = (C \cdot (C + K))/2 + 1$,

$$p_a(C_1 + C_2) = p_a(C_1) + p_a(C_2) + (C_1 \cdot C_2) - 1.$$

According to Lemma 2,

$$p_a(C_1) \geq q,$$

so that

$$p_a(C) \geq q + p_a(C_2) + (C_1 \cdot C_2) - 1.$$

If also $(C_2 \cdot F) = 1$, then $p_a(C_2) \geq q$, and since $(C_1 \cdot C_2) \geq 0$, we obtain a contradiction with (6):

$$p_a(C) \geq 2q - 1.$$

If $(C_2 \cdot F) = 0$, then $C_2 = F_b$, $(b \in B)$, $p_a(C_2) \geq 1$ and $(C_1 \cdot C_2) = (C_1 \cdot F) = 1$, so that we again obtain a contradiction with (6):

$$p_a(C) \geq q + 1.$$

In case 3) we argue in exactly the same way applying (6) and Lemma 2 to the curve $2C$. We get that

$$p_a(2C) \leqslant q,\ p_a(2C) \geqslant 2q + (C^2) - 1.$$

We obtain a contradiction if we show that $(C^2) \geq 0$. But $(C^2) < 0$ gives $(C \cdot D) < 0$, i.e. $(C \cdot (E + nK)) < 0$, and thus $(C \cdot K) < 0$, since $(C \cdot E) > 0$. Therefore $(C^2) + (C \cdot K) = 2p_a(C) - 2 < 0$, which is possible only if $(C^2) = -1$, $p_a(C) = 0$, i.e. if C is an exceptional curve of the first kind. Since we assumed that there were no such curves on V, this proves the theorem.

§4. Regular mappings of algebraic surfaces onto curves

We consider an arbitrary regular mapping $\pi: V \longrightarrow B$ of an algebraic surface V onto a nonsingular algebraic curve B with an irreducible generic fiber F. Let q and g be the genera of B and F. We will assume the following properties of such fiber spaces to be known (cf. for example, [25]).

The fiber $F_b = \pi^{-1}(b)$ is connected for all $b \in B$. For all points $b \in B$, except, perhaps, a finite number, F_b is an irreducible nonsingular algebraic curve with genus g. The set of points $\{b_1, \cdots, b_s\}$ for which this is not true will be denoted by S, and the corresponding fibers F_{b_i} will be said to be degenerate or singular.

The fiber space, $\pi: V - \pi^{-1}(S) \longrightarrow B - S$ is (if $k = \mathbf{C}$, the field of complex numbers) differentiably locally trivial.

We will denote by $\chi(L)$ the Euler characteristic of a topological space L. In particular, if

$$L = F_b = \sum_1^m n_i C_i,\ n_i > 0, \text{ then } \chi(F_b) = \chi(F'),\ F' = \sum_1^m C_i.$$

Theorem 6. *When* $k = \mathbf{C}$ *we have*

$$\chi(V) = \chi(F)\,\chi(B) + \sum_1^s (\chi(F_{b_i}) - \chi(F)). \qquad (8)$$

χ tol

Proof. (Proposed by A. B. Žižčenko.) We let $\widetilde{V} = V - \pi^{-1}(S)$. From the exact cohomology sequence determined by the space, the closed subspace, and its complement, it follows that

$$\chi(V) = \chi(\widetilde{V}) + \chi(\pi^{-1}(S)). \qquad (9)$$

It is clear that

$$\chi(\pi^{-1}(S)) = \sum_1^s \chi(F_{b_i}). \tag{10}$$

Since $\widetilde{V} \to B - S$ is locally trivial (as a differentiable fiber space), it follows from the spectral sequence of Leray for this fiber space that

$$\chi(\widetilde{V}) = \chi(F)\,\chi(B - S). \tag{11}$$

Finally,

$$\chi(B) = \chi(B - S) + s \tag{12}$$

(s is the number of points in S), as, for example, follows from the exact sequence analogous to the one considered in the derivation of (9). Comparing (9), (10), (11) and (12), we obtain (8).

Lemma 4. *If C is a connected curve (perhaps reducible) on a surface V, then*

$$\chi(C) \geqslant -(C\cdot(C + K)), \tag{13}$$

where equality holds only when C is an irreducible nonsingular curve.

Proof. Let $C = \Sigma C_i$, let $\overline{C}$ be a normalization of C, i.e. the unconnected sum of normalizations $\overline{C}_i$ of the curves C_i, and let

$$\phi: \overline{C} \to C$$

be the canonical regular mapping. At all points except a finite number, ϕ is a biregular equivalence. Therefore the usual manner of calculation of the Euler characteristic gives

$$\chi(C) = \chi(\overline{C}) - \delta,$$

where

$$\delta = \sum_{c\in C} (\deg(\varphi^{-1}(c)) - 1). \tag{14}$$

If the genus of the curve $\overline{C}_i$ is equal to g_i, then

$$\chi(\overline{C}) = \sum \chi(\overline{C}_i) = 2\sum(1 - g_i)$$
$$g_i = p_a(C_i) - \delta_i$$

(in view of formula (5) of the Introduction).

Thus,

$$\chi(C) = 2\sum(1 - p_a(C_i)) + 2\sum\delta_i - \delta$$
$$= -\sum(C_i\cdot(C_i + K)) + 2\sum\delta_i - \delta.$$

Let n_i points of the curve C_i be taken into the point $c \in C$ under the mapping ϕ. Then the corresponding term in (14) is equal to

$$\left(\sum_1^m n_i\right) - 1 = \sum_1^l (n_i - 1) + l - 1,$$

if $n_i > 1$ for $i = 1, \cdots, l$ and $n_i = 1$ for $i = l + 1, \cdots, m$. Here the term corresponding to the point c in the expression for δ_i is not smaller than $n_i - 1$, and the number $l - 1$ is not smaller than the multiplicity of the point c in the divisor $\Sigma_{i<j}(C_i \cdot C_j)$. From this it follows that

$$\delta \leqslant \sum_i \delta_i + \sum_{i<j} (C_i \cdot C_j). \tag{15}$$

Thus

$$2\delta_i - \delta \geqslant -2 \sum_{i<j} (C_i \cdot C_j), \tag{16}$$

so that

$$\chi(C) \geqslant -\sum (C_i (C_i + K)) - 2 \sum_{i<j} (C_i \cdot C_j)$$
$$= -((\sum C_i) \cdot (\sum C_i + K)) = -(C \cdot (C + K)).$$

In view of (16), equality holds in (13) only when all the $\delta_i = 0$ and $(C_i \cdot C_j) = 0$ for $i \neq j$. The first means that the curves are nonsingular, and the second that they are mutually nonintersecting. Since we are assuming a connected curve, this means that it consists of one component. The lemma is proved.

Lemma 5. *If a fiber* F_b *of the mapping* π *has the form* $\Sigma n_i C_i$, *then for* $C = \Sigma m_i C_i$, $m_i \geq 0$, we have $(C^2) \leq 0$.

We use the equality

$$(C \cdot F) = 0, \tag{17}$$

which is clear if the fiber F is taken different from F_b.

If it were true that $(C^2) > 0$, then for a hyperplane section E and a sufficiently large n

$$l(nC - E) > 0,$$

as follows directly from the Riemann-Roch inequality.

Let $nC - E \sim D > 0$, i.e.

$$nC \sim E + D.$$

Since $(D \cdot F) \geq 0$ and $(E \cdot F) > 0$, this implies that $(C \cdot F) > 0$, in contradiction with (17).

Theorem 7 (semicontinuity of the Euler characteristic). *If* F *is nonsingular and* F_0 *is a singular fiber of a mapping* π *and the surface* V *is a relatively minimal model, then*

$$\chi(F_0) \geq \chi(F), \tag{18}$$

where equality holds only when the genus of F *is equal to* 1 *and* F_0 *is a nonsingular curve of genus* 1 *taken with some multiplicity.*

Proof. It is clear that

$$\chi(F) = 2 - 2g = -(F\cdot(F+K)) = -(F\cdot K),$$

since $(F^2) = 0$.

Let $F_0 = \Sigma n_i C_i$, $n_i \geq 1$; we set $F' = \Sigma C_i$.

It follows from Lemma 4 that

$$\chi(F_0) - \chi(F) = \chi(F') - \chi(F) \geqslant -(F'^2) + ((F - F')\cdot K).$$

According to Lemma 5, $(F'^2) \leq 0$. It remains for us to show that $(F - F')\cdot K \geq 0$.

If for some C_i it were true that $(C_i \cdot K) < 0$, we would have

$$(C_i \cdot (C_i + K)) < (C_i^2),$$

and hence

$$2p_a(C_i) - 2 < (C_i^2).$$

Since $(C_i^2) \leq 0$ by Lemma 5, $p_a(C_i) - 1 < 0$, i.e. $p_a(C_i) = 0$, and $-2 < (C_i^2) \leq 0$. For C_i^2 there are consequently two values: -1 and 0. The first case would mean that C_i was an exceptional curve of the first kind, which would contradict the fact that V is a relatively minimal model. Let $(C_i^2) = 0$. From the condition $(C_i \cdot F_0) = 0$ we obtain

$$n_i(C_i^2) = -\sum_{j \neq i} n_j (C_i \cdot C_j) = 0.$$

Since $(C_i \cdot C_j) \geq 0$ for $i \neq j$, it follows from this that $(C_i \cdot C_j) = 0$ for all $i \neq j$, and this contradicts the connectedness of the fiber F_b if there exists a curve $C_j \neq C_i$. Thus, $F_0 = nC_i$. The inequality $p_a(F) \geq 0$ and the formula for p_a give us

$$0 \leqslant p_a(F) = p_a(nC_i) = np_a(C_i) + (n^2 - n)(C_i^2) + 1 - n = 1 - n,$$

i.e. $0 \leq 1 - n$, which if possible only for $n = 1$. Thus, all the $n_i = 1$. But in this case $F_0 = F'$, and $((F - F')\cdot K) = ((F_0 - F')\cdot K) = 0$. The inequality (18) is proved.

We now explain when equality may hold. For it to hold, all the inequalities met along the way must be equalities. In particular, this refers to the inequality $\chi(F') \geq -(F' \cdot (F' + K))$, which, according to Lemma 4, is an equality only if $F_0 = nC$, where C is an irreducible nonsingular curve and $n \geq 2$. The inequality

$$-(F')^2 + ((F - F')\cdot K) \geqslant 0,$$

which we proved must also be an equality. Since then every member of the left side is nonnegative, then $((F - F')\cdot K) = (n-1)(C \cdot K) = 0$, and hence $(C \cdot K) = 0$ and $(F \cdot K) = 0$. From the fact that $(F^2) = 0$ it follows that, since $F_0 = nC$, $(C^2) = 0$. Hence

$$p_a(F) = \frac{(F^2) + (F \cdot K)}{2} + 1 = 1,$$

$$p_a(C) = \frac{(C^2) + (C \cdot K)}{2} + 1 = 1,$$

which is what was asserted.

§5. The case $q = 1$

Lemma 6. *For a surface V with the invariants $p_g = 0$ and $q = 1$, we have $(K^2) \leq 0$.*

In this case, $p_a(V) = 1 - q + p_g = 0$.

From formula (4) of the Introduction it follows that $(K^2) + \chi = 1 + p_a(V) = 0$, i.e.

$$(K^2) = -\chi.$$

We consider the projection $\pi: V \longrightarrow B$ determined by the Albanese mapping. Since $\chi(B) = 0$, from Theorems 6 and 7 we obtain $\chi \geq 0$, which means that $(K^2) \leq 0$.

Remark. Another proof can be given by starting with formula (4) of the Introduction, which in our case gives

$$(K^2) = 2 - b_2.$$

It is sufficient for us to show that $b_2 \geq 2$, and for this to find two independent homology classes on V. The classes determined by the cycles E (a hyperplane section) and F (one of the fibers of the projection π), for example, will be such classes. They are independent, since $(E^2) > 0$, $(F^2) = 0$.

We will now examine separately the cases $(K^2) < 0$ and $(K^2) = 0$.

§6. The case $(K^2) < 0$

Theorem 8. *A surface with the invariants $p_g = 0$, $q = 1$, and $(K^2) < 0$ is ruled.*

Proof. We consider the same projection $\pi: V \longrightarrow B$, which now coincides with the Albanese mapping. If the genus g of a generic fiber F is equal to 0, then the surface V is ruled. If $g = 1$, then X possesses a mapping onto elliptical curves. In Chapter VII, Theorem 3 it will be proved that for such a surface $(K^2) = 0$, which contradicts the assumption $(K^2) < 0$.

There remains to be considered the case $g > 1$.

The plan of the proof of Theorem 8 is the following. For some unramified covering C of a curve B with a projection $\phi: C \longrightarrow B$ we consider the inverse image $V' = V \times_B C$ of the mapping π on C, i.e. the subvariety $C \times V$ consisting of the points (c, v) for which $\phi(c) = \pi(v)$. The projection $C \times V \longrightarrow C$ determines on V a projection $\pi': V' \longrightarrow C$ and a fiber space whose fibers are isomorphic to the fibers of π. The surface V' is itself an unramified covering of V. We will show, assuming that $g > 1$, that for a properly chosen covering C the surface V' will be the direct product $C \times F$. From this it follows, by formula (2), that on V', $K' \sim C \times K(F)$, where K' and $K(F)$ are canonical classes of V' and F. Therefore $(K'^2) = 0$. But if $f\colon V' \longrightarrow V$ is an unramified covering of degree n, then, as it is easy to verify, $K' = f^*(K)$, and therefore

$$K' \cdot K' = f^*(K) f^*(K) = f^*(K \cdot K)$$

(we are considering $K' \cdot K'$ and $K \cdot K$ as cycles here, and not as numbers). Therefore $(K'^2) = n(K^2)$ and hence $(K^2) = 0$ in contradiction with the assumption of the theorem. This proves Theorem 8.

Applying the corollary of Lemma 1 (in which one may clearly replace E by $E + F_b - F_0$), we obtain that for any point $b \in B$ there exists an $n > 0$ and a divisor $D_b > 0$ such that

$$E + F_b - F_0 + nK \sim D_b, \quad D_b \neq 0, \quad p_a(C) \leqslant 1 \tag{19}$$

for any cycle C for which $0 < C < D_b$. From this it follows that the cycle D_b cannot contain as a component a fiber F, since for it $p_a(F_b) = g > 1$. Therefore any irreducible component C of the cycle D_b is mapped by the projection π onto B, and, consequently, is not a rational curve. Hence, for it $p_a(C) = 1$, while it has no singular points and has a genus of 1.

Let β be a generic point of the curve B. We denote by C_β some irreducible (over $k(\beta)$) component of the cycle D_β. For any $b \in B$ we will designate by C_b a specialization of the curve C_β. More exactly, we will designate by ξ_β a generic point of the curve C_β and will consider in $B \times V$ an irreducible subvariety Γ with a generic point (β, ξ_β). We set

$$C_b = pr_V((b \times V) \cdot \Gamma).$$

The cycle C_b consists of components of dimension 1. Otherwise its carrier would coincide with V, i.e. Γ would contain the component $b \times V$, which would contradict its irreducibility. Let $C_\beta = \Sigma C_\beta^{(i)}$ be a decomposition into absolutely irreducible components. Similarly, we define their specializations $C_b^{(i)}$. We choose for C_β a component of the cycle D_β not defined over k (in other words, the curve C_b does not remain constant under a change in $b \in B$). In order to prove that such a component exists, it is sufficient to show that the field of definition of the cycle D_β is transcendental over k. For this we denote by β' another generic point of the curve B that is independent from β, and we show that $D_\beta \neq D_{\beta'}$. The equality $D_\beta = D_{\beta'}$ would imply the relation $F_\beta \sim F_{\beta'}$. Let $(f) = F_\beta - F_{\beta'}$. The function f is the image of some function g on B: $f\colon \pi^*(g)$. In fact, for any constant c we have $((f-c)_0 \cdot F) = 0$. From this it follows that $(f-c)_0$ consists of fibers of the mapping π, i.e. f is constant on the fibers. This means that $f = \pi^*(g)$. But then obviously $(g) = \beta - \beta'$ and this is impossible, since the genus of the curve B is equal to 1. The field determined by the curve $C_\beta^{(i)}$ is also transcendental over k. We denote by U a nonsingular model of this field. Correspondingly, we will denote the curves $C_b^{(i)}$ by $\mathfrak{L}_u$, $u \in U$. They form a one-dimensional family of elliptic curves on V.

Let $\mathfrak{L}$ be the curve $\mathfrak{L}_u$ for some fixed $u \in U$, where $\mathfrak{L}$ is irreducible, does not have singular points, and $(\mathfrak{L} \cdot F) = n$ ($n > 0$, since $\mathfrak{L} \neq F$). The mapping π determines on the curve $\mathfrak{L}$ the structure of a covering of the curve B. Since both curves are elliptic, this covering is unramified. As is known, there do not exist continuous systems of unramified coverings (cf. [32]. If $k = \mathbf{C}$, this follows from the finiteness of the number of coverings of a given degree). Therefore we have on a generic curve $\mathfrak{L}_\xi$ (ξ is a generic point of U) a covering isomorphic to the one defined on $\mathfrak{L}$. This means that there exists an isomorphism $\phi: \mathfrak{L} \to \mathfrak{L}_\xi$ such that for any $x \in \mathfrak{L}$

$$\phi(x) = \pi(\phi(x)).$$

We now consider the inverse image $\mathfrak{L} \times_B V$ of the mapping π on the covering $\mathfrak{L}$, i.e. the subvariety $V' \subset \mathfrak{L} \times V$ consisting of the pairs (y, v), $y \in \mathfrak{L}$, $v \in V$ for which $\pi(y) = \pi(v)$. The projection $\mathfrak{L} \times V \to \mathfrak{L}$ defines on V' a fiber space $\pi': V' \to \mathfrak{L}$ with base $\mathfrak{L}$, whose fibers are isomorphic to the fibers of π. We denote by $\mathfrak{L}'_\xi$ the curve on V' consisting of the points (y, v), $v \in \mathfrak{L}_\xi$, $x = \phi(y)$, $y \in \mathfrak{L}$, $v \in V$.

We note that it is possible that the mapping ϕ is defined over the larger field $k(\xi)$. The same is true of the curve $\mathfrak{L}'_\xi$. We denote by W a nonsingular curve such that the field of functions over it coincides with the field of definition of the curve $\mathfrak{L}'_\xi$. We denote by $\mathfrak{L}'_\eta$ the curve corresponding to a generic point η of the curve W, and denote its specialization for any point $w \in W$ by $\mathfrak{L}'_w$. It is easy to verify that $(\mathfrak{L}'_w \cdot F) = 1$, where F is any fiber of the mapping π'.

For any nonsingular fiber F_y, $y \in \mathfrak{L}$ of the projection π' the mapping

$$\psi(w) = \mathfrak{L}'_w \cdot F_y$$

determines a curve W as a covering of the curve F_y, or, in other words, defines $k(F_y)$ as a subfield of the field $k(W)$. As is known, there do not exist unconnected systems of subfields of order $g > 1$ (cf., for example, [15]). It follows from this that the relation of equivalence defined on W by the condition $w \sim w'$ if $\mathfrak{L}'_w \cdot F_y = \mathfrak{L}'_{w'} \cdot F_y$ does not depend on the choice of the fiber F_y. In other words, if the curves $\mathfrak{L}'_w$ and $\mathfrak{L}'_{w'}$ intersect in one point (lying on the fiber F_{y_0}), then they must have an infinite number of points of intersection (lying on all the fibers F_y), i.e. they must coincide. Since W parameterizes the family of curves $\mathfrak{L}'_w$, we obtain that $\mathfrak{L}'_w$ and $\mathfrak{L}'_{w'}$ do not intersect if $w' \neq w$.

Therefore the mapping $W \times \mathfrak{L} \to V'$ defined by the formula

$$(w, y) \to \mathfrak{L}'_w \cdot F_y, \quad w \in W, \; y \in \mathfrak{L},$$

is an isomorphism between $W \times \mathfrak{L}$ and V'. It is obvious that here F is mapped isomorphically onto W. Thus V' is isomorphic to $F \times \mathfrak{L}$. As we said in the beginning of the proof, the assertion of Theorem 8 follows from this.

§7. The case $(K^2) = 0$

The surfaces with the invariants $p_g = 0$, $q = 1$, $(K^2) = 0$ are the only surfaces (among those for which $p_g = 0$, $q > 0$) which can be not ruled. Their complete classification will be given below. The basic tool in investigating them will be the Albanese mapping $\pi : V \to B$. If the genus g of a generic fiber of the fibering π is equal to 0, then the surface is ruled. We will later consider the case $g > 0$.

We note that for surfaces of the type under consideration, $p_a(X) = 1 - q + p_g = 0$, and since $(K^2) = 0$, it is also true that $\chi(V) = 0$. The basic result which we obtain (Theorem 9) will be valid for any surfaces for which $p_a(V) = \chi(V) = 0$.

Theorem 9. *Let V be an algebraic surface such that $p_a(V) = \chi(V) = 0$, and let $\pi : V \to B$ be a regular mapping of V onto a nonsingular elliptic curve. If the genus g of a nondegenerate fiber F of the fibering π is greater than* 1, *then there exists an unramified covering $\overline{B} \to B$ such that the inverse image $\overline{V} = V \times_B \overline{B}$ of the mapping π on $\overline{B}$ is a direct product: $\overline{V} \simeq \overline{B} \times F$.*

Before proving Theorem 9 we give some useful propositions about mappings of surfaces onto curves.

Lemma 7. *If $\pi : V \to B$ is a regular mapping of the surface V onto an elliptic curve, a generic fiber of which is irreducible, and if $\chi(V) = 0$, then all the fibers of the mapping π are nonsingular or, if the genus of a generic fiber is equal to* 1, *all fibers are multiples of a nonsingular curve of genus* 1.

The lemma follows directly from Theorems 6 and 7, since in (8) $\chi(V) = \chi(B) = 0$ and thus $\chi(F_{b_i}) = \chi(F)$.

We now introduce several helpful concepts relating to fiber spaces $\pi : V \to B$ without degenerate fibers.

If f is a function on V belonging to the local ring $\mathfrak{O}_{F_b}$ of some fiber F_b, then its restriction to the fiber F_b yields a regular function on F_b. Thus the homomorphism

$$\rho_b : \mathfrak{O}_{F_b} \to k(F_b)$$

is defined.

In an analogous manner one can associate with any divisor D on the surface a divisor $D \cdot F_b$ on the curve F_b.

The intersection $D \cdot F_b$ is defined if the divisor D does not contain F_b as a component. We complete the definition for all divisors by assuming $F_b \cdot F_b = 0$ (not only as a number, but also as a divisor). Thus we obtain a homomorphism of the group of divisors on V onto the group of divisors on F_b. We designate it by ρ_b.

We finally define the concept of a differential on V over B. By this we will mean a differential in the field $k(V)/k(B)$. This field has a degree of transcendence 1, so that all its differentials have the form gDf, g, $f \in k(V)$, where by Df is meant the complete differential of the function f in the field $k(V)/k(B)$. The differentials on V over B form a module $D_{k(B)}(k(V))$ over $k(V)$. A divisor of the differential gDf is defined in the usual manner: for any divisor C not coinciding with a fiber of the mapping π, we choose a function $T \in k(V)$, such that $\nu_C(T) = 1$, we write gDf in the form $gDf = hDT$ and set $\nu_C(h) = m_C$,

$$(gDf) = \Sigma\, m_C\, C.$$

By definition, the fibers F_b do not enter into the divisor (gDf).

This definition may be given a more geometric character by considering the one-dimensional vector fiber space Θ on V, a fiber of which at the point v is a subspace of the tangent space at the point v which consists of the vectors tangent to the fiber $F_{\pi(v)}$ that passes through the point v. We denote by θ the fiber space dual to Θ. Then the differentials are the rational sections of this one-dimensional fiber space, and a divisor of a differential is a divisor of a rational section.

A differential on V over B is said to be regular at the fiber F_b if it can be written in the form gDf, g, $f \in \mathfrak{D}_{F_b}$. The set of all such differentials forms a module $D_{\mathfrak{D}_b}(\mathfrak{D}_{F_b})$ over the ring $\mathfrak{D}_{F_b}$. The homomorphism ρ_b extends to a homomorphism of the modules

$$D_{\mathfrak{D}_b}(\mathfrak{D}_{F_b}) \to D_k(k(F_b)),$$

which we will designate by ρ.

The homomorphisms introduced possess the following properties of commutativity:

$$\rho_b(f) = (\rho_b \cdot f), \quad f \in \mathfrak{D}_{F_b}, \tag{20}$$

$$\rho_b \cdot (gDf) = (\rho_b \cdot g d \rho_b f), \; g, f \in D_{\mathfrak{D}_b}(\mathfrak{D}_{F_b}). \tag{21}$$

The proof of these relationships is by direct verification.

Let β be a generic point of B. The mapping ρ_β is an epimorphism and has as kernel the group consisting of the linear combinations of fibers.

The epimorphic character of the mapping ρ_β follows from the facts that all the principal divisors of the field $k(F_\beta)/k(\beta)$ are contained in $\operatorname{Im} \rho_\beta$, that the greatest common divisor of $\rho_\beta(C)$ and $\rho_\beta(C')$ is ρ_β of the greatest common divisor of C and C', and that all the divisors can be obtained as greatest common divisors of principal divisors.

Since mutually prime effective divisors on V and those not containing fibers intersect with a generic fiber in mutually prime divisors, we obtain the assertion about the kernel of the mapping ρ_β.

From the conjunction of these assertions it follows that if $C \cdot F_\beta \sim DF_\beta$ on F_β, then $C \sim D + \Sigma m_i F_{b_i}$ on V, where the F_{b_i} are certain fibers.

We now turn to the formulation of the result that lies at the basis of the proof of Theorem 9.

Definition. *A divisor $D = \Sigma n_i C_i$ (where the C_i are mutually distinct irreducible curves) is said to be unramified if the distinct curves C_i do not intersect and for each of them the projection $\pi: C_i \to B$ defines C_i as an unramified covering of the base B.*

Lemma 8. *Let $\pi: V \to B$ be a regular mapping of a surface V onto a curve B that does not have degenerate fibers. If there exists a function $f \in k(V)$, a divisor of which is the sum of a nonzero unramified divisor and some linear combination of fibers, then for some unramified covering of the base $\overline{B} \to B$ of the fiber bundle, $\overline{V} = V \times_B \overline{B}$ is the direct sum $\overline{V} \simeq \overline{B} \times F$.*

Proof. Let

$$(f) = \Sigma n_i C_i - \Sigma n_j' C_j' + \Sigma m_k F_{b_k},$$

where $n_i > 0$, $n_j' > 0$ and $\Sigma n_i C_i - \Sigma n_j' C_j'$ is an unramified divisor. We consider a fiber bundle L over B, a fiber of which is an affine line and which corresponds to the divisor $\Sigma m_k b_k$ on B. After adding to each fiber of L an infinite point, we will consider L as a fiber bundle, a fiber of which is a projective line.

We will show that the surface V can be considered as a ramified covering of the surface L. For this we suppose that B is covered by open sets W_i such that $b_i \in W_i$, $b_j \notin W_i$ for $i \neq j$, and that there exist functions τ_i such that $\tau_i(b_i) = 0$ and $\tau_i - \tau_i(b)$ is a local parameter at any point $b \in W_i$, where $\tau_i(b) \neq 0$ for $b \in W_i$, $b \neq b_i$. Then L can be given as the union of the open sets U_i, where $U_i \simeq W_i \times P^1$, where P^1 is a projective line and the points U_i and U_j are identified by the rule

$$b \times z \sim b' \times z' \; (b \in W_i,\ b' \in W_j,\ z,\ z' \in P^1),$$

if

$$b = b' \in W_i \cap W_j;\ z' = z \cdot \frac{\tau_i^{m_i}}{\tau_j^{m_j}}(b). \tag{22}$$

We define the mapping $\phi_i: \pi^{-1}(W_i) \to U_i$, setting

$$\varphi_i(v) = (\pi(v) \times \rho_{\pi(v)}(f\pi^*(\tau_i)^{-m_i})(v)),\ v \in \pi^{-1}(W_i).$$

We note that $f\pi^*(\tau_i)^{-m_i} \in O_{F_v}$, for $v \in \pi^{-1}(W_i)$, according to the choice of τ_i, and thus the function $\rho_{\pi(v)}(f\pi^*(\tau_i)^{-m_i})$ is defined. It is clear that the ϕ_i are regular mappings.

We will show that $\phi_i = \phi_j$ on $\pi^{-1}(W_i) \cap \pi^{-1}(W_j)$, and, thus, that the collection of mappings ϕ_i determines a unique regular mapping ϕ of the surface V onto L. In fact, if $x \in \pi^{-1}(W_i) \cap \pi^{-1}(W_j)$, then

$$\varphi_j(v) = (\pi(v) \times (\rho_{\pi(v)}(f\pi^*(\tau_j)^{-m_j}))(v))$$

and

$$\rho_{\pi(v)}(f\pi^*(\tau_j)^{-m_j})(v) = \rho_{\pi(v)}(f\pi^*(\tau_i)^{-m_i})(v) \cdot \frac{\tau_i^{m_i}}{\tau_j^{m_j}}(\pi(v))$$

in accordance with (22).

Thus, under the mapping ϕ each fiber F_b is mapped onto a projective line that is a fiber of L, where for $b \in W_i$ the mapping is effected by the function $\rho_b(f\pi^*(\tau_i)^{-m_i})$. We note that here none of the fibers F_b is mapped into a point. In other words, the function $\rho_b(f\pi^*(\tau_i)^{-m_i})$ is not constant on F_b. This follows from the fact that we can even indicate its divisor – it is equal to $\Sigma n_i C_i \cdot F_b - \Sigma n_j' C_j' \cdot F_b$ and is not equal to 0, since by assumption the curves C_i and C_j' do not intersect. Consequently, the mapping ϕ determines on each fiber a mapping of a covering of the projective line, where the degree of this covering is the same for all the fibers; it is equal to $\Sigma n_i(C_i \cdot F) = \Sigma n_j'(C_j' \cdot F)$.

We denote by W_b the divisor of the ramification of the covering, which the mapping ϕ determines on the fiber F_b. Since

$$(\rho_b(f\pi^*(\tau_i)^{-m_i})) = \Sigma n_i \rho_b C_i - \Sigma n_j' \rho_b C_j',$$

it follows that

$$W_b = (d\rho_b(f\pi^*(\tau_i)^{-m_i}))_0 + \Sigma (n_j' - 1)\rho_b C_j',$$

where $(\omega)_0$ designates a zero divisor of the differential ω. By (21)

$$(d\rho_b(f\pi^*(\tau_i)^{-m_i})) = \rho_b(Df),$$

and therefore $W_b = \rho_b W$, where W is a divisor on V,

$$W = (Df)_0 + \Sigma (n_j' - 1) C_j'.$$

From the definition of the divisor (Df) it follows that

$$W = \Sigma (n_i - 1) C_i + \Sigma (n_j' - 1) C_j' + \overline{W},\ \overline{W} > 0.$$

We will now demonstrate the central part of the proof of Lemma 8 – that the divisor $\overline{W}$ does not intersect the curves C_i and C'_j. In fact, since $(f_0) = \Sigma n_i C_i + \Sigma l_s F_s$, it follows from (20) that

$$(\rho_b (f\pi^* (\tau_i)^{-m_i}))_0 = \Sigma n_i \rho_b C_i = \Sigma n_i C_i \cdot F_b.$$

By the assumption of the theorem the covering $\pi: C_i \to B$ is unramified; therefore, if $(C_i \cdot F) = m_i$,

$$C_i \cdot F_b = Q_{b,i}^{(1)} + \ldots + Q_{b,i}^{(m_i)}, \quad Q_{b,i}^{(r)} \neq Q_{b,i}^{(s)} \text{ for } r \neq s.$$

From this it follows that

$$(\rho_b (f\pi^* (\tau_i)^{-m_i}))_0 = \Sigma n_i (Q_{b,i}^{(1)} + \ldots + Q_{b,i}^{(m_i)})$$

and thus that each of the points $Q_{b,i}^{(r)}$ occurs in the divisor $(\rho_b (f\pi^* (\tau_i)^{-m_i}))_0$ with multiplicity n_i. Therefore this point occurs in the divisor $(d(\rho_b (f\pi^* (\tau_i)^{-m_i})))$ with multiplicity $n_i - 1$. But it occurs in the divisor $(n_i - 1)\rho_b C_i$ with the same multiplicity. Hence none of these points occur in the divisor $\rho_b \overline{W}$. This means that the divisors $\overline{W}$ and C_i do not have common points on any fiber F_b; thus they do not intersect in general. The divisors C'_j are considered similarly.

Under the mapping ϕ the divisors $\Sigma n_i C_i$ and $\Sigma n'_j C'_j$ are preimages of zero and the infinite section of fiber bundle L:

$$\Sigma n_i C_i = \varphi^* (S_0), \quad \Sigma n'_j C'_j = \varphi^* (S_\infty).$$

The divisor V is the preimage of some divisor S on L: $V = \phi^* (S)$. From what was just proven it follows that S does not intersect either S_0 or S_∞.

Starting from this, we now show that for some unramified covering $B' \longrightarrow B$ the preimage of L is trivial: $L' = L \times_B B' \simeq B' \times P^1$ and in L' the preimage S' of the divisor S is "constant": $S' = B' \times \Delta$, where Δ is an effective divisor on P^1. For this we denote by C an arbitrary irreducible component of $\overline{W}$. The projection defined in L, $L \longrightarrow B$, determines C as a covering of B. We consider the fiber bundle $M = C \times_B L$. In it the points (c, c), $c \in C$, form a section that intersects neither zero nor the infinite section. A one-dimensional vector fiber bundle possessing such a section must necessarily be trivial. This means that L becomes trivial on some covering $C \longrightarrow B$. For the divisor $\Sigma m_k b_k$ corresponding to L, this means that it becomes principal on the covering C. Such a divisor determines a class of divisors of finite order and then, as is known, becomes principal on some unramified covering $B' \longrightarrow B$. Thus, $L' = L \times_B B' \simeq B' \times P^1$. For the preimage S' of the divisor S, the condition that S' does not intersect the zero section means that $(S' \cdot (B' \times 0)) = 0$, and since the divisor S' is effective, it follows from this that $S' = B' \times \Delta$.

We consider the preimage V' of the fibering π on B': $V' = V \times_B B'$. We have a mapping ϕ': $V' \to L'$ which determines on each fiber $F_{b'}$ a mapping of a covering over P^1 that is ramified at the points of a fixed (not depending on b') divisor Δ. From this it already follows that the covering $V' \to L'$ is a factor of the covering $V'' \to L'$, where $V'' = \overline{B} \times N$ for $\overline{B}$ an unramified covering of B' and N a covering of P^1 which is ramified only at points of the divisor Δ. This result follows from the theorem about unramified coverings of a direct product and from a remark of Abhyankar (cf. [1]).

The remark of Abhyankar is that it is possible to find a covering $H_1 \to P^1$ that is ramified only at points of Δ and is such that any covering $F_b \times_{P^1} H_1 \to H_1$ will be unramified. For this it is sufficient to take H_1 so that its indices of ramification at the points of Δ are divisible by the corresponding indices of ramification of the coverings $F_b \to P^1$ (for this it may be necessary to extend Δ by one point). One may already apply the theorem about unramified coverings of a direct product to the covering $V: X \times_{L'} (B' \times H_1) \to B' \times H_1$ (for $k = \mathbf{C}$ it follows from the equality $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$). We obtain that it is a factor of the covering $V'' = \overline{B} \times H \to B' \times H_1$, where $\overline{B}$ and H are unramified coverings of the curves B' and H_1. Since X' in turn is a factor of the covering $V \times_{L'} (B' \times H_1) \to B' \times H_1$, V' is also a factor of the covering V''.

We denote by $\overline{V}$ the surface $V' \times_{B'} \overline{B}$ that has the projection $\overline{\pi}: \overline{V} \to \overline{B}$, and we will show that $\overline{V} \simeq \overline{B} \times F_0$, where F_0 is an arbitrary fiber of the mapping $\overline{\pi}$. From what was said above it follows that $\overline{V}$ is a factor of V'', where, as it is easy to verify, the mapping u: $V'' \to \overline{V}$ commutes with the projections of both the surfaces onto $\overline{B}$.

Going if necessary from H to a larger covering, we can assume that H is a normal covering of P^1 with the Galois group G. From the construction it follows that we have a sequence of coverings

$$\overline{B} \times H \xrightarrow{u} \overline{V} \xrightarrow{v} \overline{B} \times P^1,$$

where the mappings u and v commute with the projections of all the members of this sequence onto $\overline{B}$. The covering $\overline{V} \to \overline{B} \times P^1$ belongs to some subgroup G_1 of the Galois group G of the covering $\overline{B} \times H \to \overline{B} \times P^1$. Since the automorphisms $g \in G$ act according to the law

$$g(\overline{b} \times h) = \overline{b} \times g(h),$$

the automorphisms $g_1 \in G_1$ also act in the same way. In view of this, $\overline{V} \simeq (\overline{B} \times H)/G_1 \simeq \overline{B} \times H/G_1$. Lemma 8 is proved.

In order to apply Lemma 8 to the proof of Theorem 9, we need to construct on a surface V satisfying the assumptions of the theorem, a function, a divisor of which consists of an unramified divisor and a linear combination of fibers. The following lemmas are connected with the construction of such a function.

We will designate by $C \approx D$ the algebraic, and by $C \stackrel{\approx}{\approx} D$ the numerical equivalence of divisors on a surface (if $k = \mathbf{C}$ one is concerned with homologousness and weak homologousness).

Lemma 9. *Let V be an algebraic surface with $(K^2) = 0$, $p_a(V) = 0$, and let $\pi: V \to B$ be its mapping onto a curve B, where all the fibers of the mapping are not degenerate and have a genus $g > 1$. For a sufficiently large n (for example $n \geq 3$), there exists for any divisor D for which $D \stackrel{\approx}{\approx} nK$ a divisor C such that $C > 0$, $C \approx D$.*

Proof. It is sufficient for us to find a divisor C such that $C \approx D$, $l(C) > 0$. We will look for C in the form $D + F_b - F_0$, $b \in B$. We will show that $l(D + F_b - F_0) > 0$ for some $b \in B$, which is what we need.

Let $l(D + F_b - F_0) = 0$ for all $b \in B$. From the exact sequence

$$0 \to \mathcal{F}(D + F_b - F_0) \to \mathcal{F}(D + F_b) \to \mathcal{F}_{F_0}(D + F_b) \cdot F_0) \to 0$$

follows the exactness of the sequence

$$0 \to \mathcal{L}(D + F_b - F_0) \to \mathcal{L}(D + F_b) \to \mathcal{L}_{F_0}((D + F_b) \cdot F_0). \tag{23}$$

From the assumption made it follows that the restricted homomorphism

$$\rho_0 : \mathcal{L}(D + F_b) \to \mathcal{L}_{F_0}((D + F_b) \cdot F_0) \tag{24}$$

is an injection for any $b \in B$.

We calculate the dimensions of both of the spaces in (24). Since

$$((D + F_b) \cdot F_0) = ((nK + F_b - F_0) \cdot F_0) = 2n(g - 1),$$

the number $l((D + F_b) \cdot F_0)$ can be found by the Riemann-Roch theorem applied to the curve F_0:

$$l((D + F_b) F_0) = (2n - 1)(g - 1). \tag{25}$$

On the other hand, the Riemann-Roch theorem on V gives

$$l(D + F_b) \geqslant \frac{((D + F_b) \cdot (D + F_b - K))}{2} = \frac{(2n - 1)(K \cdot F_b)}{2} = (2n - 1)(g - 1). \tag{26}$$

From (25), (26), and the fact that the ρ_0 in (24) is an injection it follows that both spaces in (24) have the same dimension, and thus that ρ_0 is an isomorphism for any $b \in B$.

Thus, for any effective divisor Δ contained in the complete linear system

$|(D + F_b) \cdot F_0|$ on F_0, there exists exactly one effective divisor $C_b \in |D + F_b|$ such that $C_b \cdot F_0 = \Delta$.

We will show that at least one curve C_b passes through each point $v \in V$. For this we consider in $V \times B$ the subset Γ consisting of the points (v, b) for which $v \in C_b$. It is clear that Γ is an algebraic subvariety in $V \times B$. Under the mapping $\Gamma \longrightarrow B$ induced by the projection $V \times B \longrightarrow B$, each point $b \in B$ has for a preimage a curve C_b, from which it follows that $\dim \Gamma = 2$. The assertion that we wish to prove is that $pr_V \Gamma = V$. In view of the completeness of all the varieties considered, $pr_V \Gamma$ is an algebraic subvariety of V, and, consequently, coincides with V if $\dim pr_V \Gamma = 2$. If $pr_V \Gamma = U$, $\dim U = 1$, then $\Gamma = U \times B$, and this in turn means that $C_b = U$ for all $b \in B$. This cannot be true, since $C_b \neq C_{b'}$ for $b \neq b'$. In fact, it would follow from the equality $C_b = C_{b'}$ that $F_b \sim F_{b'}$. We saw in the proof of Theorem 8 that this leads to a contradiction: if $(f) = F_b - F_{b'}$, then $f = \pi^*(g)$, $g \in k(B)$, $(g) = (b) - (b')$, i.e. $b \sim b'$, and this contradicts the fact that the genus of the curve B is equal to 1.

We choose for the point v, in particular, an arbitrary point of the curve F_0 that is not contained in the divisor Δ. By assumption there exists a point $b \in B$ such that $C_b \ni v$. But then C_b intersects F_0 in more than $(C_b \cdot F_0)$ points, and this is possible only when C_b contains F_0 as a component. If $C_b = H + F_0$, $H > 0$, then $D + F_b \sim H + F_0 > 0$, i.e. $D + F_b - F_0 \sim H > 0$, which contradicts the assumption $l(D + F_b - F_0) = 0$. The lemma is proven.

Lemma 10. *If V is a surface satisfying the conditions of Lemma 9, then the divisor C, whose existence is established in that lemma, is distinct from 0 and is unramified.*

Let $C \approx nK$, $C > 0$. Since $(K \cdot F) > 0$, $nK \not\approx 0$, and thus $C \neq 0$. Let us assume that

$$C = \Sigma n_i C_i,\ C_i \neq C_j \text{ for } i \neq j,\ n_i > 0,$$

where the C_i are irreducible curves.

Since every irreducible curve lying on the surface V is either a covering of the base B or coincides with a fiber F, where the genus of B is equal to 1 and the genus of F is greater than 1, an irreducible rational curve cannot lie on V. Therefore

$$(C_i^2) + (C_i \cdot K) \geq 0. \tag{27}$$

We will show that $(C_i \cdot K) \geq 0$. From $(C_i \cdot K) < 0$ it would follow by (27) that $(C_i^2) > 0$. Since $C \approx nK$, then $(C_i \cdot K) < 0$ would give

$$(C_i \cdot \sum n_j C_j) = n_i (C_i^2) + \sum_{j \neq i} n_j (C_i \cdot C_j) < 0,$$

which is impossible since $(C_i \cdot C_j) \geq 0$ for $i \neq j$, and $(C_i^2) > 0$.

In view of the fact that $C \approx nK$, the equality $(K^2) = 0$ can be rewritten as

$$\Sigma n_i (C_i \cdot K) = 0 \tag{28}$$

or

$$(\sum n_i C_i)^2 = \sum n_i^2 (C_i^2) + \sum_{i \neq j} n_i n_j (C_i C_j) = 0. \tag{29}$$

From (28) it now follows that $(C_i \cdot K) = 0$, and from (27) and (29) that $(C_i^2) = 0$ and $(C_i \cdot C_j) = 0$ for $i \neq j$. Therefore

$$p_a(C_i) = \frac{(C_i^2) + (C_i \cdot K)}{2} + 1 = 1,$$

and since C_i is a nonrational curve, it is a nonsingular curve of genus 1. It cannot coincide with a fiber because the genus of a fiber is greater than 1, and is thus a covering of the base. Since the genus of each curve, both C_i and B, is equal to 1, this covering is unramified. Finally, the relation already shown, $(C_i \cdot C_j) = 0$ for $i \neq j$, completes the proof that the divisor C is unramified.

Lemma 11. *Let $\pi: V \longrightarrow B$ be a fiber space without degenerate fibers, let β be a generic point of the curve B and let F_β be a generic fiber. Every class of divisors of finite order on the curve F_β is defined over some unramified extension of the field $k(\beta)$.*

Proof. Let J_β be the Jacobian variety of the curve F_β. It is defined over the field $k(\beta)$ and for any $b \in B$ has as a specialization the Jacobian variety J_b of the curve F_b. We denote by $\mathfrak{a}_n(\beta)$ the cycle consisting of the points of order n on J_β. As is known, this cycle consists of n^{2g} points with multiplicity of one. Under the specialization of β into b the cycle $\mathfrak{a}_n(\beta)$ is specialized into the cycle $\mathfrak{a}_n(b)$ consisting of the points of order n on J_b. Since the fiber J_b is nondegenerate, the cycle $\mathfrak{a}_n(b)$ consists of n^{2g} distinct points. By the generalized lemma of Hensel (cf. [34]) it then follows that the cycle $\mathfrak{a}_n(\beta)$ is rational over the complete local ring of the point b. Since this is true for any point b, it follows from this that the field of definition of the cycle $\mathfrak{a}_n(\beta)$ is unramified over the field $k(\beta)$. The lemma is proved.

Remark. One may give this argument the following geometric form. We denote by J_b the Jacobian variety of the fiber F_b. In the set of all the varieties J_b, $b \in B$ it is possible to introduce the structure of an algebraic variety J equipped with a projection $\phi: J \longrightarrow B$. For each $b \in B$ the variety J_b contains n^{2g} points of order n. The collection of all these points for all the $b \in B$ forms, as it is easy to verify, a one-dimensional effective divisor $\mathfrak{a}_n \subset J$. The projection ϕ determines on $\mathfrak{a}_n$ the structure of a covering of B. Since for any $b \in B$ the cycle

$\alpha_n \cdot J_b$ consists of n^{2g} dsitinct points, this covering is unramified. This is the statement of the lemma.

Lemma 12. *Let $K = k(B)$ be the field of functions over an algebraic curve B, U be an algebraic curve over K, and C be the class of divisors on U to which there corresponds a point, rational over K of the Jacobian variety of the curve U, where $l(C) > 0$. Then there exists in C an effective divisor defined over K.*

This assertion essentially coincides with the so-called criterion of rationality of Cartier.

Let L/K be a normal extension of the field K over which the class C is already defined, let G be the Galois group L/K, and let $\mathfrak{D}_L$, P_L, $\mathfrak{C}_L$, be the groups of divisors, of principal divisors, and of classes of divisors of U over L. All these groups are G-modules. By assumption $C^\sigma = C$ for $\sigma \in G$, i.e. $C \in H^0(G, \mathfrak{C}_L)$. We will show that there exists a $D' \in \mathfrak{D}_K$, $D' \in C$. For this we consider the exact sequence

$$H^0(G, \mathfrak{D}_L) \to H^0(G, \mathfrak{C}_L) \to H^1(G, P_L),$$

which is obtained from the exact sequence

$$(1) \to P_L \to \mathfrak{D}_L \to \mathfrak{C}_L \to (1).$$

It is clear that our assertion will be proved if we show that $H^1(G, P_L) = (1)$. For this we consider the exact sequence

$$(1) \to L^* \to L(U)^* \to P_L \to (1),$$

where L^* and $L(U)^*$ are the multiplicative groups of the fields L and $L(U)$, and the exact sequence

$$H^1(G, L(U)^*) \to H^1(G, P_L) \to H^2(G, L^*).$$

Since G is the Galois group $L(U)/K(U)$, $H^1(G, L(U)^*) = (1)$ by a well-known algebraic fact. On the other hand, $H^2(G, L^*) = 0$ by a theorem of Tsen, for $K = k(B)$ and the field k is algebraically closed. From this it follows that $H^1(G, P_L) = (1)$.

It remains to be shown that there exists an effective divisor D defined over K and equivalent to the divisor D'. But this follows from the fact that $l(C) > 0$ and that the dimension of a divisor is not reduced under a separable extension of the field of constants (cf. for example, [58], Chapter V, §6, Corollary 1 to Theorem 4).

Proof of Theorem 9. We choose a temporarily arbitrary number m and consider on the curve F_β the class of divisors $\mathfrak{S}$ whose order is equal to m. By Lemma 11, this class is defined over some unramified extension field $k(\overline{B})$. If $\overline{B}$ is a nonsingular model of this unramified extension, then we have an unramified covering $\phi: \overline{B} \to B$, and on the surface $\overline{V} = V \times_B \overline{B}$ the class of divisors $\mathfrak{S}$ of the curve $F_{\overline{\beta}}$ is defined already over the field $k(\overline{B})$. By Lemma 12, there exists an effective divisor $\mathfrak{D}$ of the curve $F_{\overline{\beta}}$ belonging to the class $K(F_{\overline{\beta}})+\mathfrak{E}$ and also defined over $k(\overline{B})$. To this divisor there corresponds an effective divisor D on $\overline{V}$ such that

$$D\cdot F_{\overline{\beta}}=\mathfrak{D}\in K(F_{\beta})+\mathscr{E}.$$

Then

$$(D+(n-1)K)\cdot F_{\overline{\beta}}\in nK(F_{\overline{\beta}})+\mathscr{E}, \tag{30}$$

$$m(D+(n-1)K)\cdot F_{\overline{\beta}}\in mnK(F_{\overline{\beta}}),$$

and hence on $F_{\overline{\beta}}$

$$m(D+(n-1)K)\cdot F_{\overline{\beta}}\sim mnK\cdot F_{\overline{\beta}}. \tag{31}$$

From this it follows that on V

$$m(D+(n-1)K)\sim mnK+\Sigma m_i F_{b_i},$$

and since

$$\Sigma m_i F_{b_i}\approx lF_b,$$

where $l=\Sigma m_i$, and b is any point on B,

$$m(D+(n-1)K)\approx mnK+lF_b. \tag{31a}$$

We will show that for a suitable choice of m, l must be divisible by m. For this it is sufficient to consider the indices of intersection of the separate parts (31a) with K. We obtain

$$m(D\cdot K)=2l(g-1),$$

and if m is relatively prime to $2(g-1)$, then it divides l. Until now m was arbitrary. Therefore we can choose it relatively prime to $2(g-1)$ and assume that $l=l'm$. Then

$$m(D+(n-1)K-l'F_b)\approx mnK,$$

and hence

$$D+(n-1)K-l'F_b\approxeq nK.$$

We can apply Lemma 9 to the divisor $D+(n-1)K-l'F_b$. We obtain a divisor $C'>0$ such that

$$C'\sim D+(n-1)K-l'F_b+F_{b_1}-F_0.$$

On the other hand, by the same Lemma 9 there exists a divisor $C>0$ such that

$$C\sim nK+F_{b_2}-F_0.$$

From this it follows from (31) that

$$mC'\sim mC+\Sigma s_i F_{b_i},$$

i.e. there exists a function f on X for which

$$(f) = -mC' + mC + \Sigma s_i F_{b_i}.$$

By Lemma 10 the divisor $-mC' + mC$ is not ramified. It is not equal to 0, since $C' \neq C$. In fact, by (30), it is even true that

$$C' \cdot F_{\bar{\beta}} \not\sim C \cdot F_{\beta}.$$

Applying Lemma 8, we obtain the assertion of Theorem 9.

We now consider the case when the genus g of a generic fiber of the Albanese mapping is equal to 1. As we will see later, in this case the exact analogue of Theorem 9 is not true. The following weaker statement, however, will follow from Lemma 7 and Corollary 3 of Theorem 7, Chapter VII.

Theorem 10. *If $\pi: V \to B$ is a regular mapping, a generic fiber of which is an elliptic curve, and $\chi(V) = 0$, then there exists a (perhaps ramified) covering $C \to B$ such that the inverse image of π on C is a direct product*

$$V \times_B C \simeq C \times F.$$

§8. Surfaces with $p_g = 0$, $q > 0$

(Classification and Theorem of Enriques)

In the preceding sections it was proven that a surface with the invariants $p_g = 0$, $q > 0$ is ruled, except perhaps for the case $q = 1$, $(K^2) = 0$ (Theorems 5 and 8). The basic analysis of this last case is Theorem 9. Starting from this theorem we now give a complete classification of those surfaces and verify that those of them which satisfy the condition $P_{12} = 0$ are ruled (the theorem of Enriques).

We consider separately the cases when the genus g of a generic fiber of the mapping π is greater than 1 and is equal to 1. We start with the case $g > 1$.

According to Theorem 9, in this case the surface V with invariants $p_g = 0$, $q = 1$, $(K^2) = 0$ have as an unramified covering the surface $\overline{V} = \overline{B} \times F$. We consider how it is possible to obtain the surface V from $\overline{V}$.

Theorem 9 shows that $\overline{V} = V \times_B \overline{B}$, where B is the Albanese variety of the surface V (in the given case, an elliptic curve), and $\overline{B} \to B$ is an unramified covering. We will assume without loss of generality that $\overline{B} \to B$ is a normal covering. We denote its Galois group by G. It is clear that $X = \overline{X}/G$. Under the isomorphism

$$\overline{V} \simeq \overline{B} \times F, \tag{32}$$

to the automorphism $\sigma \in G$ of the covering $\overline{B} \to B$ there corresponds the automorphism

$$\sigma(\bar{b}, f) = (\sigma\bar{b}, f\sigma^{-1}), \tag{33}$$

where $f \to f\sigma^{-1}$ is some automorphism of the curve F. This follows from the fact operation of the automorphism $\sigma \in G$ on $\overline{V}$, like that of the isomorphism of (32), commutes with the projection $\overline{V} \longrightarrow \overline{B}$.

Thus, we have a homomorphism

$$\phi: G \longrightarrow \mathrm{Aut}(F)$$

of the group G into the group of automorphisms of the curve F. Conversely, the existence of such a representation of the group G (which is itself given as the Galois group of the covering $\overline{B} \longrightarrow B$) determines its operation as the group of automorphisms of the surface $\overline{V}$:

$$\sigma(\overline{b}, f) = (\sigma\overline{b}, f\varphi(\sigma)^{-1}).$$

It is immediately clear that we thus obtain a group operating on $\overline{V}$ without fixed points, and that the surface $V = \overline{V}/G$ does not have singular points. We see that we can obtain in this way all the surfaces interesting to us, but perhaps also many others. It is thus necessary for us to give the conditions which the covering $\overline{B} \longrightarrow B$, the curve F, and the representation ϕ must satisfy in order that the surface V will have the invariants $p_g = 0$, $q = 1$. In order to formulate the result, we designate by L the curve $F/\phi(G)$.

Lemma 13. *For the surface* $V = \overline{V}/G$

$$q = 1 + \gamma, \tag{34}$$

$$p_g = \gamma, \tag{35}$$

where γ *is the genus of the curve* L.

Proof. The projection $\psi: \overline{V} \longrightarrow V$ determines the mapping

$$\psi_i^*: \Omega^i(V) \longrightarrow \Omega^i(\overline{V})$$

of the i-dimensional differential forms of first order. It is clear that ψ_i^* is an imbedding and that $\psi_i^*(\Omega^i(V)) = \Omega^i(\overline{V})^G$. Therefore it is necessary for us first to find the differential forms of first order on $\overline{V}$ and then to choose among them those that are invariant with respect to the operation of the automorphism from G.

We begin with the one-dimensional forms. It is obvious that

$$\Omega'(\overline{V}) = p_1^*(\Omega'(\overline{B})) \oplus p_2^*(\Omega'(F)), \tag{36}$$

where p_1 and p_2 are projections of $\overline{V}$ onto $\overline{B}$ and F. We now tell how an automorphism $\sigma \in G$ acts on a form from $\Omega'(V)$. By the definition (33) of the operation of σ on $\overline{V}$ the spaces $p_1^*\Omega'(B)$ and $p_2^*\Omega'(F)$ remain invariant under the operation of σ and are transformed in the same way as the space $\Omega'(B)$ under the operation of σ and the space $\Omega'(F)$ under the operation of $\phi(\sigma)^{-1}$. We now note that σ operates on $\Omega'(\overline{B})$ trivially, so that $\Omega'(\overline{B})^G = \Omega'(\overline{B})$. On the other hand, as is easily seen, $\Omega'(F)^{\phi(G)} \simeq \Omega'(L)$.

From this it follows that

$$\Omega'(V)^G \simeq \Omega'(\bar{B}) \oplus \Omega'(L),$$

which yields (34).

We now consider the forms of second degree. According to (2)

$$K(\bar{V}) = K(\bar{B}) \times F + \bar{B} \times K(F),$$

and since $K(\bar{B}) = 0$,

$$K(\bar{V}) = \bar{B} \times K(F). \tag{37}$$

Therefore

$$\Omega^2(\bar{V}) \simeq \Omega'(\bar{B}) \otimes \Omega'(F) \simeq \Omega'(F) \tag{38}$$

and

$$\Omega^2(\bar{V})^G \simeq \Omega'(F)^{\varphi(G)} \simeq \Omega'(L), \tag{39}$$

from which (35) follows.

Corollary. *The surface* $V = \bar{V}/G$ *has the invariants* $q = 1$, $p_g = 0$ *if and only if* $g = 0$, *i.e. the curve* $F/\phi(G)$ *is rational.*

We now turn to the construction of the surfaces with the invariants $p_g = 0$, $q = 1$ (again assuming that $g > 1$, i.e. that these surfaces are described by Theorem 9). We first note that the covering $\bar{B} \to B$, as an unramified covering of an elliptic curve, has an abelian Galois group with one or two generators:

$$G = \{\sigma_1\} \times \{\sigma_2\},\ \sigma_1^{m_1} = 1,\ \sigma_2^{m_2} = 1.$$

The field $k(F)/k(L)$ has as a Galois group the group $\phi(G)$. Consequently this Galois group has two generators; $\phi(\sigma_1)$ and $\phi(\sigma_2)$, the orders of which are divisors of the numbers m_1 and m_2. Since the curve L must be rational, the field $k(L)$ is isomorphic with the field of rational functions $k(L) = k(x)$, and the field $k(F)/k(L)$ can be obtained in the form $k(x, \sqrt[m_1]{P_1(x)}, \sqrt[m_2]{P_2(x)})$, (this easily follows from Galois theory), where $P_1(x)$ and $P_2(x)$ are any polynomials in x. Conversely, it is possible to define automorphisms s_1 and s_2 in the field $k(x, \sqrt[m_1]{P_1(x)}, \sqrt[m_2]{P_2(x)}$ by setting

$$s_1(\sqrt[m_1]{P_1(x)}) = \varepsilon_1 \sqrt[m_1]{P_1(x)},$$
$$s_1(\sqrt[m_2]{P_2(x)}) = \sqrt[m_2]{P_2(x)},$$
$$s_2(\sqrt[m_2]{P_2(x)}) = \varepsilon_2 \sqrt[m_2]{P_2(x)},$$

where ϵ_1 and ϵ_2 are primitive roots of 1 whose degrees are equal to the degrees of the fields $k(x, \sqrt[m_1]{P_1})/k(x)$ and $k(x, \sqrt[m_1]{P_1}, \sqrt[m_2]{P_2})/k(x, \sqrt[m_1]{P_1})$. Taking for F

a nonsingular model of the field $k(x, \sqrt[m_1]{P_1}, \sqrt[m_2]{P_2})$, we can define a homomorphism ϕ of the group G into the group $\mathrm{Aut}\, F$ by setting $\phi(\sigma_1) = s_1$, $\phi(\sigma_2) = s_2$, and on the surface $\overline{V} = \overline{B} \times F$ giving the operation of G by the formula (33). Since the curve $F/\phi(G) = L$ is rational, the surface $V = \overline{V}/G$ will have the invariants $p_g = 0$, $q = 1$. We have thus proven the following assertion.

Theorem 11. *For the case* $g > 1$ *all the surfaces* V *with the invariants* $p_g = 0$, $q = 1$ *can be obtained in the form* $V = \overline{V}/G$, $\overline{V} = \overline{B} \times F$, *where* $\overline{B} \to B$ *is an unramified covering of an elliptic curve,* F *is a nonsingular model of the field* $k(x, \sqrt[m_1]{P_1(x)}, \sqrt[m_1]{P_2(x)})$, G *is isomorphic to the Galois group of the covering* $\overline{B} \to B$ *and operates on* $\overline{X}$ *according to the rule* $\sigma(\overline{b}, f) = (\sigma\overline{b}, f\phi(\sigma)^{-1})$, $\sigma \in G$, *and* $\phi(\sigma)$ *is a homomorphism of the Galois group of the covering* $\overline{B} \to B$ onto the *Galois group of the field* $k(x, \sqrt[m_1]{P_1}, \sqrt[m_2]{P_2})/k(x)$.

We now consider the case when the genus of a generic fiber of the fibering π is equal to 1.

According to Theorem 10, there exists a covering $C \to B$ of the base such that

$$C \times_B V \simeq C \times F,$$

where F is a nonsingular curve of genus 1. We can of course assume that C is a normal covering with a Galois group G. It is clear that on $C \times F$ the group G operates according to the rule

$$\sigma(c \times f) = \sigma(c) \times f\varphi(\sigma)^{-1}, \sigma \in G, \tag{40}$$

where

$$\phi\colon G \to \mathrm{Aut}\, F$$

is a homomorphism of G into the group of biregular automorphisms of the curve F. It is obvious that

$$V = (C \times_B V)/G = (C \times F)/G.$$

Here we can limit ourselves to the case when ϕ is a monomorphism. In fact, if N is the kernel of ϕ, $G_1 = G/N$, $C_1 = C/N$, then it is easy to see that

$$V \times_B C_1 \simeq C_1 \times F, \quad V = (C_1 \times F)/G_1.$$

We now indicate which fixed points the automorphisms $\sigma \in G$ have.

It is clear that the point $\sigma = (c \times f)$ is fixed for an automorphism if and only if $\sigma c = c$, $f\phi(\sigma) = f$. Since for $\sigma \neq 1$ it is also true that $\phi(\sigma) \neq 1$, both σ and $\phi(\sigma)$ have a finite number of fixed points on C and F respectively, and consequently σ has only a finite number of fixed points on $C \times F$. Thus the covering $C \times F \to V$ can have only isolated branch points. From this and from the fact that neither $C \times F$ nor V have singular points, it follows that this covering does not generally have branch points. This follows, for example, from the theorem of Zariski about varieties of ramification [17], according to which the variety of ramification of any covering of a nonsingular variety by a nonsingular one has codimension 1. It is also possible to verify this directly, considering the subring $\mathfrak{D}_x^H$ consisting of the H-invariant elements of the local ring $\mathfrak{D}_x$ of a point $x \in C \times F$, where H is a stationary subgroup of the point x, and showing that this ring is nonregular (the ring $\mathfrak{D}_x^H$ is isomorphic, as it is easy to see, to the local ring of the image of the point x on V).

Conversely we take a normal covering $C \longrightarrow B$ with a Galois group G and a monomorphism $\phi: G \longrightarrow \operatorname{Aut} E$ of the group G into the group of automorphisms of the curve F of genus 1. We define the operation of G on $C \times F$ by formula (40) and explain when G operates on $C \times F$ without fixed points and when $(C \times F)/G$ has the invariants $p_g = 0$, $q = 1$.

The group $\phi(G)$ is a group of the automorphisms of a nonsingular curve F of genus 1. We introduce in F the structure of a one-dimensional abelian variety. Then, as is known, the finite group of the biregular transformations of F is a semidirect product

$$\phi(G) = H \cdot \mathfrak{A}, \tag{41}$$

where $\mathfrak{A}$ consists of translations and H of the automorphisms of the abelian variety F, while $\mathfrak{A}$ has one or two generators and H is a cyclic group of order 1, 2, 3, 4, or 6. Here, naturally, $\mathfrak{A}$ is an H-operator group.

It is easy to verify that the elements of $\mathfrak{A}$ and only they do not have fixed points on F. Consequently, for the group G acting on $C \times F$ according to (33) not to have fixed points it is necessary and sufficient that in the group G of automorphisms of the covering $C \longrightarrow B$ only the automorphisms of the form $\phi^{-1}(a)$, $a \in \mathfrak{A}$ have fixed points. This in turn means that the covering $C_1 \longrightarrow B$, belonging to the subgroup $\phi^{-1}(\mathfrak{A})$ by Galois theory, is unramified.

We assume that $F/\phi(G) = L$ and we designate the genus of the curve L by γ. From formula (36) it follows that if $\overline{V} = C \times F$, $V = (C \times F)/G$,

$$\Omega'(V) \simeq \Omega'(\overline{V})^G \simeq \Omega'(C)^G \oplus \Omega'(F)^G \simeq \Omega'(B) \oplus \Omega'(L),$$

and therefore for V

$$q = 1 + \gamma,$$

so that $q = 1$, if and only if the curve L is rational. This in turn is equivalent to the fact that in (41) $H \neq 1$.

It follows from formula (38) that

$$\Omega^2(\overline{V}) \simeq \Omega'(C) \otimes \Omega'(F).$$

In order to calculate $\Omega^2(\overline{V})^G$, we first define $\Omega^2(\overline{V})^{\mathfrak{A}}$. Since the elements $a \in \mathfrak{A}$ are translations of the abelian variety F, $\Omega'(F)^{\mathfrak{A}} = \Omega'(F)$ and

$$\Omega^2(\overline{V})^{\mathfrak{A}} \simeq \Omega'(C)^{\mathfrak{A}} \otimes \Omega'(F) \simeq \Omega'(C_1) \otimes \Omega'(F),$$

where C_1 is a covering $C_1 \to B$ belonging to the subgroup $\phi^{-1}(\mathfrak{A})$ of the Galois group of the covering $C \to B$. The covering $C_1 \to B$ is by assumption unramified, and we can apply formula (39) to it. Therefore

$$\Omega^2(\overline{V})^G = (\Omega^2(\overline{V})^{\mathfrak{A}})^H \simeq \Omega'(L),$$

and, consequently, $p_g = 0$ if and only if the curve L is rational, and hence $H \neq 1$. We have proved the following result.

Theorem 12. *For the case $g = 1$ all the surfaces V with the invariants $p_g = 0$, $q = 1$ can be obtained in the form $V = \overline{V}/G$, $\overline{V} = C \times F$, for $C \to B$ a covering of an elliptic curve B with a Galois group G of the type (41) with $H \neq 1$, where G is isomorphic to the group of biregular transformations of the curve F of genus 1, and in this isomorphism the elements of $\mathfrak{A}$ correspond to translations, and H to the automorphisms of the one-dimensional abelian variety F. Here the covering $C_1 \to B$, belonging in the sense of Galois theory to the subgroup $\mathfrak{A}$ of the Galois group G of the covering $C \to B$, must be unramified.*

It would have been possible to give an explicit construction of the extensions of the field $k(B)$ that have a Galois group of type (41) and satisfy all the conditions of Theorem 12, and at the same time give an explicit construction for the surfaces with the invariants $p_g = 0$, $q = 1$ for the case $g = 1$ also. A more elegant classification of such surfaces, however, will be obtained in the theory of surfaces with a pencil of elliptic curves (cf. Theorem 12, Chapter VII).

Theorem 13 (Criterion of Enriques). *A surface V is ruled if and only if $P_{12} = 0$ for it.*

Proof. First of all we verify that for a ruled surface $P_{12} = 0$. In view of the birational invariance of the number P_{12} it is sufficient to establish the equality $P_{12} = 0$ for the surface $V = B \times P^1$, where P^1 is a projective line. For this surface $P_n = 0$ for all $n > 0$. In fact, it follows from (2) that on V

$$(K \cdot (b \times P^1)) = -2 < 0.$$

Therefore if $P_n(V)$ were positive for some $n > 0$, we would have $nK \sim D > 0$,

$$(D \cdot (b \times P^1)) = (nK \cdot (b \times P^1)) = -2n < 0,$$

which is impossible.

We will show that it follows from the equation $P_{12} = 0$ that the surface V is ruled. First of all we note that because of this condition $p_g = 0$ and $p_2 = 0$.

Therefore, if $q = 0$, then the surface V is rational by the criterion of Castelnuovo and moreover is ruled. In the same way, if $q > 1$, the surface is ruled according to Theorem 5. It remains for us to consider the case $q = 1$.

In this case we can use Theorems 11 and 12 and represent V in the form $\overline{V}/G$, where $\overline{V} = C \times F$, $C \to B$ is a covering, perhaps ramified, and the group G defined in Theorems 11 and 12 also operates on $\overline{V}$ without fixed points. We have to determine $P_n(X)$, i.e. the dimension of the space of differentials of degree n and of first order on X.

If x and y are two functions that are algebraically independent on X, then they will be independent also on $\overline{X}$, and any differential of nth degree on $\overline{X}$ can be written in the form $f(dx \wedge dy)^n$, $f \in k(\overline{X})$. The invariance of this differential with respect to the operation of the automorphisms of G is equivalent to the invariance of the function f, and this in turn means that $f \in k(X)$, and hence the differential itself is also an image of a differential on X under the mapping of the differentials that is induced by the mapping $\pi: \overline{X} \to X$.

We have thus shown that the differentials of nth degree on $\overline{X}$ that are invariant with respect to the automorphisms of G coincide with the differentials of the form $\pi^*(\omega)$, where ω is a differential on X.

We now note that since the covering $\overline{X} \to X$ is unramified a differential ω is of first order on X if and only if $\pi^*(\omega)$ is of first order on $\overline{X}$. Thus,

$$\Omega_n(X) \simeq \Omega_n(\overline{X})^G,$$

where $\Omega_n(X)$ and $\Omega_n(\overline{X})$ are the spaces of the differentials of nth degree and first order on X and $\overline{X}$ respectively. It is clear that

$$\Omega_n(\overline{V})^G \supset \Omega_n(C)^G \otimes \Omega_n(F)^G \supset \Omega_n(B) \otimes \Omega_n(F)^G.$$

Since B is an elliptic curve, $\Omega_n(B) \simeq k$. Hence, in view of what we proved earlier,

$$P_n(V) \geqslant \dim_k \Omega_n(F)^{\phi(G)}.$$

It remains to find the dimension of the space $\Omega_n(F)^{\phi(G)}$. For this we recall that $\phi(G)$ is the Galois group of the extension $k(F)/k(L)$ and that L is a rational curve. Since the differentials of nth degree that are invariant with respect to the automorphisms of $\phi(G)$, as we saw, have the form $\psi^*(\omega)$ where ψ is a covering $F \to L$ and ω is a differential on L, the space $\Omega_n(F)^{\phi(G)}$ coincides with the space of all differentials of nth degree on L for which $\psi^*(\omega)$ is of first order.

The surface V possesses a regular mapping $V \to B$ whose fibers are isomorphic to F (this mapping is induced by the mapping $\overline{V} \to \overline{B}$). Therefore if we show that the genus of the curve F is equal to 0, we will have proved at the same time that the surface V is ruled.

We see that the assertion of Theorem 11 reduces to the following lemma about algebraic curves.

Lemma 14. *Let* ψ: $F \longrightarrow L$ *be a normal covering of a rational curve. If any differential* ω *of* 12*th degree on* L *is such that* $\psi^*(\omega)$ *of first order on* F *is equal to* 0, *then* F *is a rational curve.*

Proof. We choose a coordinate t on the projective line L so that all the branch points of the covering ψ are finite. Let $\omega = f(dt)^n$ be a differential of nth degree on L. We give the conditions to which the function f must be subjected in order for the differential $\psi^*(\omega)$ to be of first order.

Let a point $P_i \in F$, $\psi(P_i) = Q_i \neq \infty$ and let P_i be a branch point of multiplicity e_i for the covering ψ. Then $t = \tau_i^{e_i}$ on F, where τ_i is a local parameter at the point P_i, while $\nu_{P_i}(u) = 0$, and

$$\psi^*(\omega) = \psi^*(f)(\tau_i^{e_i-1}v)^n, \quad \nu_{P_i}(v) = 0.$$

From this it follows that ω is regular in P_i if and only if

$$\nu_{P_i}(\psi^*(f)) \geqslant -n(e_i - 1),$$

and since $\nu_{P_i}(\psi^*(f)) = e_i \nu_{Q_i}(f)$, the same condition can be written in the form

$$\nu_{Q_i}(f) \geqslant -n\left(1 - \frac{1}{e_i}\right),$$

or

$$\nu_{Q_i}(f) \geqslant -\left[n\left(1 - \frac{1}{e_i}\right)\right].$$

An analogous consideration at the infinite point (∞) yields the condition

$$\nu_\infty(f) \leqslant -2n.$$

We thus see that the differential $\psi^*(\omega)$ is a differential of first order if and only if

$$f \in \mathcal{L}(D), D = \Sigma\ [n(1 - 1/e_i)]\ Q_i - 2n(\infty).$$

We have to explain when there exists a differential $(\omega) \neq 0$ such that $\psi^*(\omega)$ is of first order on F. In other words, we must explain when

$$l(D) > 0.$$

Since on a curve of order 0 the dimension $l(D) > 0$ if and only if $\deg D \geq 0$, the case of interest to us takes the form

$$\Sigma\ [n(1 - 1/e_i)] \geqslant 2n. \tag{42}$$

We now assume that the genus of the curve F is different from 0, and we will show that the relationship (42) can be satisfied for $n = 2, 3, 4$, or 6. This then gives us the existence of a differential ω of degree $n = 2, 3, 4$, or 6 such that $\psi^*(\omega)$ is of first order of F. The differential $\omega^{12/n}$ will then satisfy all the conditions of the lemma.

Let the covering $\psi: F \to L$ have degree m. Then m/e_i branch points of multiplicity e_i lie over each of the points Q_i (because of the normality of the covering). Since the genus g of the curve F is by assumption ≥ 1,

$$\frac{1}{2}\Sigma\,(e_i - 1)\frac{m}{e_i} - m + 1 \geqslant 0,$$

i.e.

$$\Sigma\,(1 - 1/e_i) \geqslant 2. \tag{43}$$

Let $e_1 \leq e_2 \leq \cdots \leq e_k$. Since $e_i \geq 2$, if $k \geq 4$ condition (42) is satisfied already for $n = 2$:

$$2\,(1 - 1/e_i) = 2 - 2/e_i \geqslant 1,$$

$$\sum_1^k [2\,(1 - 1/e_i)] \geqslant k \geqslant 4.$$

If $3 \leq e_1 \leq e_2 \leq e_3$, then the relationship (42) is satisfied with $n = 3$:

$$3\,(1 - 1/e_i) \geqslant 2, \quad i = 1, 2, 3,$$

$$\sum_1^3 [3\,(1 - 1/e_i)] \geqslant 6.$$

If $e_1 = 2$, $4 \leq e_2 \leq e_3$, then (42) is satisfied for $n = 4$:

$$[4\,(1 - 1/2)] + [4\,(1 - 1/e_2)] + [4\,(1 - 1/e_3)] \geqslant 2 + 3 + 3 = 8.$$

If $e_1 = 2$, $e_2 = 3$, $e_3 \geq 6$, then (42) is satisfied for $n = 6$:

$$[6\,(1 - 1/2)] + [6\,(1 - 1/3)] + [6\,(1 - 1/e_3)] \geqslant 3 + 4 + 5 = 12.$$

Thus there remain the unexamined cases $e_1 = 2$, $e_2 = 3$, $e_3 = 3$, 4, or 5. In these cases (corresponding to the groups of right polyhedra) the relationship of (43) is not satisfied:

$$1 - 1/2 + 1 - 1/3 + 1 - 1/e_3 \leqslant 1\tfrac{29}{30} < 2 \quad \text{for} \quad e_3 \leqslant 5.$$

Lemma 14, and with it the theorem of Enriques, is proved.

Remark. If the genus of the curve F is greater than 1, then for a sufficiently large n the dimension of the space of the differentials ω of nth degree on L such that $\psi^*(\omega)$ is of first order on F takes as large a value as is desired. In other words, if $V = (F \times B)/G$, then $\max P_n(V) = \infty$.

In fact, in this case instead of (43) we have the inequality

$$\theta = \Sigma\,(1 - 1/e_i) - 2 > 0,$$

and if $n \equiv 0 \pmod{e_i}$ for all e_i, then

$$\deg D = \Sigma\,[n\,(1 - 1/e_i)] - 2n = n\theta$$

and hence $l(D)$ grows indefinitely large along with n.

CHAPTER V

MINIMAL MODELS OF RULED AND RATIONAL SURFACES

§1. Basic results

In this section the base field k is assumed to be algebraically closed and of arbitrary characteristic.

The three-tuple (V, π, B), where V is a complete nonsingular surface, B is a a complete nonsingular curve, and $\pi: V \dashrightarrow B$ is a regular epimorphism, is said to be a *geometrically ruled surface* if V over B is birationally equivalent to the product $B \times P^1$ over B (P^1 is a projective line), and $\pi^{-1}(Q) \simeq P^1$ for any point $Q \in B$. For simplicity we will sometimes simply call V the geometrically ruled surface, B its base, π a projection, and the curve $\pi^{-1}(Q) \subset V$ a fiber over the point $Q \in B$.

A *trivial ruled surface* will be a direct product $B \times P^1$ with a canonical projection onto the first factor.

Let V be any surface, $P \in V$ be an arbitrary point, $l \subset V$ be an exceptional curve of the first kind. By the symbols $\mathrm{dil}_P: V \longrightarrow V'$ and $\mathrm{cont}_l: V \longrightarrow V''$ we mean respectively birational mappings of dilation of the point P and contradiction of the curve l.

Let V be a ruled surface, $P \in V$ be an arbitrary point. We denote by l a proper image of the fiber passing through the point P on the surface $V'' = \mathrm{dil}_P V$. It is clear that $l^2 = -1$ and that l is therefore an exceptional curve of the first kind on V''; since the image of l under the composite mapping

$$V'' \xrightarrow{\mathrm{dil}_P^{-1}} V \xrightarrow{\pi} B$$

is a point on B, the rational mapping π' of the surface $V' = \mathrm{cont}_l V''$ on B defined from the commutativity condition of the diagram

$$\begin{array}{ccc} V'' & \xrightarrow{\mathrm{dil}_P^{-1}} & V \\ {\scriptstyle \mathrm{cont}_l}\downarrow & & \downarrow{\scriptstyle \pi} \\ V' & \xrightarrow{\pi'} & B \end{array}$$

is regular. Clearly (V', π', B) is a geometrically ruled surface. We will say that V' is obtained from V by means of an elementary transformation elm_P with center at the point P: $\mathrm{elm}_P = \mathrm{cont}_l \circ \mathrm{dil}_P$: $V \to V'$.

Our main goal is the proof of the following result describing relatively minimal models of ruled surfaces (i.e., models on which there are no exceptional curves of the first kind).

Theorem 1. a) *Let V be an irrational ruled surface, π: $V \to B$ its canonical mapping onto its image in the Albanese variety. The surface V is a relatively minimal model if and only if (V, π, B) is a geometrically ruled surface.*

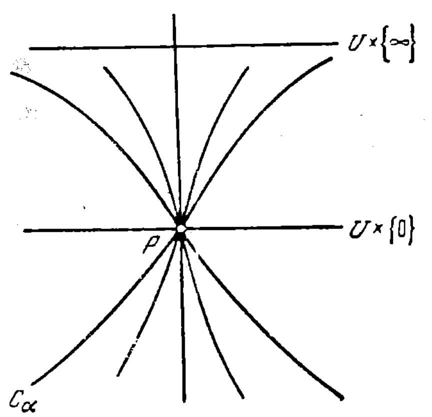
b) *Let V be a rational surface. V is a relatively minimal model if and only if it is isomorphic to either a projective plane P^2, or to some geometrically ruled surface with a projective line as a base.*

c) *Every geometrically ruled surface (rational or irrational) can be obtained from a trivial surface by the successive application of a finite number of elementary transformations.*

This theorem will be proved in the following sections; the cases of irrational and rational surfaces will be treated separately, for they require different methods.[1] Before proceeding to the proof, we note a simple property of elementary transformations, which, together with point b) of Theorem 1, gives us more exact information about ruled surfaces.

Proposition 1. *Let (V, π, B) be a (locally trivial) fiber bundle with fiber P^1 and with a projective structure group. Then the same is true for the surface (V', π', B), where $V' = \mathrm{elm}_P V$, $P \in V$ being any point.*

Proof. It is clear that it is sufficient to verify that, for some Zariski neighborhood U of the point $Q = \pi(P) \in B$ for which $\pi^{-1}(U) \simeq U \times P^1$, the open set $\mathrm{elm}_P(\pi^{-1}(U)) \in V'$ (in an obvious definition) remains isomorphic to $U \times P^1$, where this isomorphism is compatible with the projection onto the first factor $U \times P^1 \to U$. We choose affine coordinates on P^1 such that for a point $P \in U \times P^1 = \pi^{-1}(U)$ on its fiber the second coordinate is zero. We consider some local parameter τ at the point $Q \in B$ and we choose U such that in the neighborhood $U \ni Q$ the function τ does not have any zeroes and poles. We consider on $U \times P^1$ the system of curves



[1] The part of Theorem 1 relating to rational surfaces was proven by Nagata in the article [38]. Elementary transformations were first introduced there. We basically follow his method. The validity of the analogous result for irrational surfaces is apparently new.

$\{C_\alpha\}$, $\alpha \in k \cup \{\infty\}$, setting $C_\alpha = \{(Q', a_T(Q')) \mid Q' \in U\}$, $\alpha \in k$, $C_\infty = U \times \{\infty\}$.

We denote by C'_α the proper image of the curve C_α in the open set $\mathrm{elm}_P(\pi^{-1}(U))$ with respect to the mapping elm_P. It is easy to see that the system of curves $\{C'_\alpha\} \in \mathrm{elm}_P(U \times P^1)$ possesses the following properties:

a) $C'_\alpha \cap C'_\beta = \emptyset$ for $\alpha \neq \beta$;

b) some curve C'_α passes through each point of any fiber $(\pi')^{-1}(P') \in \mathrm{elm}_P(U \times P^1)$, $P' \in U$;

c) every curve C'_α is a section (the image of a section) with respect to the canonical projection π': $\mathrm{elm}_P(U \times P^1) \to U$.

The choice of the sections $\{C'_\alpha\}$ determines an isomorphism $\mathrm{elm}_P(U \times P^1) \simeq U \times P^1$, which is clearly compatible with the projection onto U. The proposition is proved. From it, by Theorem 1, we obtain

Corollary. *Every relatively minimal model of a ruled surface, with the exception of the projective plane, possesses the structure of a fiber bundle with base B, fiber P^1, and a projective structure group.*

Our last result will apply only to the rational surfaces for which the classification of ruled surfaces by Theorem 1c allows one to give a complête description.

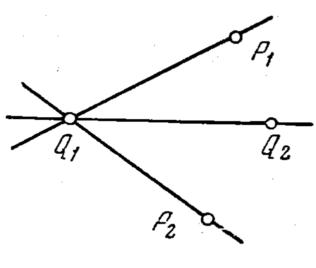
Lemma 1. *Let $P_1, P_2 \in P^1 \times P^1$ be points not lying simultaneously on either a curve of the form $P \times P^1$ or $P^1 \times Q$. Considering $P^1 \times P^1$ as a ruled surface with a projection onto the second factor, we have*

$$\mathrm{elm}_{P_2} \circ \mathrm{elm}_{P_1}(P^1 \times P^1) \simeq P^1 \times P^1.$$

Proof. Let P^2 be the projective plane, Q_1 and Q_2 two points on it, l the proper image of the line Q_1Q_2 on the surface $\mathrm{dil}_{(Q_1,Q_2)}P^2$. The surface $P^1 \times P^1$ is isomorphic to the surface

$$\mathrm{cont}_l \circ \mathrm{dil}\, P_{(Q_1,Q_2)}P^2.$$

Under this isomorphism the points of the lines on P^2 passing through the points Q_1, Q_2 are identified with two systems of generators on the quadric $P^1 \times P^1$. We identify $P^1 \times P^1$ with $\mathrm{cont}_l \circ \mathrm{dil}_{(Q_1,Q_2)}P^2$, and we shall designate by P_1, P_2 both the points figuring in the condition set by the lemma and their biregular images on P^2 with respect to this identification. Then, as it is easy to see,



$$\mathrm{elm}_{(P_2,P_1)}(P^1 \times P^1) = \mathrm{cont}_{(l,l_1,l_2)} \circ \mathrm{dil}_{(Q_1,Q_2,P_1,P_2)}P^2,$$

where l_1, l_2 are images of the lines Q_1P_1, Q_1P_2 respectively. Moreover

$$\mathrm{cont}_{(l,\, l_1,\, l_2)} \circ \mathrm{dil}_{(Q_1,\, Q_2,\, P_1,\, P_2)} = \mathrm{cont}_l \circ \mathrm{dil}_{(Q_1,\, Q_2)} \circ c\,(Q_1, P_1, P_2),$$

where $c(Q_1, P_1, P_2)$ is the standard quadratic transformation of the plane with centers Q_1, P_1, P_2. This yields the required isomorphism and proves the lemma.

Now we are able to establish the following result. On the trivial surface $P^1 \times P^1$ we choose some section $b_0 = P \times P^1$ and a set of points $Q_1, \cdots, Q_n \in b_0$. We set

$$F_0 = P^1 \times P^1,\ F_n = \mathrm{elm}_{(Q_1, \cdots, Q_n)}(P^1 \times P^1),\ \ n \geq 1.$$

We denote by the symbol b_n the proper image of the section b_0 on F_n.

Proposition 2. *Every surface V which can be obtained from $P^1 \times P^1$ by a sequence of elementary transformations is isomorphic to one of the surfaces F_n, $n \geq 0$. The surface F_n is determined by the number n up to isomorphism.*

Proof. It is clear, in view of the uniformity of the surface F_0, that $\mathrm{elm}_Q F_0 = F_1$ does not depend on the position of the point Q.

We perform an induction on the number of elementary transformations applied to F_0. We assume that the application of not more than n transformations leads to a surface F_r, $r \leq n$; we shall show that the application of another transformation to F_r leads either to the surface F_{r-1} or to F_{r+1}, thus completing the induction.

It is possible to assume that $r \neq 0$. If $Q \in b_r$, then $\mathrm{elm}_Q F_r \simeq F_{r+1}$. If $Q \notin b_r$, then, using Lemma 1 and the permutability of elementary transformations at distinct points, we obtain that $\mathrm{elm}_Q F_r \simeq F_{r-1}$.

This argument is not valid if Q lies on one of the fibers obtained as a result of preceding birational transformations. Then it is necessary to apply another pair of elementary transformations with centers Q_1 on b_r and Q_2 outside of b_r lying on distinct fibers and not on those which appeared as a result of preceding elementary transformations; then we apply Lemma 1 to the transformation $\mathrm{elm}_{(Q_1, Q_2)}$. This argument shows that F_n depends only on n and not on the choice of the points $Q_1, \cdots, Q_n$. The proposition is proved.

Proposition 3. *The curve b_n is the only irreducible curve on the surface F_n with a negative index of self-intersection: $b_n^2 = -n$.*

Corollary. *Any relatively minimal ruled surface is isomorphic to some F_n, $n \neq 1$; the surfaces F_n, F_m for $m \neq n$ are not isomorphic.*

Proof. The equality $b_n^2 = -n$ follows from the definition of F_n. Let s be any fiber of F_n; the classes b_n and s generate the group of one-dimensional cycles F_n up to linear equivalence; this is obvious for F_0, and the situation does not change under the application of elm_P. Let $C \subset F_n$ be any irreducible curve that does not coincide with b_n or s; we set $(C, s) = d$, $(C, b_n) = d_0$. Since $s^2 = 0$, we have

$$C \sim db_n + (d_0 + dn)\, s,$$

from which

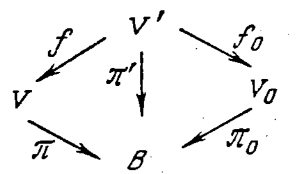
$$C^2 = 2d\,(d_0 + n) \geqslant 0,$$

follows since $d \geq 0$, $d_0 \geq 0$. The assertion is proved.

Remark. Since any rational ruled surface $F_n \xrightarrow{\pi} P^1$ has a canonical section with image b_n, it can be regarded as a fiber bundle with an affine structure group.

§2. Proof of Theorem 1 for irrational surfaces

Let $V \xrightarrow{\pi} B$ be a nonsingular ruled surface together with the mapping onto its image in its Albanese variety; let $V_0 = B \times P^1 \longrightarrow B$ be a trivial model of the field $k(V)$. There exists a nonsingular model V' of the field $k(V)$ such that the following diagram is commutative:



(1)

where the maps of V, V', and V_0 on B are natural, and the maps $V' \longrightarrow V$ and $V' \longrightarrow V_0$ are regular; in fact, one may take as V' a nonsingular model dominating the graph of the birational correspondence between V and V_0.

Lemma 2. *Every exceptional curve l of the first kind on V' is an irreducible component of the preimage of some point $Q \in B$.*

Proof. This is clear, for l is a rational curve and the genus of B is greater than or equal to 1. (We note that it is essential to use the irrationality of the surface V here; for a rational surface not only the proof but also the result ceases to be valid, which is the main reason for the complication of the situation.)

Corollary. *A ruled irrational surface V contains only a finite number of exceptional curves of the first kind.*

(This corollary is also invalid for rational surfaces.)

Lemma 3. *The preimage $(\pi')^{-1}(Q) \in V'$ of every point $Q \in B$ is connected and is the union of nonsingular rational curves.*

Proof. In fact, $(\pi')^{-1} = f_0^{-1} \circ \pi_0^{-1}$; moreover, $\pi_0^{-1}(Q) \simeq P^1$, and f_0^{-1} decomposes into the product of a finite number of dilations, each of which adds a nonsingular rational curve and does not destroy the connectedness. The lemma is proved.

By a *weighted tree* we shall mean a finite tree (graph without cycles), to each vertex of which is ascribed an integer, its weight. To each preimage of a point $(\pi')^{-1}(Q) \subset V'$ we set in correspondence the graph D'_Q in which the vertices are in one-to-one correspondence with irreducible components of the fiber $(\pi')^{-1}(Q)$, two vertices are joined by a simplex if and only if the corresponding components intersect, and with each vertex there is associated a negative integer — the index of intersection of the corresponding component taken with the opposite sign. It is clear that D'_Q is a weighted tree (here the weights are negative). In the future we shall consider only weighted trees and shall simply call them trees.

Let D be some tree. We define two operations allowing us to obtain from D a new weighted tree D'.

We add to D one new vertex with a weight of 1; we join it by one simplex to an old vertex of the tree D and we increase the weight of this old vertex by one. We shall say that the new tree D' thus constructed is obtained from D by an elementary dilation of the first kind, or that D is obtained from D' by an elementary contraction of the first kind.

We add to D one new vertex with a weight of 1, we join it by two simplexes to two vertices of the tree D that were joined in D by a simplex, we remove this simplex and increase the weight of each of its vertices by one. We shall say that the new tree D' is obtained from D by an elementary dilation of the second kind, or that D is obtained from D' by an elementary contraction of the second kind. The justification of these definitions lies in the following obvious result.

Lemma 4. *Let f: $V' \to V$ be a regular mapping contracting an exceptional curve l on V' of the first kind that lies over a point $Q \in B$; let D'_Q, D_Q be the trees of the preimages of the point Q on V' and V respectively. Then D'_Q is obtained from D_Q by an elementary dilation of the first or second kind depending upon whether the image $f(l)$ lies on one irreducible component of the preimage of Q or on the intersection of two such components. Conversely, applying an elementary dilation or contraction to the tree D'_Q we obtain the tree corresponding to the preimage of Q on the surface obtained from V' by a dilation of a point or the contraction of some component of a fiber over Q.*

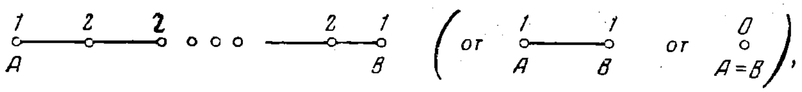
The product of elementary dilations will simply be called a dilation of a tree; analogously, the product of elementary contractions is called a contraction.

Let $Q \in B$; the tree D'_Q of the preimage of Q on V' is obtained by a dilation of the tree consisting of one vertex with a weight of zero which corresponds to the preimage of Q on V_0. The mapping f then induces a contraction of the tree D'_Q, where, if the model V is minimal, there cannot be a vertex with a weight of one in the image D_Q of the tree D'_Q. We shall show in particular that then D_Q again consists of one vertex with a weight of zero, i.e., the preimage of the point Q on V is a line. Our following arguments will use weighted trees and we shall interpret them with the aid of Lemma 4.

A *simple tree* is defined to be a connected tree, the weight of each of whose vertices is equal to the number of simplexes emanating from the vertex. A terminal vertex is a vertex of weight 1, and a terminal simplex, a simplex that has a vertex of weight 1. A simple tree is completely defined by its graph.

Lemma 5. a) *A tree is simple if and only if it is obtained from a one-vertex simple tree by a sequence of elementary dilations of the first kind. Each contraction of a simple tree yields a simple tree.*

b) *Let D be a simple tree, and A and B be two of its vertices. Then there exists a simple tree of the form*



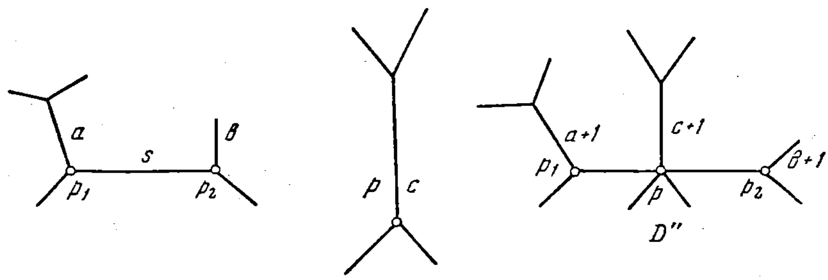
(2)

which is obtained from D by a contraction and from which it is possible to obtain A and B by contractions.

Proof. The first two statements follow quickly from the definitions by induction on the number of vertices.

The last assertion is also almost obvious: we join A with B by a sequence of simplexes and apply to D successively elementary contractions until there remain no terminal vertices other than A and B in the tree obtained. This is possible, for by the definition of a simple tree and of an elementary contraction, the latter destroys terminal complexes and only those. But every simple tree having only the terminal vertices A and B has one of the forms shown in diagram (2). The lemma is proved.

Now let D be an arbitrary tree and D' a simple tree. We choose any simplex s of the tree D and any vertex p of the tree D'. We join the vertex p by two simplexes to the ends p_1, p_2 of the simplex s, we remove the simplex s and increase the former weights of the vertices p, p_1, and p_2 by one each. We shall say that the new tree D'' thus constructed is obtained by a grafting of the tree D' to D; the vertex p will be called the *grafted* vertex.



Lemma 6. *The tree D is obtained by a dilation of a one-vertex simple tree if and only if it is obtained from some simple tree D_0 by a sequence of graftings of simple trees.*

Proof. Let D be obtained by a sequence of graftings of simple trees. We contract the last grafted tree to the one of its terminal vertices to which it was grafted; then we remove this terminal vertex by an elementary contraction of the second kind. The new tree is a contraction of D and is obtained by a sequence of a smaller number of graftings; induction on the number of graftings permits one to contract D to a simple tree, which contracts to a one-vertex simple tree by Lemma 5b.

To prove the converse we use induction on the number of vertices. A one-vertex simple tree satisfies the assertion. Let it be true for trees D with a number of vertices r. A tree D' with $r+1$ vertices is obtained from some tree D by an elementary dilation. If it is of the second kind, then this is a grafting of a one-vertex simple tree.

We now assume that D' is obtained from D by an elementary dilation of the first kind. Then the new vertex is joined by a simplex with an old one belonging precisely to one of the simple trees that was successively grafted. It is easy to see that it is possible to add this vertex at the beginning, and then graft the whole tree: essentially this reduces to the associativity of addition of the integers. The lemma is proved.

We denote by Δ the class of trees obtained by a dilation from a one-vertex simple tree.

Lemma 7. a) *The contraction of any tree of the class Δ belongs again to Δ.*

b) *If a tree of the class Δ does not have a vertex of weight 1, it is a one-vertex simple tree.*

Proof. It is necessary to verify the first assertion for elementary contractions; for this it is sufficient to use the argument of the second half of the proof of Lemma 6 in the opposite direction. The second assertion follows easily from Lemma 6.

In fact, the tree grafted last always has at least one vertex of weight 1 unless this last tree consists of only one vertex. The lemma is proved.

Lemma 8. *Let a tree D of the class Δ be obtained by a sequence of graftings to the tree D_0. If the tree D can be contracted to its vertex A, then A belongs to D_0.*

Proof. Let us assume that A belongs to D_1, one of the grafted trees of the sequence different from D_0. Let $p \in D_1$ be a grafted vertex of this tree. If $p \neq A$, the weight of p in the tree D is not larger than two, for D belongs to the class Δ; since this remains true by Lemma 7a for any contraction of the tree D containing A, we can never contract the vertex p without first contracting A, so this case is impossible.

Let us now assume that the tree D consists of a single vertex A. In the process of contracting D to the vertex A we must at some point arrive at a tree which, in view of Lemma 7a, belongs to the class Δ and is obtained by a graft of A to a simple tree D_0' that is a contraction of the tree D_0. But the tree D_0' contains a simplex onto which the vertex A is grafted, and the weight of the vertices of this simplex is greater than or equal to 2, so it is necessary to contract them without first contracting A. The contradiction obtained proves the lemma.

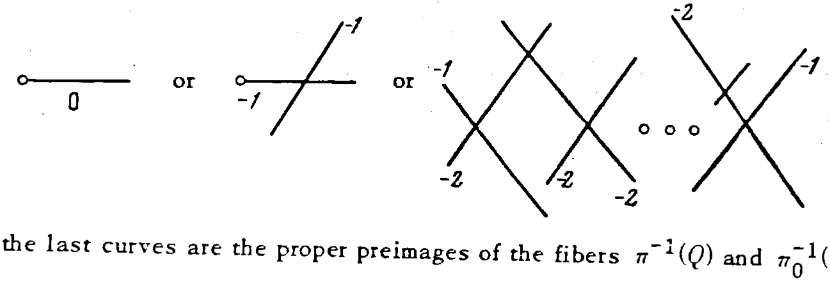
Corollary. *Let a tree D belong to the class Δ and be contractable to two of its vertices A, B. Then D is contractable to a tree of form* (2).

Proof. In view of Lemma 8 the vertices A and B belong to a simple tree D_0. We can contract the tree D to D_0, and then apply Lemma 5b to D_0.

We shall now show that the geometrical interpretation of the results obtained leads to a proof of the part of Theorem 1 that refers to irrational surfaces.

In fact, let the surface V in diagram (1) be a relatively minimal surface. Then the tree of each fiber $\pi^{-1}(Q)$ cannot contain vertices of weight one. On the other hand, such a tree is the contraction of a tree of the fiber $(\pi')^{-1}(Q)$ belonging to the class Δ. By Lemma 7b the tree of the fiber $\pi^{-1}(Q)$ has only one vertex, i.e., $\pi^{-1}(Q) \simeq P^1$.

Further, in the tree D_Q' of the fiber $(\pi')^{-1}(Q)$, let A and B be the vertices corresponding to the proper preimage of the fibers $\pi^{-1}(Q)$ and $\pi_0^{-1}(Q)$. Then by the corollary of Lemma 8 and by Lemma 4, the surface V' can be chosen so that the fiber $(\pi')^{-1}(Q)$ will have the form



where the last curves are the proper preimages of the fibers $\pi^{-1}(Q)$ and $\pi_0^{-1}(Q)$

respectively. But this is clearly the restriction over Q of the graph of the product of the elementary transformations applied to V_0.

All our assertions are thus proved.

§3. Cycles over a surface

We introduce several definitions.[1] Let us agree temporarily to identify every surface V with the subset of the local subrings of the field $k(V)$ whose elements are local rings of the points of V (rational over the field k). In this way distinct models of the field $k(V)$ are identified with distinct sets of local subrings; as is known, it is possible to characterize the permissible sets axiomatically.

We shall say that a point P' of one of the models *dominates* a point P of another model, and shall write $P' > P$, if the relationship of dominance is satisfied for the corresponding local rings. Every point dominating some point of a model V of the field $k(V)$ (in particular every point of V) will be said to be a point (lying) *over* V.

For every point P over V there exists a unique point $Q \in V$ for which $P > Q$. If $P = P_n > P_{n-1} > \cdots > P_0 = Q$ is a sequence of maximal paths, then according to classical terminology the point P is infinitely close to Q of order n. We shall say that the point P lies over the point Q.

A cycle over V is an element of the free abelian group generated by the irreducible curves on V and by the points over V. Thus the group of cycles is a direct sum of the group of divisors on V and the group of zero-dimensional cycles over V.

For any cycle $C - \Sigma m_i P_i$ (in the notation of this form we shall always assume that C is a divisor and P_i is a point over V) we define the *arithmetic genus*, setting

$$p_a(C - \Sigma m_i P_i) = p_a(C) - \Sigma m_i (m_i - 1)/2.$$

For each pair of cycles $C - \Sigma m_i P_i$, $D - \Sigma n_i P_i$ we define their index of intersection, setting

$$(C - \Sigma m_i P_i, D - \Sigma n_i P_i) = (C, D) - \Sigma m_i n_i.$$

(In other words, the index of intersection is bilinear, the subgroups of divisors and zero-dimensional cycles are orthogonal, and finally $P^2 = -1$ and $(P, Q) = 0$ for $P \neq Q$.)

Let $Z(V)$ be the group of cycles over V (from now on all the surfaces are

[1] The presentation of the theory of ruled systems with the base conditions prescribed in this section is a variant of a method of Nagata [39].

complete and nonsingular). The group $Z(V)$ possesses strong functional properties with respect to the birational transformations, which one may summarize in the following way.

Let $f = \mathrm{dil}_P\colon V \to V'$. We define a homomorphism (not preserving the dimension of a pure cycle!) $f_*\colon Z(V) \to Z(V')$, setting $f_*(P) = \mathrm{dil}_P P$, $f_*(Q) = f(Q)$ for $Q \neq P$, and denoting by $f_*(C)$ the total image of the curve C with respect to f. The following assertion is true.

Proposition 4. *For every birational mapping $f\colon V \to V'$ there exists a homomorphism $f_*\colon Z(V) \to Z(V')$ which can be defined by decomposing f into a product of mappings of the form dil_P and cont_l and using the formula $(g \circ h)_* = g_* \circ h_*$. The homomorphism f_* is determined uniquely; $Z(V)$ is transformed by it into a covariant functor on the category of complete nonsingular surfaces over k in which the morphisms are birational mappings.*

Lemma 9. *For any birational mapping $f\colon V \to V'$ and any cycles $Z, Z' \in Z(F)$, the following identities are true:*

$$(Z, Z') = (f_*(Z), f_*(Z')), \tag{3}$$

$$p_a(Z) = p_a(f_*(Z)). \tag{4}$$

Proof. By Proposition 4 we may assume that $f = \mathrm{dil}_P$. Then both formulas are easily obtained from the formulas describing the behavior of the arithmetic genus and the index of intersection of *divisorial* cycles under the dilation of a point.

We now note that for any function $\phi \in k(V)$ and any birational mapping $f\colon V \to V'$ the relationship

$$f_*((\phi)) = (f_*\phi),$$

holds, where $f_*\phi$ is the function on V' induced by ϕ. Consequently, the subgroup of the principal divisors $Z_l(V)$ in the group $Z(V)$ is mapped isomorphically under all the homomorphisms f_*. By the same token, f_* is defined on the factor group $Z(V)/Z_l(V)$. The situation is analogous with the arithmetic genus and the index of intersection, which, as it is easy to see, depend only on the class of cycles $\mathrm{mod}\, Z_l(V)$. The coset of the cycle Z over the subgroup $Z_l(V)$ will be designated by $\|Z\|$ and is sometimes called a linear system. Linear equivalence of cycles is denoted as usual by $\sim$.

As an example we describe the operation $(\mathrm{elm}_P)_*$ on F_n; we shall need this result in the future. We note that the group $Z(F_n)/Z_l(F_n)$ is generated by the classes of the base b_n, the fiber s_n, and all the points over F_n.

Lemma 10. *Let $f = \mathrm{dil}_P\colon F_n \to F_{n\pm 1}$. Then we have:*

a) *For $P \in b_n$:*

$$f_*(b_n) \sim b_{n+1} + s_{n+1} - P^*,$$
$$f_*(s_n) \sim s_{n+1},$$
$$f_*(P) \sim s_{n+1} - P^*,$$

where P^* *is the proper image of the fiber* F_n *that passes through the point* P;

b) *for* $P \notin b_n$:

$$f_*(b_n) \sim b_{n-1} - P^*,$$
$$f_*(s_n) \sim s_{n-1},$$
$$f_*(P) \sim s_{n-1} - P^*,$$

where P^* *is the proper image of the fiber* F_n *that passes through the point* P.

One proves this lemma by direct applications of the definitions.

§4. Proof of Theorem 1 for rational surfaces

We shall now prove the following result, which, in conjunction with Propositions 2 and 3, establishes the validity of Theorem 1 for rational surfaces.

Proposition 5. *Every nonsingular rational surface that does not have exceptional curves of the first kind is isomorphic either to the projective plane* P^2 *or to one of the surfaces* F_n, $n \neq 1$.

We first establish a series of useful lemmas.

Lemma 11. *Let* $L \subset P^2$ *be the linear equivalence class of the line on the projective plane. The linear system* $M = dL - \Sigma m_i P_i$ $(d \neq 0)$ *of cycles over* P^2 *cannot simultaneously possess the following properties:*

1) $M^2 = -1$, $p_a(M) = 0$,
2) $m_0 \geq m_1 \geq \cdots \geq m_k \geq 0$,
3) $m_0 \geq \Sigma_{i=1}^h m_i$ *for some* h, $1 \leq h \leq k$,
4) $m_0 + m_1 \leq d$,
5) $m_0 + 2m_{h+1} \leq d$.

Proof. The relationships 1) can be rewritten in the form

$$d^2 - \sum_{i=0}^{k} m_i^2 = -1,\ d(d-1) - \sum_{i=0}^{k} m_i(m_i - 1) = 0.$$

From this follows the equation

$$3d - \sum_{i=0}^{k} m_i = 1.$$

Using properties 2) and 3), we find

$$1 = 3d - \sum_{i=0}^{k} m_i \leqslant 3d - m_0 - m_1 - \sum_{i=h+1}^{k} m_i,$$

from which we have

$$\sum_{i=h+1}^{k} m_i + 1 \leqslant 3d - m_0 - m_1. \tag{5}$$

But $m_{h+1} \geq 0 > -1$, so $d^2 < \Sigma_{i=0}^k m_i^2 + m_{h+1}$; using properties 2), 3), and 4) and the inequality (5), we find

$$\begin{aligned} d^2 < \sum_{i=0}^{k} m_i^2 + m_{h+1} &\leqslant m_0^2 + m_1 \sum_{i=1}^{h} m_i + m_{h+1} \sum_{i=h+1}^{k} m_i + m_{h+1} \\ &\leqslant m_0^2 + m_1 \sum_{i=1}^{h} m_i + m_{h+1}(3d - m_0 - m_1) \\ &\leqslant m_0^2 + m_1 m_0 + m_{h+1}(3d - m_0 - m_1) \\ &= (m_0 + m_1)(m_0 - m_{h+1}) + 3dm_{h+1} \\ &\leqslant d(m_0 - m_{h+1}) + 3dm_{h+1} = d(m_0 + 2m_{h+1}), \end{aligned}$$

which contradicts property 5), because $d \neq 0$.

Lemma 12. *Let* $C - \Sigma_{i=1}^k m_i P_i$ *be a linear system over* F_n *possessing the properties*

1) $(C, b_n) = d_0 \geq 0$, $(C, s_n) = d \geq 0$,

2) $0 \leq m_i \leq d/2$ *for all* i, $0 \leq i \leq k$,

3) $C^2 = -1$, $p_a(C) = 0$.

Then either $n = 1$, C *is the class* $\|b_1\|$, *and all the* $m_i = 0$; *or* $C = 0$, $k = 0$, *and* $m_0 = 1$.

Proof. We may assume that the points P_i are numbered so that $m_i \geq m_{i+1}$, $0 \leq i \leq k-1$. Let us assume first that $m_0 \neq 0$ and either $d \neq 0$ or $d_0 \neq 0$.

1) Let $n \geq 2$. We perform elementary transformations with centers in the $(n-1)$-st point in general position (not lying on the base b_n). Here, by Lemma 10, the system $C - \Sigma_{i=0}^k m_i P_i$ goes into the following system on the surface F_1:

$$C' = \| (dn + d_0) s_1 + db_1 - \sum_{i=0}^{n-1} dP_i^* - \sum_{i=0}^{k} m_i P_i \|$$

(we denote the point P_i and its proper image by the same symbol). After a transformation cont_{b_1} the system C' goes into the system

$$C'' = (nd + d_0) L - ((n-1)d + d_0) Q - \sum_{i=0}^{n-1} dP_i^* - \sum_{i=0}^{k} m_i P_i$$

on the plane P^2, where Q is the image of b_1, and biregularly corresponding points, as always, are designated by the same letter. The system C'' possesses all the properties listed in Lemma 11, with $h = n$. Consequently the case under consideration is elimated.

2) We now assume that $n = 0$. By the symmetry of F_0 with respect to a permutation of the direct factors, it is possible to assume $d_0 \geq d$. Under the operation of the transformation $(\mathrm{elm}_P)_*$, $P \neq P_i$, $0 \leq i \leq k$ the system

$$C - \sum_{i=0}^{k} m_i P_i = \| db_0 + d_0 s_0 - \sum_{i=0}^{k} m_i P_i \|$$

goes into the system

$$\| (d + d_0)\, s_1 + db_1 - dP^* - \Sigma m_i P_i \| \text{ on } F_1.$$

We then apply the transformation $(\mathrm{cont}\, b_1)_*$; as a result we arrive at the system

$$(d + d_0)\, L - d_0 Q - dP^* - \sum_{i=0}^{k} m_i P_i$$

on the plane P^2, which satisfies the conditions of Lemma 11 with $h = 1$. The contradiction obtained elimates this case also.

3) It remains to consider the possibility $n = 1$.

If $m_0 \leq d_0$, we apply the transformation cont_{b_1} to our system. The resulting system

$$(d + d_0)\, L - d_0 Q - \sum_{i=0}^{k} m_i P_i$$

satisfies the conditions of Lemma 11 with $h = 1$ and leads to a contradiction.

The case $m_0 > d_0$ is reduced to the case $m_0 \leq d_0$ by the transformation $\mathrm{dil}_{P_0} \circ \mathrm{cont}_{b_1}$. In fact, our system then goes into the system

$$\| (d + d_0 - m_0)\, s_1 + m_0 b_1 - d_0 P - \sum_{i=1}^{k} m_i P_i \|$$

on the surface F_1.

Thus we are not lead to a contradiction only in the case when $n = 1$ and either $m_0 = 0$ or $d = d_0 = 0$.

If $m_0 = 0$, we have $C = \|b_1\|$ by condition 1) of Lemma 12 and Proposition 3.

If $d = d_0 = 0$, then $m_0 = 1$, $m_i = 0$ for $i \geq 1$, since $\Sigma m_i^2 = 1$. The lemma is proved.

Lemma 13. *Every linear system on a surface* F_n

$$\| db_n + d_0 s_n - \sum_{i=0}^{k} m_i P_i \|, \; m_i \geqslant 0,$$

can be taken by a sequence of elementary transformations into a system on a surface F_r *of the form*

$$\| db_r + d_0' s_r - \sum_{i=0}^{k} m_i' P_i' \|, \quad d \geqslant 0,$$

where $0 \le m_i' \le d/2$ *for all* i, $0 \le i \le k'$ *for which* $P_i' \in F_r$.

Proof. Let us assume that $m_0 > d/2$. We apply the transformation elm_{P_0}; by Lemma 10 our system then goes into the system

$$\| db_{n+1} + (d + d_0 - m_0)\, s_{n+1} - (d - m_0)\, P^* - \sum_{i=1}^{k} m_i P_i \|,$$

if $P_0 \in b_n$, or into the system

$$\| db_{n-1} + (d_0 - m_0)\, s_{n-1} - (d - m_0)\, P^* - \sum_{i=1}^{k} m_i P_i \|,$$

if $P_0 \notin b_n$. Since $d - m_0 < d/2$, the collection of coefficients m_i with $m_i > d/2$ is reduced by one. Continuing in the same way, we finally arrive at the required system. The lemma is proven.

For the formulation of the last lemma we introduce the following definition. A cycle $C - \Sigma m_i P_i$ over V is *faithful*, if the following conditions are satisfied:

a) if $P_i > P_j$, then $m_i \le m_j$,

b) $C > 0$, and for any point $P_i \in V$, $m_i \le m(P_i, C)$ (the multiplicity of the point P_i on the cycle C).

Lemma 14. *Let* $f\colon V \to V'$ *be a regular mapping and* $C - \Sigma m_i P_i$ *a faithful cycle over* V, *where none of the components of* C *is contracted by the mapping* f. *Then* $f_*(C - \Sigma m_i P_i)$ *is a faithful cycle over* V'.

Proof. It is sufficient to verify this for the case $f = \mathrm{cont}_l$. Then we have $f_*(C) = C' - (C, l)P$, where $P = \mathrm{cont}_l\, l$ and C' is the proper image of the cycle C.

Condition a) is satisfied, because if $P_i > P$, then P_i lies over the point $P_k \in l$, and thus, $(C, l) \ge m(P_k, C) \ge m_k \ge m_i$. Condition b) is satisfied since $(C, l) = m(P, C)$. The lemma is proved.

It will be necessary for us to apply this lemma in the case when the cycle over V is simply an effective divisor of C. We note that in this case the lemma is true even if the f is not assumed to be regular, if fundamental points of f do not lie on C: in fact, $f_*(C) = f_{2*} \circ f_{1*}(C)$, where f_1 is a dilation such that $f_{1*}(C)$ is an effective divisor, and f_2 is a regular mapping of contraction.

We now turn to the proof of Proposition 5. Let us assume that there exists a nonsingular surface V birationally dominating neither P^2 nor F_n. Then there exists a similar surface $\overline{V}$ such that $\overline{V} = \mathrm{dil}_P V$ for some point $P \in V$ already dominates either P^2 or F_n. It is always possible to assume that $\overline{V}$ dominates F_n, for if $\overline{V}$ dominates P^2, then there is on P^2 a fundamental point Q that does not belong to the image $l = \mathrm{dil}_P(P)$, so that $\overline{V}$ dominates $\mathrm{dil}_Q P^2 \simeq F_1$.

Let $f\colon \overline{V} \to F_n$ be the mapping of dominance under consideration. We consider the system

$$f_{*}(\|l\|) = (dn + d_0)\, s_n + db_n - \sum_{i=0}^{h} m_i P_i \tag{6}$$

on the surface F_n. All the points P_i lying on F_n are fundamental with respect to f. Applying a transformation elm_P with center at such a point and defining a mapping g by the commutativity of the diagram

$$\begin{array}{ccc} \overline{V} & \searrow^{g} & \\ f\downarrow & & \\ F_n & \xrightarrow{\mathrm{elm}\, P} & F_{n\pm 1} \end{array}$$

we obtain a new dominance mapping g. Using Lemma 13 we can thus prove the inequalities $m_i \le d/2$ for all i for which $P_i \in F_n$; then by Lemma 14 these inequalities are valid for all i. We shall assume that this is already true for the mapping f.

Using the invariance of the arithmetic genus and the index of intersection under birational transformations, we can then apply Lemma 12 to the system (6) (the inequalities $d \ge 0$ and $(dn + d_0) \ge 0$ follow from the fact that the system $\|(dn + d_0)s_n + db_n\|$ contains the proper f-image of the curve l).

If $n = 1$, $\|(dn + d_0)s_n + db_n\| = \|b_1\|$, and $m_i = 0$, then $\overline{V}$ dominates $\mathrm{cont}_{b_1} F_1 = P^2$, and thus V dominates P^2 in spite of the premise.

If $d = d_0 = 0$, then the mapping $f: \overline{V} \to F_n$ contracts l into a point, and thus V dominates F_n, also contradictory to the premise.

The proof of Proposition 5 and Theorem 1 is completed.

§5. Numerical invariants

In this section and the following one we shall prove the well-known theorem of Noether about the structure of the group of birational automorphisms of a projective plane P^2. The proof given below is a variant of a proof by Alexander [3].

Theorem 2. *Every birational automorphism of a projective plane over an algebraically closed field of constants can be represented in the form of the product of a projective automorphism and standard quadratic Cremona transformations (we recall that the standard quadratic Cremona transformation $c(P_1, P_2, P_3)$ with center in the triple of points on the plane P^2 takes each point into the line passing through the other two points).*

First of all we describe the effect of the transformation $c(P_1, P_2, P_3)$ on the linear system $dL - \Sigma_{i=1}^{k} m_i P_i$ of cycles over P^2 (L is the system of lines on P^2). The result below is verified by reference to the definition of f_*: it is necessary to represent $c(P_1, P_2, P_3)$ in the form of the product of three dilations and three contractions and to consider how each of them effects the points and the lines.

Lemma 15. *Setting* $a = d - (m_1 + m_2 + m_3)$, $c = c(P_1, P_2, P_3)$, *we have*

$$c_* \left(dL - \sum_{i=1}^{k} m_i P_i\right) = (d + a) L - \sum_{i=1}^{3} (m_i + a) P_i + \sum_{i \geqslant 4} m_i P_i^*,$$

where P_i^* *is the proper image of the point* P_i.

Now let $f: P^2 \to P^2$ be some birational automorphism of the plane. The basic object with which we shall work is the linear system

$$f_* (L) = dL - \sum_{i=0}^{k} m_i P_i. \tag{7}$$

The points P_i, $i = 0, \cdots, k$ will be called fundamental points of the system (7). The choice of the numbers $(d, m_0, \cdots, m_k)$ is a numerical characteristic of the automorphism f, and we shall first study its properties. The number d is called the degree of the automorphism f and of the system $f_*(L)$.

First, taking into account that f_* preserves the arithmetic genus and the index of self-intersection, we obtain

$$\sum_{i=0}^{k} m_i^2 = d^2 - 1, \tag{8}$$

$$\sum_{i=0}^{k} m_i (m_i - 1) = (d - 1)(d - 2). \tag{9}$$

We renumber the points P_i so that the inequalities $m_0 \geq m_1 \geq \cdots \geq m_k$ are satisfied, and we call m_0 the highest multiplicity of the system (7). From equation (8) it follows that for $d = 1$ all the m_i are equal to zero, so f is a projective automorphism.

From now on we shall always assume that $d \geq 2$, and, consequently, $m_0 \neq 0$ and $m_1 \neq 0$. By Lemma 14 and the application to it of the equations of (7), it follows that the point P_0 lies *on the plane* P^2, and, moreover, that $d > m_0$.

We define the number $j \geq 1/2$ by the formula

$$2j = d - m_0$$

and the integer h by the conditions

$$m_h > j \geq m_{h-1}.$$

Lemma 16.

$$\text{a) } 2j \geq m_1; \quad \text{b) } h \geq 2.$$

Proof. The inequality a) is equivalent to the inequality $m_0 + m_1 \leq d$. For the proof of the latter we note that by (7)

$$d - m_0 - m_1 = (f_*(L), L - P_0 - P_1),$$

while the index of intersection on the right is nonnegative, because it can be calculated on the surface $\mathrm{dil}_{(P_0, P_1)} P^2$, where the total image of the cycle $l - P_0 - P_1$ (l is the line passing through the points P_0, P''_1) coincides with the proper image l' of the line l; i.e., it is a curve, and in the image of the system L there exists a cycle whose divisorial part is effective and does not contain l' as a component.

For the proof of the inequality b) we multiply equation (9) by some number s and equation (8) by $1 - s$, and we add the relationships obtained:

$$\sum_{i=0}^{k} m_i (m_i - s) = (d - 1)(d - 3s + 1). \tag{10}$$

In equation (10) we set $s = j$, we remove from the left all the terms with $i > h$, which are nonpositive, and we subtract $3j - 1$ on the right:

$$\sum_{i=0}^{h} m_i (m_i - j) > d(d - 3j) = d(m_0 - j).$$

Taking the first member to the right-hand side, we obtain $\Sigma_{i=1}^{h} m_i (m_i - 1) > 2j(m_0 - j)$. Taking into consideration that $2j \geq m_1 \geq m_i$ we find

$$\sum_{i=1}^{h} (m_i - j) > m_0 - j. \tag{11}$$

Since $m_1 - j \leq m_0 - j$, it follows from this that $h \geq 2$. The lemma is proved.

Remark. It follows from (11) that $\Sigma_{i=1}^{h} m_i > d - (h - 3)j$, and hence

$$\sum_{i=1}^{h} m_i > d, \quad \text{if} \quad h \geqslant 3. \tag{12}$$

Further, since $h \geq 2$ and m_r, $m_s \geq j$ for any r, $s \leq h$, we have

$$m_0 + m_r + m_s > m_0 + 2j = d. \tag{13}$$

We quickly see that this inequality is the motivation of the introduction of the number h.

§6. Simplification algorithms

We shall now study the effect on the numerical characteristic of the system (7) of a standard Cremona transformation, one of whose centers is the point P_0.

If there were among the points $P_0, \cdots, P_h$ two distinct from P_0 and lying on P^2, say P_r and P_s, then, applying the mapping $(c(P_0, P_r, P_s))_*$ to $f_*(L)$ and using Lemma 15 and inequality (13), we would quickly arrive at a system whose degree would be less than d. This condition, however, is far from being always satisfied, and for an unfavorable distribution of the points $P_0, \cdots, P_h$ on and over the plane it is only possible to "simplify" the system $f_*(L)$ by using a specially chosen sequence of quadratic transformations. Here it is more convenient to follow the change of the parameters j and h than that of d.

We shall now describe in a series of lemmas the simplifying transformations depending on the distribution of the points P_i. We first note that under the application of a standard quadratic transformation c with one of its centers P_0 (which are the only kinds of transformations we shall use), the parameter j can either retain its original value or be reduced.

In fact, let d' and m' be respectively the degree and the highest multiplicity of the new linear system. Then

$$2j = d - m_0 = (f_*(L), L - P_0) = (c_*(L), c_*(L - P_0)).$$

But by Lemma 15

$$c_*(L - P_0) = L - P_0.$$

Thus the last index of intersection is equal to $d' - m_0'$, where m_0' is the new multiplicity of the point P_0. But

$$d' - m_0' \geqslant d' - m' = 2j'.$$

This establishes the validity of our remark. We note that $j' < j$ if and only if P_0 ceases to be the point of the highest multiplicity.

We shall say that a system of the form (7) with parameters (j', h') is simpler than a system with parameters (j, h) if $(j', h') < (j, h)$ with respect to the lexicographical order. In all the following lemmas we shall apply a standard quadratic transformation to the system (7) and shall assume that the degree of the system obtained is ≥ 2, without mentioning this specifically.

Lemma 17. *Let us assume that in the system* (7) *two distinct points of the set* $P_1, \cdots, P_n$, *say* P_r *and* P_s, *lie on the plane* P^2. *Then the system* $c_* \circ f_*(L)$, $c = c(P_0, P_r, P_s)$ *is simpler than the system* (7).

Proof. By Lemma 15 we have

$$c_* \circ f_*(L) = (d + a)L - (m_0 + a)P_0 - (m_r + a)P_r - (m_s + a)P_s - \Sigma m_i P_i^*,\ i \neq 0, r, s,$$

where $a = d - m_0 - m_r - m_s$. If P_0 ceases to be the point of maximal multiplicity, then the new system is simpler than the old one by the remark made above, for then $j' < j$. Otherwise, $j' = j$, but then $h' = h - 2$, because $m_r + a = d - m_0 - m_s = 2j - m_s < j = j'$, and analogously $m_s + a < j'$, whereas the remaining multiplicities do not change. The lemma is proved.

Lemma 18. *Assume that it is impossible to find any points* P_r, P_s *satisfying the conditions of Lemma* 17. *Let* A *and* B *be two points on* P^2 *such that none of* *the points* P_i, $1 \le i \le k$, *lies on the lines* P_0A, P_0B *and* AB, *and the directions of* P_0A, P_0B *do not correspond to any of the points lying over* P_0. *Set* $c = c(P_0, A, B)$. *Then:*

a) *for the system* $c_* \circ f_*(L)$ *we have* $j' = j$ *and* $h' = h + 2$;

b) *no other fundamental point of the system lies over the point of maximal multiplicity* P_0 *of the system* $c_* \circ f_*(L)$.

Proof. a) Since $d - m_0 > 0$, P_0 remains the point of maximal multiplicity. The fundamental points of the system $c_* \circ f_*(L)$ will be the proper images of the points $P_1, \cdots, P_k$ with their previous multiplicities, and the points A and B, each with the multiplicity $d - m_0 = 2j$ (by Lemma 15). But $j' = j$, and therefore the set of fundamental points with multiplicities $\ge j'$ will consist of the images of the points $P_1, \cdots, P_h$ and of the points A and B. Consequently, $h' = h + 2$.

b) The point P_0 remains the point of maximal multiplicity. The points A and B are chosen so that the proper images of the fundamental points of the system (7) lying over P_0 after the transformation c will lie on and over the line AB; the remaining fundamental points cannot lie over P_0. The lemma is proved.

Lemma 19. *Assume that in the system* (7) *none of the fundamental points lies over* P_0, *but for some* $r, s \le h$ *the point* P_r *lies over* P_s, *of order one. We choose a point* C *on the plane such that no fundamental point lies on the lines* P_0C *and* P_sC, *and such that the direction of* P_sC *does not correspond to the point* P_r. *Set* $c = c(P_0, P_s, C)$. *Then:*

a) *no fundamental point of the system* (7) *of multiplicity greater than* j *lies on the line* P_0P_s, *and the direction of this line does not correspond to any point over* P_s;

b) *the system* $c_* \circ f_*(L)$ *is either simpler than the system* (7) *or* $j' = j$, $h' = h$, *and the set of fundamental points of the system* $c_* \circ f_*(L)$ *lying on the plane and having multiplicity* $\ge j$ *is greater than the analogous number for the system* (7).

Proof. a) If such a point P_t (possibly lying over P_s) were to lie on the line $l = P_0P_s$, then, as in similar cases above, we would have

$$d - m_0 - m_s - m_t = (L - P_0 - P_s - P_t, f_*(L))$$
$$= (g_*(l - P_0 - P_s - P_t), g_* \circ f_*(L)) \geqslant 0, g = \mathrm{dil}_{(P_0, P_s, P_t)},$$

contradicting inequality (13).

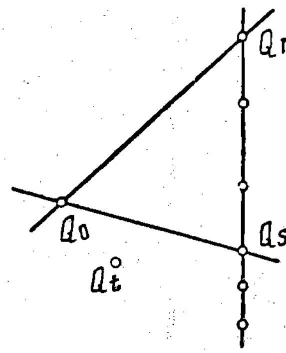
b) It is possible to assume that P_0 remains the point of maximal multiplicity. By Lemma 15 the multiplicity of C is $d - m_0 - m_s < j = j'$, and the multiplicities of the points P_0 and P_1 remain respectively the numbers $d - m_s > j = j'$ by the assumption about the maximality, and $d - m_0 = 2j > j'$. Thus $h' = h$.

All the distinct fundamental points on a plane of the system (7) will have distinct images, again lying on the plane because of the choice of the point C and the assertion of a). Moreover, there is a fundamental point on the line P_0C – the image of the point P_r – and it does not coincide with either P_0 or C. This establishes the second assertion and completes the proof of the lemma.

We shall now describe the simplification algorithm for the system (7).

If the system (7) satisfies the condition of Lemma 17, then either it can be simplified by the transformation described in that lemma, or one can immediately arrive with the help of this transformation at a projective automorphism corresponding to a system of degree 1.

In the contrary case, we apply to the system (7) the transformation of Lemma 18, and then several times in succession the transformation of Lemma 19. If at any point we attain a simplification of the system, the cycle is ended; if we arrive at a system of degree 1, the algorithm is terminated. If neither of these events occurs, by Lemmas 18 and 19 we arrive at a system with parameters $(j, h+2)$ for which all the fundamental points $Q_1, \cdots, Q_{h+2}$ of multiplicity $> j$ lie on a plane. Since $h + 2 \geq 4$, by inequality (12) not all of the points $Q_1, \cdots, Q_{h+2}$ lie on the same line. Similarly by inequality (13) not all of the points $Q_0, Q_r, Q_s, r, s \leq h + 2$, lie on the same line.



Then it is possible to find points $Q_r, Q_s, r, s \leq h + 2$, such that there are two more points Q_u, Q_v, $u, v \leq h + 2$, not lying on any of the lines Q_0Q_r, Q_rQ_s, Q_0Q_s. (If all except say one, Q_t, of the points Q lie on the line Q_rQ_s, then we choose the pair Q_s, Q_t instead of the pair Q_r, Q_s.) After this we apply successively the pair of quadratic transformations $c(Q_0, Q_r, Q_s)$ and $c(Q_0, Q_u, Q_v)$. By Lemma 17 we arrive either at a system of degree 1, or at a system with degree less than j, or at a system with parameters $(j, h-2)$, i.e., we obtain the desired simplification.

We have described the steps of an algorithm allowing one to go from the system (7) to a simpler system by means of a sequence of quadratic transformations. Iterating these steps and keeping in mind that the sequence of linear systems in which each successive one is simpler than the preceding breaks off, we arrive at a system of degree 1, that is at a relationship of the form

$$c_r \circ c_{r-1} \circ \dots \circ c_1 \circ f = g,$$

where the c_i are quadratic automorphisms of the plane and g is a projective automorphism. It is clear that this assertion establishes the validity of Theorem 2.

§7. Biregular classification of ruled surfaces

The basic step in the biregular classification of rational surfaces is the provision of any rational minimal model with the structure of a ruled surface. Ruled surfaces (not only rational ones) are described with the aid of elementary transformations (cf. Theorem 1, §1). This method gives a precise classification of rational surfaces, but is insufficient for irrational ruled surfaces. The classification of irregular ruled surfaces is based on studying them as locally trivial bundles over an algebraic curve with a fiber being a projective line or the group of linear transformations.

The set of biregular equivalence classes of algebraic one-dimensional projective fiber bundles is $H^1(X, \mathrm{PGL}(1))$. This is true for fiber bundles with any group G, i.e., the set of classes of fiber bundles is in one-to-one correspondence with $H^1(X, \mathbf{G})$, where $\mathbf{G}$ is the sheaf of germs of maps of the base X into G. Every projective line P can be considered as a variety of one-dimensional subspaces of a vector space E. For each projective fiber bundle F it is possible to construct a vector bundle E such that each fiber F_X of the bundle F is the variety of one-dimensional subspaces of the fiber E_X of the bundle E. The projective bundle F is determined uniquely by such a vector bundle E and for each F such an E exists. In order to prove this it is sufficient to consider the exact sequence of sheaves

$$(1) \longrightarrow k^* \longrightarrow \mathrm{GL}(2) \longrightarrow \mathrm{PGL}(1) \longrightarrow (1) \tag{1}$$

where $\mathrm{GL}(2)$ is the general linear group of second order, and the exact sequence corresponding to it can be written in the following form

$$H^1(X, k^*) \longrightarrow H^1(X, \mathrm{GL}(2)) \longrightarrow H^1(X, \mathrm{PGL}(1)) \longrightarrow (1).$$

Here $H^2(X, k^*) = 0$, since $\dim X = 1$. Thus, if we denote by $P(E)$ the projectivization of a vector bundle E, then any ruled surface has the form $P(E)$ for some E. It is clear that if L is a line bundle, then $P(E \otimes L) = P(E)$ and $\det(E \otimes L) = \det E \otimes L^2$. Therefore, if $\det E$ is fixed, then $P(E) = P(E')$ if and only if $E' = L \otimes E$ and $L^2 = 1$. There are a finite number, 2^{2g}, of such L. We give the classification with exactness up to this number.

For $\det E$ we may take either the trivial bundle or a fixed bundle of degree 1.

Definition. A surface is said to be even (odd) if it can be obtained from a two-dimensional affine bundle of even (odd) degree.

Thus, the problem of the classification of ruled surfaces is reduced to the problem of the classification of two-dimensional algebraic vector bundles. We note first that for a curve of genus 0 the classification of two-dimensional bundles, given by Grothendieck [14], coincides with the classification of rational ruled surfaces. In order to demonstrate the ideas and methods of working with bundles, we briefly reproduce here the results of Grothendieck for our particular cases.

Thus, let X be a nonsingular rational line and let E be a two-dimensional vector bundle over X with a trivial determinant (i.e., the matrices ϕ_{ij} giving the transition functions of E have a determinant of value 1). On an algebraic curve each section yields a one-dimensional sub-bundle (because of the existence of a local uniformizing set; cf. [5]). We shall show that the bundle E has a section. We use the Riemann-Roch Theorem for a bundle on a curve (cf. [15]):

$$\dim H^0(X, \mathrm{E}) - \dim H^1(X, \mathrm{E}) = \deg E + r(1-g),$$

where $\deg E$ is the degree of the determinant of E, in our case 0 or 1 and r is the dimension of a fiber, in our case 2, and g is the genus of the base curve, in our case 0. Thus

$$\dim H^0(X, \mathrm{E}) \geqslant 2.$$

We take any section $S \in \Gamma(X, \mathrm{E})$ and the effective sub-bundle L_S determined by it. Each one-dimensional fibering is determined by an integer n, the degree of the divisor that it determines. Therefore L_S is determined by a positive integer n; $L_S = L(n)$. We have the exact sequence

$$0 \to L(n) \to E \to L(-n) \to 0 \quad \text{if } \det E = L(0)$$
$$0 \to L(n) \to E \to L(-n+1) \to 0 \quad \text{if } \det E = L(i).$$

The obstruction to the decomposition of this triple is (cf. [6]) an element of $H^1(X, \mathrm{Hom}(L(-n), L(n)))$, where $\mathrm{Hom}(L, L')$ is the sheaf of germs of homomorphisms of L into L'; it is clear that $\mathrm{Hom}(L(-n), L(n)) = L^*(-n) \otimes L(n) = L(2n)$, and since $n \geq 0$, by the Riemann-Roch Theorem we have $H^1(X, \mathrm{L}(2n)) = 0$, i.e., $E = L(n) \otimes L(-n)$.

We shall now show that each E has only one effective sub-bundle. In fact, let there be a second sub-bundle $L'(n')$. Then we have the commutative diagram:

$$\begin{array}{ccccccc} 0 \to L(n) & \xrightarrow{i} & E & \xrightarrow{j} & L^*(n) \to 0, \\ & & \uparrow i' & \nearrow_{ji'} & \\ & & L'(n') & & \\ & & \uparrow & & \\ & & 0 & & \end{array} \qquad (2)$$

i.e., there is defined a homomorphism of $L'(n')$ into $L^*(n)$ or a section $S \in H^0(X, \mathrm{Hom}(L'(n'), L^*(n)))$. But $\mathrm{Hom}(L'(n'), L^*(n)) = L'^*(n') \otimes L^*(n) = L(-m)$, $m < 0$, and therefore either $E = L(0) \otimes L(0)$, or ji' is the zero homomorphism, i.e., $L'(n')$ belongs to the kernel of j and thus coincides with $L(n)$, which is what is necessary to show. We have thus proved the following theorem:

Theorem 1. *On a curve of genus 0 any two-dimensional fiber bundle E with a trivial determinant is uniquely determined by a nonnegative integer $n(E)$, the* *degree of its unique effective sub-bundle. Namely, if* $n(E) = n$, *then* $E = L(\eta) \otimes L(-n)$. *It is not difficult to verify that the given classification coincides with the classification of Nagata, i.e., that the number* $2n(E)$ *coincides with the number of elementary transformations determining the surface* F *corresponding to* E. *If* $\det E = L(i)$, *then the number of Nagata is* $2n + 1$.

Now let X be an elliptic curve. The classification of fiber bundles of any dimension over such curves was obtained by Atiyah [5]. As in the rational case, we immediately obtain an infinite series of classes of bundles if we consider all the bundles of the form $E = L(n) \otimes L^*(\eta)$, where η is the divisor determining $L(\eta)$. In fact, both terms of the direct product are uniquely determined, and thus distinct classes determine distinct bundles. But the variety J_n of the classes of divisors of degree n is the principal homogeneous space of the Jacobian variety J of the curve X. We thus obtain a denumerable number of Jacobian varieties. In the rational case the direct products or the decomposable bundles consist of the whole set of classes. In the elliptical case indecomposable bundles first appear along with a series of decomposable bundles. We turn to the description of the indecomposable bundles. We recall a basic principle, which follows easily from diagram (2): each bundle E with a trivial determinant can only have one effective sub-bundle. We shall show that an indecomposable bundle E always has an effective sub-bundle. Let $O \in X$ be a point such that $O^2 \not\sim C^2$. Then by the Riemann-Roch Theorem $\dim H^0(X, E \otimes L(O)) \geq 2$, i.e., there are at least two linearly independent sections S and S'. There exists a point $C \in X$ such that $L_{S_C} = L_{S'_C}$, since otherwise $E = L_S \otimes L_{S'}$. But then there exists a linear combination $S = \alpha_0 S + \alpha_1 S'$, $\alpha_i \in k$, such that $S(C) = 0$, i.e., E has a sub-bundle of the form $L(\eta C/O)$, where η is effective. We have the expansion

$$0 \to L(\eta C O^{-1}) \to E \to L^*(\eta C O^{-1}) \to 0,$$

which is given by $h \in H^1(S, \mathbf{L}(\eta^2 C^2 O^{-2}))$. But it is clear that $H^1(X, \mathbf{L}(\eta^2 C^2 O^{-2})) = 0$ in all cases except when $C = O$, $\eta = 1$, i.e., if E has a trivial sub-bundle, $\dim H^1(X, \mathbf{L}^2(1)) = 1$ in this case, and since proportional cocycles determine the same bundle E (cf. [20]), there exists a unique indecomposable bundle E_2 with a trivial determinant. We have

Theorem 2. *For a curve of genus* 1 *the set of classes of two-dimensional bundles* E *with a trivial determinant consists of a series of decomposable bundles* (J_n, $n > 0$, J_n *being biregularly equivalent to the Jacobian variety of* X) *and another point.*

We make another remark about a series of decomposable bundles. The ruled surfaces that they determine have the following form. Let X' and X'' be biregular images of the curve X in some P^N, where X' and X'' do not intersect. Let

$\phi: X' \to X''$ be a biregular mapping of X' onto X''. We pass through $P \in X'$ and $\phi(P) \in X''$ a line in P^N, so that no two such lines intersect. Then we obtain a ruled surface F for which the corresponding two-dimensional bundle is decomposable. Since for rational curves all the bundles are decomposable, it is possible to obtain all ruled surfaces in this way. It was in this manner that Andreotti actually obtained his classification [4].

The classification of bundles over a curve of arbitrary genus requires a deeper study of the invariants of a fibering.

As always, we have a series of decomposable bundles; and the problem is the classification of the undecomposable ones.

The study of two-dimensional vector bundles over an algebraic curve of arbitrary genus is based on the description of their invariants. The definition of these invariants and the basic results connecting them appear in the works [53] and [54]. We introduce them here, emphasizing the descriptive side of the new concepts and the idea of classification.

Thus, let X be an arbitrary nonsingular algebraic curve of genus g, over the set of classes of bundles which we wish to describe.

There is not one complete set of invariants for all classes of bundles. A bundle has an invariant called the defect of the bundle, denoted by the symbol $d(E)$, which can take either the value 0 or the value 1. Depending on the value of this invariant we have two different systems of invariants, which we denote by

$$n(E),\ c(E),\ d(E)\begin{array}{l} \overset{0}{\nearrow}\ \xi(E),\ O(E) \\ \underset{1}{\searrow}\ h(E). \end{array}$$

Definitions are given below.

1) To define $n(E)$ one fixes any point O of the curve X which is not a Weierstrass point. Any divisor ξ is equivalent to a divisor of the form ηO^{-n}, where η is an effective divisor. If we choose such a representation with a minimal exponent n, then both n and η will be uniquely determined (cf. [53], Chapter I). The exponent n thus determined is called the height of the divisor. Since each line bundle determines a class of divisors, an invariant of the bundle L, namely $n(L)$, the height of the bundle, is correctly defined. We denote by the symbols $\mathrm{Sp}(E)$ the set of one-dimensional sub-bundles of a two-dimensional bundle E. Then $\mathrm{Sp}(E)$ is not empty ([53], Chapter I, §2). The invariant $n(E)$ is then defined by

$$n(E) = \min_{L \in \mathrm{Sp}(E)} n(L).$$

It is easy to see that this definition is equivalent to

$$\dim \Gamma (E \otimes L (O^{n(E)-1})) = 0, \dim \Gamma (E \otimes L (O^{n(E)})) = i > 0.$$

2) The invariant $c(E)$ is a divisor on X and is defined as the greatest common divisor of zero of all the sections of the bundles $E \otimes L(O^{n(E)})$.

3) It is easy to see that the integer $i > 0$ can either be 1 or 2. The defect of E, $d(E) = 2 - i$.

It is clear that if $d(E) = 1$, then E contains a unique line bundle L of the form $L(c(E) O^{-n(E)})$.

4) If $d(E) = 1$, we have the following expansion

$$0 \to L(c(E) O^{-n(E)}) \to E \to L(c^{-1}(E) O^{n(E)}) \to 0,$$

which is determined by the cocycle [6]

$$a \in H^1(X, L^2(c(E) O^{-n(E)}).$$

Since such a sub-bundle is unique, it is clear that the point h corresponding to a in $P(H^1(X, L^2(c(E) O^{-n(E)})))$, the projective space corresponding to the vector space, is an invariant of E namely $h(E)$. It is clear that if $d(E) = 1$, then $n(E)$, $c(E)$ and $h(E)$ determine the class E.

5) Now let $d = 0$. For the definition of $\xi(E)$ we consider two distinct line sub-bundles L_1 and L_2 of the bundle E. Let L be a line bundle such that $L_1 \otimes L = L'_1$ and $L_2 \otimes L = L'_2$ correspond to effective divisors. Then each L'_i determines in $E \otimes L$ a section S_i. We consider the points $P \in X$ for which

$$\alpha_1 S_1 (P) = \alpha_2 S_2 (P)$$

and α_1 or $\alpha_2 \neq 0$. We associate with such a point P the point $(\alpha_1 : \alpha_2) \in P^1$ of the projective line, which we denote by $Z(P)$. We denote by $\xi(L_1, L_2)$ the divisor that is the product of all such points P. It is not difficult to see by using diagram (2) that $\xi(L_1, L_2) \in |L_1^* \otimes L_2^*|$, where we denote by $|L|$ the linear series of divisors that determines L. It is possible to prove ([53], Chapter I) that if L_1 and L_2 are chosen so that $n(L_1) = n(L_2) = n(E)$, then the divisor $\xi(L_1, L_2)$ does not depend on the choice of L_1 and L_2. It is designated by $\xi(E)$. It is clear that $\xi(E) \in |O^{2n(E)} c(E)^{-2}|$ and that $\deg \xi(E) = 2n(E) - 2 \deg c(E)$.

6) Let $m = 2n(E) - 2 \deg c(E)$ and $\xi(E) = \Pi_{i=1}^m P_i$. We set

$$V(L_1, L_2) = (Z(P_1), \ldots, Z(P_m)) \in (P^1)^m,$$

where $n(L_1) = n(L_2) = n(E)$. Since any such sub-bundle determines a section of the bundle $E \otimes L(O^{n(E)})$ and is uniquely determined by a one-dimensional space on this section (cf. [53], Chapter I), the transfer to another pair L'_1, L'_2 is made with the aid of a bilinear transformation γ of the space $P(\Gamma(E \otimes L(O^{n(E)})))$.

One may easily prove the following formula:

$$V(\gamma(L_1), \gamma(L_2)) = (\hat{\gamma}(Z(P_1)), \ldots, \hat{\gamma}(Z(P_m))),$$

where $\hat{\gamma}$ is some automorphism of the group $PGL(1)$ of bilinear transformations into itself.

Let the group $G = PGL(1)$ operate on $(P^1)^m$ according to the law

$$g(Z_1, \ldots, Z_m) = (g(Z_1), \ldots, g(Z_m)).$$

We denote by U the factor space $(P^1)^m/G$. Thus, each bundle determines uniquely a point in the space U. This point is an invariant of the class and is denoted by $O(E)$. We note that U is a preschema but not a schema. But it is possible to filter it (i.e., construct a sequence of closed subpreschema $U = F_1 \supset F_2 \supset \cdots \supset F_N$ such that $\dim F_i > \dim F_{i+1}$ and $F_i - F_{i+1}$ is an algebraic variety [36]). We have the following theorem (cf. [54]):

Theorem 3. *The invariants* $n(E)$, $c(E)$, $\xi(E)$ *and* $O(E)$ *uniquely determine the class of a bundle without a defect.*

In view of this theorem a classification of bundles will be obtained if we describe what values these invariants run through when E runs through all the bundles of the type being considered.

We first indicate what values may be taken by the integer invariants, $n(E)$, $\deg c(E)$, and $m = \deg \xi(E) = 2n(E) - 2\deg c(E)$, which depend on $d(E)$.

1) $d(E) = 1, g - 1 \geqslant n(E) \geqslant -\frac{g-1}{2}, 0 \leqslant \deg c(E) \leqslant g.$

2) $d(E) = 0, g \geqslant n(E) \geqslant -\frac{g-1}{2}, 0 \leqslant \deg c(E) \leqslant g - 1.$

We now describe what values may be taken by the invariants for a fixed defect.

Let $\deg c(E) = k$. Here $c(E)$ can be taken to be any effective divisor η of degree k such that $\dim |\eta| = 0$ and $\eta \neq O$ ([53], Chapter I, §1). If we denote by J_k the variety of the classes of divisors of degree k and by $S^k(X)$ the symmetric product of the base S, then there exists a regular mapping $P_k: S^k(X) \to J_k$ identifying equivalent divisors. If we denote by $\widetilde{S}^k(X)$ the maximal open set of $S^k(X) - O \times S^{k-1}(X)$ on which the mapping P_k is biregular, then $\widetilde{S}^k(X)$ will be the region of variation of $c(E)$. We note that $\dim \widetilde{S}'^k(X) = k$.

1) Let $d(E) = 1$, $c(E) = \eta$ and $n(E) = n$; then $h(E)$ runs over the space $P(n, \eta) = P(H^1(X, L(O^{-2n}\eta^2))$.

It can be proved that if this space is nonempty, there exists in it a set open in the sense of Zariski that is the region of variation of $h(E)$ ([53], Chapter III). Thus, a calculation of the constants shows that the variety $C(n, k, d)$ of the classes of fiberings with $c(E) = n$, $\deg c(E) = k$ has dimension $g + 2n - k$.

2) Let $d(E) = 0$, $n(E) = n$, $c(E) = \eta$. Then $\xi(E)$ runs over the affine space $|O^{2n}\eta^{-2}| - |O^{2n-1}\eta^{-2}|$. There exists an open subset $\widetilde{U}$ that runs over $O(E)$ [54]. Thus, the dimension of the variety of the classes of the bundles with $n(E) = n$, $\deg c(E) = k$ is equal to $4n - 3k - g - 3$.

It is clear from this that the maximal dimension has a component of fiberings with $n(E) = g$, $k = 0$ and $d = 0$. This dimension is equal to $3g - 3$. We describe this component in more detail. We shall call a variety (not necessarily complete) *rational* if it is possible to remove from it a proper subvariety so that the remaining variety is biregularly equivalent to an affine space.

We can show that the preschema $\widetilde{U}$ is a schema, and therefore the component of maximal dimensionality of $\widetilde{U} \times (|O^{2g}| - |O^{2g-1}|)$ is a rational variety. We thus have

Theorem 4. *There exist two components of maximal dimensionality of a variety of ruled surfaces $V \subset A(V)$ of a fixed curve of genus g. They correspond to even and odd surfaces and are of dimension $3g - 3$.*

A variety of odd surfaces is rational. A variety of even surfaces is an isotrivial bundle with rational base and rational fiber.

In [54] there is constructed a filtration of varieties $C(n, k, d) \supset C(n, k, d, 1) \supset \cdots \supset C(n, k, d, N)$ such that $C(n, k, d, i)$ is a base of the family of bundles and a bundle with the given invariants lies over each point. There exists, however, an obstacle to the representability of a basis functor and to the solution of the universal problem. The families constructed are universal objects only if one also assumes analytic mappings of the bases (local sections of finite coverings). For algebraic representability it is necessary to introduce a functor of rigidity, after which the universal problem becomes solvable. Here a covering category is the category of quasi-bundles [53].

CHAPTER VI

SURFACES OF FUNDAMENTAL TYPE

It is well known that to linear systems on an algebraic variety there correspond mappings of the variety into a projective space. One is especially interested in the mappings corresponding to the canonical system and its multiples. The fact is that if $|nK|$ gives a birational imbedding for two varieties V_1 and V_2, then a necessary and sufficient condition for the birational equivalence of V_1 and V_2 is the projective equivalence of the images of these varieties under the mapping corresponding to $|nK|$.

In the theory of algebraic curves it is proved that if the genus of a curve $p_g > 1$, then the canonical system yields a birational imbedding or a mapping of degree two onto a rational surface; here if $p_g > 2$ and the curve is not hyperelliptic, $|K|$ yields a birational imbedding; if $p_g > 2$ and the curve is hyperelliptic, then $|2K|$ yields a birational imbedding and for $p_g = 2$ $|3K|$ yields a birational imbedding.

In this chapter we consider analogous questions for algebraic surfaces over the field of complex numbers.

Curves of genus greater than one have surfaces corresponding to them that we will call "surfaces of fundamental type"; a simpler definition of them consists in the requirement of the absence of an elliptic pencil and in the existence of a plurigenus greater than one.

Among the results of this chapter we note the following:

1) for a surface of fundamental type with $p_g > 3$ the system $|3K|$ gives a birational imbedding (Theorem 2);

2) there exists a surface of fundamental type with $p_g = 3$ for which $|3K|$ does not give a birational imbedding (Theorem 6);

3) for any surface of fundamental type the system $|9K|$ gives a birational imbedding.

Throughout this chapter the topology is understood in the sense of Zariski.

§1. Lemmas

In this section we will prove some very loosely connected lemmas that will be important in the future.

Lemma 1. *Let V be a nonsingular algebraic surface and $|R|$ be a linear system on V not having fixed curves and base points and such that the mapping corresponding to it of V into a projective space is a mapping onto a curve.*

Then there exists an irreducible one-dimensional algebraic system $\{R^*_1\}$ on V and an open subset U in the space of parameters for $|R|$ such that to any point $t^{(0)} \in U$ there corresponds a cycle $R(t^{(0)}) \in |R|$ for which all the irreducible components, nonmultiple, nonsingular, are members of the system $\{R^*_1\}$ and, moreover, are fibers of some differentiable fiber space $V' \to E'$, where V' is obtained from V by the removal of a finite number of curves and E' is obtained from E by parameterizing the system $\{R^*_1\}$ by the removal of a finite number of points.

Proof. By the theorem of Bertini the regular mapping f corresponding to the system $|R|$: $V \to C$ (we denote by C the image of the mapping) can be obtained as the composition of two regular mappings: f_1: $V \to E$, f_2: $E \to C$, where E is an algebraic curve and f_1: $V \to E$ is a mapping whose generic fiber is irreducible. We denote the algebraic system of the fibers of the mapping f_1: $V \to E$ by $\{R^*_1\}$.

Following Kodaira [25], it is easy to show that if there is a regular mapping f_1 of the surface V onto a curve E, then it is possible to remove from E a finite number of points so that if the remaining set is denoted by E' and $f_1^{-1}(E')$ by V', then the regular mapping $V' \to E'$ possesses the following properties:

1) it does not have multiple fibers,

2) each fiber is a nonsingular irreducible curve,

3) the mapping $V' \to E'$ gives V' the structure of a differentiable fiber bundle.

It is possible to find on C a finite number of points, whose preimages under f_2 are in $E - E'$ or contain multiple components. We remove from C all such points and denote the remaining set by C''. Let

$$E'' = f_2^{-1} C'', V'' = f_1^{-1}E''. \tag{1}$$

It is clear that a preimage of any point $Q \in C''$ is a complete curve without multiple components on V'' whose irreducible components are certain fibers of the mapping $V' \to E'$.

The space of the parameters of the system $|R|$ coincides with the space of the parameters of the system of hyperplane sections of the curve C under the imbedding of it into a projective space corresponding to the mapping f. But it is obvious that those hyperplane sections that pass through one of the points of $C - C''$ give a subvariety of a smaller dimension in the space of parameters. Moreover, those hyperplane sections of the curve C whose decomposition into simple divisors contain multiple components also give a subvariety of a smaller dimension in the space of parameters (this follows from the fact that such sections correspond to hyperplanes tangent to C at some point).

We can now find an open set U in the space of the parameters of the system $|R|$ such that for any point $t^{(0)} \in U$ the corresponding cycle $R(t^{(0)}) \in |R|$ is the union of the preimages of a finite number of points on C'', i.e. $R(t^{(0)})$ does not contain multiple components and has irreducible components, nonsingular curves that are fibers of the differentiable fibering $V' \to E'$.

Lemma 2. *Let V be a nonsingular algebraic surface, and $|L|$ and $|M|$ be infinite linear systems on V, where M is irreducible, the geometric genus of a generic curve*

$$M \in |M| P_M \geqslant 2, |L| = |K + M + C|, \quad \textit{where} \quad C \geqslant 0, \tag{2}$$

$$\dim H^0(V, O[K + C]) > 0, \tag{3}$$

$$H^1(V, O[K + C]) = 0. \tag{4}$$

Then the mapping of V into a projective space corresponding to the system $|L|$ is either a birational imbedding or is a mapping of degree two onto a rational surface, and for $(C \cdot M) \geq 3$ the mapping is always a birational imbedding.

Proof. Using the fact that $\dim H^0(V, O[K + C]) > 0$, we choose a nonnegative[1)] cycle $C' \in |K + C|$. Having removed from V the curve C' and the base points of the system $|M|$, we obtain an open set U'. It is clear that the mappings f_L and f_M corresponding to the systems $|L|$ and $|M|$ are regular on U'.

By a well-known theorem of Bertini [8] almost all the curves of the system $|M|$ have singular points only in the base points of $|M|$. Let $\mathbf{P}^l$ be the space of the parameters of the system $|M|$ and let W be the subvariety in $\mathbf{P}^l$ corresponding to the curves of $|M|$ that have singular points also outside of the base points or are reducible, or have a genus less than $P_{|M|}$. To each point $P_0 \in U'$ there corresponds a hyperplane $H(P_0) \subset \mathbf{P}^l$ parameterizing the curves of $|M|$ passing through P_0. We note that a fiber of the mapping f_M containing the point P_0 is a subvariety of smaller dimension on V, and therefore the set of points $P' \in U'$ for which $H(P') = H(P_0)$ lies on a subvariety of smaller dimension in U'.

Since W can contain only a finite number of hyperplanes of the form $H(P)$,

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1) We call a cycle D nonnegative if $D \geq 0$.

it is possible to find on U' only a finite number of subvarieties of smaller dimension for whose points the corresponding hyperplanes $H(P)$ belong to W. We remove these subvarieties from U'. We obtain an open set U'' such that for any point $P \in U''$ the corresponding $H(P) \not\subset W$. This means that for any point $P \in U''$ there exists a curve $M \in |M|$ passing through P, that is irreducible, of genus $P_{|M|}$, and has singularities only in the base points of $|M|$, i.e. outside of U''.

Let us assume that there exist points $P_1, P_2, P_3 \in U''$ which are distinct and such that $f_L(P_1) = f_L(P_2) = f_L(P_3)$.

Since P_1, P_2, and P_3 do not lie on C', from the existence of a curve $M \in |M|$ passing through P_1 but not through P_2 and P_3 would follow the existence of a curve $C' + M \in |L|$ passing through P_1 but not through P_2 and P_3, and the points $f_L(P_i)$, $i = 1, 2, 3$ would not coincide. Thus, any curve $M \in |M|$ passing through P_1 also passes through P_2 and P_3. We pass through P_1 a curve $M \in |M|$ that is irreducible, of genus $P_{|M|}$, and lacking singularities on U''. Let $\widetilde{M}$ be a nonsingular model of the curve M. It is known [49] that there exists a positive divisor $\mathfrak{c}$ on $\widetilde{M}$ with a carrier in the preimages of the singular points of M on $\widetilde{M}$ such that if we denote by $[L]_{\widetilde{M}}$ the fiber space of lines over $\widetilde{M}$ induced by the fibering $[L]$ over V under the mapping $\widetilde{M} \to V$ and by $\overline{O(-\mathfrak{c})[L]_{\widetilde{M}}}$ the direct image of the sheaf $O_{\widetilde{M}}(-\mathfrak{c})[L]_{\widetilde{M}}$ under the mapping $\widetilde{M} \to M$, then one has the inclusion

$$\overline{O(-\mathfrak{c})\,[L]_{\widetilde{M}}} \subset O\,[L]_M; \tag{5}$$

from which we have

$$0 \to H^0(M, \overline{O(-\mathfrak{c})\,[L]_{\widetilde{M}}}) \to H^0(M, O\,[L]_M). \tag{6}$$

Since $M \cap U''$ does not contain singular points, $M \cap U''$ is biregularly equivalent to some open set $\widetilde{U}''$ on $\widetilde{M}$ that does not contain points of $\mathfrak{c}$. It follows from this that the mapping of the set U'' given by the sections of the group $H^0(M, \overline{O(-\mathfrak{c})[L]_{\widetilde{M}}})$, coincides with the mapping of the set $\widetilde{U}''$ given by the sections of the group $H^0(\widetilde{M}, O_{\widetilde{M}}[-\mathfrak{c}][L]_{\widetilde{M}})$. But $O_{\widetilde{M}}[-\mathfrak{c}][L]_{\widetilde{M}} = O_{\widetilde{M}}[(K+M)_{\widetilde{M}} - \mathfrak{c} + (C)_{\widetilde{M}}]$. Kodaira shows in [25] that $[(K+M)_{\widetilde{M}} - \mathfrak{c}]$ is the canonical fibering over M. On the other hand, in the theory of algebraic curves it is well known that if the genus of a curve $p_g \geq 2$, then the mapping into a projective space corresponding to the canonical class is either a birational imbedding or a mapping of degree two onto a rational line.

It follows from this that the mapping of the set $U'' \cap M$ given by the sections of the group $H^0(M, \overline{O(-\mathfrak{c})[L]_{\widetilde{M}}})$ have a degree not greater than two. But then there exists a section $\phi_{P_1} \in H^0(M, \overline{O(-c)[L]_{\widetilde{M}}})$ equal to zero at P_1 and not equal to zero simultaneously at P_2 and P_3. We will consider ϕ_{P_1} an element of the group $H^0(M, O[L]_M)$. Since $H^1(V, O[K+C]) = 0$, we have the epimorphism

$$H^0(V, O[K+M+C]) \to H^0(M, O[K+C+M]_M) \to 0. \tag{7}$$

Taking the section in $H^0(V, O[K+M+C])$ mapped under this epimorphism into ϕ_{P_1}, we obtain that in $H^0(V, O[K+M+C])$ there exists a section mapped into zero at P_1, but not equal to zero simultaneously at P_2 and P_3. We thus see that there do not exist three points on U'' having the same image under the mapping f_L. Thus f_L maps U'' onto an algebraic surface and the degree of the mapping does not exceed two. If this degree is equal to one, then the mapping of the set U'', and with it also V, is a birational imbedding.

Now let the degree of f_L be equal to two. It is clear that U'' may be reduced to an open set U''' such that for each point $P \in U'''$ there exists a unique point $P' \in U'''$ with the properties $P' \neq P$, $f_L(P') = f_L(P)$. We call the point P' the associated point of the point P. It is clear that almost all the curves of $|M|$ do not lie in $V - U'''$, are irreducible, have a genus $P_{|M|}$, and have singular points only in the base points of $|M|$. Let M be some curve with these properties. Clearly, for any point $P \in M \cap U'''$ the associated point $P' \in M \cap U'''$. It follows from this that f_L gives on M a mapping $f_L|M$ of degree two. But by (7), f_L coincides on $M \cap U'''$ with the mapping given by the group of sections $H^0(M, O[K+M+C]_M)$.

We remove from $\widetilde{U}'''$ the fixed points of the system $|(K+M+C)_{\widetilde{M}} - c|$ (if there are any), and from U''' their images. After the removal we obtain biregularly equivalent open sets $\widetilde{U}^{\mathrm{IV}}$ and $U^{\mathrm{IV}} \cap M$. Since the sections of the group $H^0(M, \overline{O(-c)[L]_{\widetilde{M}}})$ do not have common zeros on $U^{\mathrm{IV}} \cap M$, there exists a regular mapping g of the variety $f_L(M \cap U^{\mathrm{IV}})$ onto the variety $\widetilde{f}_{L,M}(M \cap U^{\mathrm{IV}})$, where $\widetilde{f}_{L,M}$ is the mapping corresponding to the group $H^0(M, \overline{O(-c)[L]_{\widetilde{M}}})$. Here we have the commutative diagram

$$M \cap U^{\mathrm{IV}} \begin{array}{l} \xrightarrow{f_L|M} f_L(M \cap U^{\mathrm{IV}}) \\ \qquad\qquad \downarrow g \\ \xrightarrow[\widetilde{f}_{L,M}]{} \widetilde{f}_{L,M}(M \cap U^{\mathrm{IV}}) \end{array}.$$

For the degrees of the mappings we have the equation

$$d(\widetilde{f}_{L,M}) = d(g) \cdot d(f_L | M). \tag{8}$$

It is clear that $\widetilde{f}_{L,M}$ on $M \cap U^{\mathrm{IV}}$ is just the same as the mapping of the set $\widetilde{U}^{\mathrm{IV}}$

with the aid of the group $H^0(\widetilde{M}, O[(K+M+C)_{\widetilde{M}} - c])$, from which it follows that $d(\widetilde{f}_{L,M}) \leq 2$. But $d(f_L|M) = 2$; hence $d(g) = 1$, $d(\widetilde{f}_L|M) = 2$ and $\widetilde{f}_{L,M}(M \cap U^{IV})$ is a rational curve. From $d(g) = 1$ we obtain that $f_L(M \cap U^{IV})$ is a rational curve.

Thus M is mapped onto a rational curve. We obtain that the surface $f_L(V)$ contains an infinite linear system of rational curves, i.e. is a rational surface.

It remains to consider the case $(C \cdot M) \geq 3$. We will show that in this case the degree cannot be equal to two. For this we take any point $P \in U'''$ and curve $M \in |M|$ passing through the point P, irreducible, of genus $P_{|M|}$ and having singular points only in the base points of $|M|$. The associated point P' of P lies on $M \cap U'''$.

We denote the points on $\widetilde{U}'''$ corresponding to the points P and P' by $\widetilde{P}$ and $\widetilde{P}'$. We consider the exact sequence

$$0 \to H^0(\widetilde{M}, O(-\widetilde{P} - \widetilde{P}')[K_{\widetilde{M}} + (C)_{\widetilde{M}}]) \to H^0(\widetilde{M}, O(-\widetilde{P}')[K_{\widetilde{M}} + (C)_{\widetilde{M}}]) \to \mathbf{C}_{\widetilde{P}} \to H^1(\widetilde{M}, O(-\widetilde{P} - \widetilde{P}')[K_{\widetilde{M}} + (C)_{\widetilde{M}}]), \quad (9)$$

here $\mathbf{C}_{\widetilde{P}}$ is the sheaf on $\widetilde{M}$, whose stalk is a complex line at the point $\widetilde{P}$ and is zero at the remaining points. Since $(C \cdot M) \geq 3$, the degree of the divisor $-\widetilde{P} - \widetilde{P}' + K_{\widetilde{M}} + (C)_{\widetilde{M}} \geq 2P_{\widetilde{M}} - 2 + 1$ ($P_{\widetilde{M}}$ is the genus of the curve $\widetilde{M}$). From this we have

$$H^1(\widetilde{M}, O(-\widetilde{P} - \widetilde{P}')[K_{\widetilde{M}} + (C)_{\widetilde{M}}]) = 0.$$

We obtain the epimorphism

$$H^0(\widetilde{M}, O(-\widetilde{P}')[K_{\widetilde{M}} + (C)_{\widetilde{M}}]) \to \mathbf{C}_{\widetilde{P}} \to 0$$

Hence in the group $H^0(\widetilde{M}, O[K_{\widetilde{M}} + (C)_{\widetilde{M}}])$ there exists a section $\phi_{\widetilde{P}}$ equal to zero at the point $\widetilde{P}'$ and different from zero at the point $\widetilde{P}$. But then there exists in the group $H^0(M, \overline{O(-c)[(K+M)_{\widetilde{M}} + (C)_{\widetilde{M}}]})$ a section $\phi_{P'}$ equal to zero at P' and different from zero at P. $\phi_{P'}$ can be considered as an element of the group $H^0(M, O[K+M+C])$. Then the epimorphism (7) shows that in the group $H^0(V, O[K+M+C])$ there exists a section equal to zero at P' and different from zero at P.

Thus P' and P have distinct images under the mapping f_L and we arrive at a contradiction to the fact that the degree of f_L is equal to two. The lemma is proved.

Lemma 3. *Let V be a nonsingular algebraic surface, $\{R_1\}$ be a one-dimensional irreducible algebraic system on V that is a system of fibers of a regular mapping of V onto a curve E, and let S_1 be a connected curve on V such that $(R_1 S_1) > 0$. Then for almost all curves of the system $\{R_1\}$*

$$H^1(V, O[K + R_1 + S_1]) = 0.$$

Proof. Using the proof of Lemma 1 we obtain that E and V can be reduced to open subsets E' on E and V' on V such that there exists a regular mapping of V' onto E', whose fibers are complete nonsingular curves of the system $\{R_1\}$, where no fiber is a component of S_1, and, moreover, this mapping gives V' the structure of a differentiable fiber space with base E' and fibers that are curves of the system $\{R_1\}$. We now note that for any points $(C_1), (C_2) \in E'$ the corresponding curves $R_1(C_1)$ and $R_1(C_2)$ are homeomorphic and any one-dimensional cycle on $R_1(C_2)$ is homologous on V' to some one-dimensional cycle on $R_1(C_1)$. It follows from this that if α is a one-dimensional differential of first order on V that vanishes on $R_1(C_1)$, then the integrals of α on all the one-dimensional cycles on $R_1(C_2)$ are zero, and consequently α also vanishes on $R_1(C_2)$. We obtain that if α vanishes on some curve $R_1(C_0)$, $(C_0) \in E'$, then α is zero on all the curves $R_1(C)$, $(C) \in E'$. Let us now assume that α vanishes not only on $R_1(C_0)$, but also on S_1. We consider the many-valued function $f(P) = \int_{P_0}^{P} \alpha$. It is clear that on all the curves $R_1(C)$, $(C) \in E'$, $f(P)$ is a constant.

Since $f(P)$ is also a constant on S_1 and $(R_1 \cdot S_1) > 0$, $f(P)$ must take the same value on all the $R_1(C)$, $(C) \in E'$, i.e. $f(P)$ is a constant on all of V', and thus on all of V. It follows from this that $\alpha = 0$. We have thus shown that there does not exist a nontrivial one-dimensional differential of first order on V that vanishes on $R_1(C_0)$ and on S_1. To finish the proof it remains to note that $R_1(C_0) + S_1$ is a connected curve and to refer to the following result of Kodaira [25]: for a curve C on a nonsingular algebraic surface V, $\dim H^1(V, O[K + C]) = m - 1 + k$, where m is the number of connected components of the curve C and k is the number of linearly independent one-dimensional differentials of first order on V that vanish on C.

Lemma 4. *Let V be a nonsingular algebraic surface without exceptional curves of the first kind and with $(K^2) > 0$, $p_g = 0$. Then the irregularity of V, $q = 0$.*

Proof. The arithmetic genus of V, $p_a = 1 - q + p_g = 1 - q$, and the second Betti number, $b_2 = h^{2,0} + h^{1,1} + h^{0,2} = 2p_g + h^{1,1} = h^{1,1}$. The formula of Noether gives us

$$p_a = \frac{1}{12}((K^2) + 2 - 4q + b_2),$$

or

$$1 - q = \frac{1}{12}((K^2) + 2 - 4q + h^{1,1}).$$

Hence

$$(K^2) + h^{1,1} = 10 - 8q.$$

Since $(K^2) > 0$ and $h^{1,1} > 0$, we have $10 - 8q > 0$, $q \leq 1$.

Let us assume that $q = 1$. Then $(K^2) + h^{1,1} = 2$, $(K^2) = 1$, $h^{1,1} = 1$. The equation $h^{1,1} = 1$ means that two-dimensional (in the real sense) algebraic cycles on V are homologous over the field of rational numbers to multiples of a hyperplane section, i.e. to cycles of the form rH (where r is a rational number and H is a hyperplane section of V). Thus in particular there are no effective algebraic cycles C on V with $(C^2) = 0$.

On the other hand, the equation $q = 1$ means that the albanese mapping is a regular mapping of V onto an elliptic curve. Any fiber C of this mapping is an effective curve, and clearly $(C^2) = 0$.

The assumption $q = 1$ led us to the contradiction. Thus $q = 0$. The lemma is proved.

Lemma 5. *The following three conditions for an algebraic surface are equivalent.*

1) *For some* $n > 0$, $P_n \geq 2$ *and* V *does not have a pencil of elliptic curves.*

2) *For some* $n' > 0$, $P_{n'} > 0$ *and on a minimal model* V, $(K^2) > 0$.

3) *For some* $n'' > 0$, *the system* $|n''K|$ *maps* V *onto an algebraic surface.*

Proof. We deduce 2) from 1). It is given that V is nonsingular, does not have exceptional curves of the first kind, does not have an elliptic pencil, and for some $n > 0$, $P_n \geq 2$. It is necessary to show that $(K^2) > 0$.

We will first show that for any irreducible curve D on V, $(K \cdot D) \geq 0$. In fact, if D is irreducible, then $(K \cdot D) + (D^2) \geq -2$ or $(K \cdot D) \geq -2 - (D^2)$. For $(D^2) < 0$, we have $(K \cdot D) \geq -2 + 1 = -1$; here equality will hold only if $(D^2) = -1$, which, together with $(K \cdot D) = -1$, characterizes an exceptional curve of the first kind. This contradicts the minimality of V. Now let $(D^2) \geq 0$. Then, taking an effective curve from $|nK|$, we obtain $(nK \cdot D) \geq 0$, i.e. $(K \cdot D) \geq 0$.

Since $P_n \geq 2$, $|nK|$ is an infinite system. Let $|R|$ be its nonfixed and S its fixed part. If we had $(K^2) \leq 0$, we would have $n(K \cdot (R + S)) \leq 0$. Since $(K \cdot R) \geq 0$, $(K \cdot S) \geq 0$, we have from this that $(K \cdot S) = 0$, $(K \cdot R) = 0$.

The last equation gives $(R \cdot R) + (R \cdot S) = 0$. Since $|R|$ is a system without fixed parts, $(R \cdot S) \geq 0$, $(R \cdot R) \geq 0$, and we have $(R \cdot S) = 0$, $(R \cdot R) = 0$. The equation $(R \cdot R) = 0$ shows that it is possible to apply Lemma 1 to the system $|R|$ ($|R|$ does not have fixed points and maps V onto a curve). By Lemma 1 there exists a one-dimensional irreducible algebraic system $\{R_1^*\}$ such that for almost every $R \in |R|$ we have $R = \Sigma_i R_1^{(i)}$, where the $R_1^{(i)}$ are nonsingular irreducible curves of the system $\{R_1^*\}$. The equations $(R \cdot S) = 0$ now give us $(R_1 \cdot S) = 0$, $(R_1 \cdot R) = 0$, $(R_1^2) = 0$. From this we have $n(K \cdot R_1) = 0$, $(K \cdot R_1) = 0$, $2p_{R_1} - 2 = ((K + R_1) \cdot R_1) = 0$. This shows that $\{R_1^*\}$ is a pencil of elliptic curves. Thus the assumption $(K^2) \leq 0$ led us to a contradiction with condition 1).

We deduce 3) from 2). We are given that for some $n' > 0$, $P_{n'} > 0$ and $(K^2) > 0$. It is necessary to show that there exists an $n'' > 0$, such that $|n''K|$ maps V onto an algebraic surface, i.e. $|n''K|$ is an infinite system not composed of a pencil. Since $(K^2) > 0$ and $P_{n'} > 0$, there must be a positive curve in $|n'K|$. For a hyperplane section H of the surface V we have $n'(K \cdot H) > 0$ and $(K \cdot H) > 0$. Then it is clear that for $m > 1$ we have

$$\dim H^2(V, O[mK]) = \dim H^0(V, O[-(m-1)K]) = 0.$$

The Riemann-Roch theorem gives us

$$\dim H^0(V, O[mK]) \geqslant \frac{(mK(mK-K))}{2} + 1 - q + p_g > bm^2 \quad \text{for} \quad m > m_0,$$

where b is some positive constant. This inequality shows that for $m > m_0$, $|mK|$ is an infinite system.

Let us now assume that the system $|mK|$ for some $m > m_0$ consists of a pencil, and let C_m be the algebraic curve onto which $|mK|$ maps the surface V.

Let $|R_m|$ be the nonfixed part of the system $|mK|$. Using σ-processes it is possible to replace V with a surface $\overline{V}$ such that the proper image $|\overline{R}_m|$ of the system $|R_m|$ does not have base points on $\overline{V}$ and the system $|\overline{R}_m|$ corresponds to a regular mapping of the surface $\overline{V}$ onto C_m. Let H_C be the hyperplane section of the curve C_m in its projective imbedding corresponding to the mapping $\overline{V} \to C_m$. It is clear that the curves of the system $|\overline{R}_m|$ are preimages of the sections H_C of the curve C_m, from which it follows that

$$\dim H^0(\overline{V}, O_{\overline{V}}[\overline{R}_m]) \leqslant \dim H^0(C_m, O_{C_m}[H_C]). \tag{10}$$

If we denote by $a(H_C)$ the degree of the divisor H_C on C, then we deduce from the Riemann-Roch theorem for a curve that

$$\dim H^0(C_m, O_{C_m}[H_C]) \leqslant a(H_C) + 1.$$

But $a(H_C)$ does not exceed the number of irreducible components of a curve $\overline{R}_m$ on $\overline{V}$, which clearly coincides with the number of irreducible components of the corresponding curve R_m on V. The latter number can exceed neither $(R_m \cdot H)$ nor $m(K \cdot H)$. Finally we obtain that for $m > m_0$

$$bm^2 < \dim H^0(V, O[mK]) \leqslant m(K \cdot H) + 1; \tag{11}$$

here we have used the equations

$$\dim H^0(V, O[mK]) = \dim H^0(V, O[R_m]) = \dim H^0(\overline{V}, O_{\overline{V}}[\overline{R}_m]).$$

It is possible to take m_0 large enough so that $bm^2 > m(K \cdot H) + 1$ for $m > m_0$, and then assuming that $|mK|$ for $m > m_0$ consists of a pencil leads to a contradiction.

To finish the proof of the lemma it remains to deduce 1) from 3). Here one must assume that there exists an $n'' > 0$ for which $|n''K|$ gives a mapping of V onto an algebraic surface, and then prove that there is no pencil of elliptic curves on V (the fact that $P_{n''} \geq 2$ follows immediately from the premise).

We will give a proof by contradiction. Let there be on V a pencil of elliptic curves $\{R_1\}$, i.e. a one-dimensional irreducible algebraic system of elliptic curves. Let E_1 be a curve parameterizing the system $\{R_1\}$. In the product $E_1 \times V$ there exists an irreducible cycle Γ such that the curve $R_1(t) \in \{R_1\}$ corresponding to any point $(t) \in E_1$ is obtained as

$$R_1(t) = P_{r_V}[\Gamma((t) \times V)].$$

It is clear that Γ is an algebraic surface with a pencil of elliptic curves $\{R'_1\}$ that are fibers of the regular mapping $\Gamma \to E_1$.

Let $\widetilde{\Gamma}$ be a nonsingular model of the surface Γ. Then there exist regular mappings $f_1: \widetilde{\Gamma} \to E_1$ and $f_2: \widetilde{\Gamma} \to V$. It is clear that there is a pencil of elliptic curves (we denote it by $\{\widetilde{R}_1\}$) on $\widetilde{\Gamma}$, such that a generic fiber coincides under the mapping $\widetilde{\Gamma} \to E_1$ with a generic curve of the pencil $\{\widetilde{R}_1\}$. It follows from this that $(\widetilde{R}_1^2) = 0$ and that almost all the curves of the pencil $\{\widetilde{R}_1\}$ are nonsingular [25]. Since $\{\widetilde{R}_1\}$ is a pencil of elliptic curves, it follows from the nonsingularity of almost all of the curves of the pencil that $((K_{\widetilde{\Gamma}} + \widetilde{R}_1) \cdot \widetilde{R}_1) = 2p_{\widetilde{R}_1} - 2 = 0$. From $(\widetilde{R}_1^2) = 0$ we obtain $(K_{\widetilde{\Gamma}} \cdot \widetilde{R}_1) = 0$ We will now show that the system $|n''K_{\widetilde{\Gamma}}|$ does not consist of a pencil.

Let $\phi_0^{(n'')}, \phi_1^{(n'')}, \ldots, \phi_N^{(n'')}$ be a basis of the space of regular binary differentials of degree n'' on V. The image of V under the mapping corresponding to the system $|n''K_V|$ is a variety with a generic point

$$\left(1, \frac{\varphi_1^{(n'')}}{\varphi_0^{(n'')}}, \ldots, \frac{\varphi_N^{(n'')}}{\varphi_0^{(n'')}}\right).$$

Since this image is a surface there are two algebraically independent functions among $\phi_1^{(n'')}/\phi_0^{(n'')}, \ldots, \phi_N^{(n'')}/\phi_0^{(n'')}$, say $\phi_1^{(n'')}/\phi_0^{(n'')}$ and $\phi_2^{(n'')}/\phi_0^{(n'')}$.

The differentials $\phi_0^{(n'')}, \phi_1^{(n'')}, \phi_2^{(n'')}$ induce under the mapping $f_2: \widetilde{\Gamma} \to V$ the differentials $f_2^*\phi_0^{(n'')}, f_2^*\phi_1^{(n'')}, f_2^*\phi_2^{(n'')}$ on $\widetilde{\Gamma}$, where

$$f_2^*\left(\frac{\varphi_1^{(n'')}}{\varphi_0^{(n'')}}\right) = \frac{f_2^*\varphi_1^{(n'')}}{f_2^*\varphi_0^{(n'')}}; \quad f_2^*\left(\frac{\varphi_2^{(n'')}}{\varphi_0^{(n'')}}\right) = \frac{f_2^*\varphi_2^{(n'')}}{f_2^*\varphi_0^{(n'')}}.$$

The algebraic independence of the functions $f_2^* \phi_1^{(n'')}/f_2^* \phi_0^{(n'')}$ and $f_2^* \phi_2^{(n'')}/f_2^* \phi_0^{(n'')}$ now follows from the algebraic independence of the functions $\phi_1^{(n'')}/\phi_0^{(n'')}$ and $\phi_2^{(n'')}/\phi_0^{(n'')}$. These functions belong to that subfield of the field of functions over $\widetilde{\Gamma}$ that corresponds to the image of $\widetilde{\Gamma}$ under the mapping coming from the system $|n''K_{\widetilde{\Gamma}}|$. We obtain that this subfield has a degree of transcendency of two, and consequently that $|n''K_{\widetilde{\Gamma}}|$ does not consist of a pencil. Using the infiniteness of the system $\{\widetilde{R}_1\}$ it is very easy to deduce from this that $n(K_{\widetilde{\Gamma}} \cdot R_1) > 0$ and $(K_{\widetilde{\Gamma}} \cdot R_1) > 0$. We arrive at a contradiction with the equation $(K_{\widetilde{\Gamma}} \cdot R_1) = 0$. The lemma is proved. Lemma 5 leads us to the definition of a surface "of fundamental type."

Definition. *An algebraic surface is said to be a surface of fundamental type if it satisfies any of the three conditions whose equivalence was proved in Lemma 5.*

Remark. A surface of fundamental type cannot contain a pencil of rational curves, for a surface with a pencil of rational curves has all the $P_n = 0$ (is a ruled surface).

§2. Mappings of surfaces of fundamental type with the aid of the multiples of the canonical class

Theorem 1. *Let V be an algebraic surface of fundamental type and n a natural number satisfying the condition $P_n \geq 2$. Then*

1) *for $P_n = 2$ the system $|(3n+2)K|$ yields a birational imbedding of V,*

2) *for $P_n = 3$ the system $|(2n+2)K|$ (if nK consists of a pencil) or the system $|(3n+1)K|$ (if $|nK|$ does not consist of a pencil) yields a birational imbedding of V,*

3) *for $P_n = 3$, $P_{n-1} > 0$ (it is assumed that $P_0 = 1$, since $H^0(V, O[0 \cdot K]) = 1$) the system $|(2n+1)K|$ yields either a birational imbedding of V or a mapping of V of degree two onto a rational surface,*

4) *for $P_n > 3$ the system $|(n+2)K|$ (if $|nK|$ consists of a pencil) or the system $|(2n+1)K|$ (if $|nK|$ does not consist of a pencil) yields a birational imbedding of V.*

Proof. Since $P_n \geq 2$, $|nK|$ is an infinite system. We note that it is sufficient to prove the theorem for any surface birationally equivalent to V. Therefore we can assume that V is a nonsingular surface and, moreover, that the nonfixed part of the system $|nK|$ (which we denote by $|R|$) does not have base points. (The last can always be obtained by applying a finite number of σ-processes to V.) We can also assume that there are no exceptional curves of the first kind on V whose index of intersection with R is equal to zero.

We first consider the case when $|R|$ consists of a pencil. Then it is clear that $|R|$ satisfies all the requirements of Lemma 1. Applying this lemma, we obtain that there exists an irreducible algebraic system $\{R_1^*\}$ and an open set U in the space of the parameters for $|R|$ such that to any point $t^{(0)} \in U$ there corresponds a curve $R(t^{(0)}) \in |R|$ whose irreducible components are all nonsingular, are not multiples in $R(t^{(0)})$, belong to $\{R_1^*\}$, and are fibers of a differentiable fiber space $V' \to E'_1$, where V' and E'_1 are open subsets on V and on the curve E_1 respectively, E_1 being a curve parameterizing the system $\{R_1^*\}$. Since $\{R_1^*\}$ cannot be a pencil of elliptic curves or rational curves (V is a surface of fundamental type), we have $((K^* + R_1^*) \cdot R_1^*) > 0$, and from $(R_1^{*2}) = 0$ we obtain $(K \cdot R_1^*) > 0$, $n(K \cdot R_1^*) > 0$. Comparing this with $(R \cdot R_1^*) = 0$, we conclude that the fixed part of the system $|nK|$ has an irreducible component S_1 with $(S_1 \cdot R_1^*) > 0$.

By Lemma 3 the set E'_1 can be reduced to an open E''_1 in such a way that for any curve $R_1(t) \in \{R_1^*\}$ parameterizible by the point (t) of E''_1,

$$H^1(V, O[K + R_1(t) + S_1]) = 0. \tag{12}$$

The proof of Lemma 2 shows, moreover, that the number of linearly independent one-dimensional differentials of first order on V vanishing on $R_1(t) + S_1$ is equal to zero.

Let $R_1^{(1)}, R_1^{(2)}, R_1^{(3)}$ be three distinct irreducible nonsingular curves of $\{R_1^*\}$ parameterizible by points of E''_1.

We will show that the system $|2K + \Sigma_{i=1}^3 R_1^{(i)} + S_1|$ yields a birational imbedding of the surface V.

We consider the exact sequence

$$\begin{aligned} 0 &\to H^0(V, O[K + R_1^{(1)}]) \to H^0(V, O[K + R_1^{(1)} + R_1^{(2)}]) \\ &\to H^0(R_1^{(2)}, O_{R_1^{(2)}}[K + R_1^{(1)} + R_1^{(2)}]_{R_1^{(2)}}) \to H^1(V, O[K + R_1^{(1)}]) \\ &\to H^1(V, O[K + R_1^{(1)} + R_1^{(2)}]) \\ &\to H^1(R_1^{(2)}, O_{R_1^{(2)}}[K + R_1^{(1)} + R_1^{(2)}]_{R_1^{(2)}}). \end{aligned} \tag{13}$$

Since $R_1^{(1)} \cap R_1^{(2)}$ is empty, the divisor $R_1^{(1)} \cdot R_1^{(2)}$ on $R_1^{(2)}$ is linearly equivalent to zero, and thus the divisors on $R_1^{(2)}$, $(K + R_1^{(1)} + R_1^{(2)}) \cdot R_1^{(2)}$ and $(K + R_1^{(2)}) \cdot R_1^{(2)}$, are linearly equivalent. But $(K + R_1^{(2)}) \cdot R_1^{(2)}$ is a canonical divisor on $R_1^{(2)}$ ($R_1^{(2)}$ is a nonsingular curve) and thus $(K + R_1^{(1)} + R_1^{(2)}) \cdot R_1^{(2)}$

is also a canonical divisor on $R_1^{(2)}$. Hence

$$\dim H^1 (R_1^{(2)}, O_{R_1^{(2)}} [K + R_1^{(1)} + R_1^{(2)}]_{R_1^{(2)}})$$
$$= \dim H^1 (R_1^{(2)}, O_{R_1^{(2)}} [K_{R_1^{(2)}}]) = \dim H^0 (R_1^{(2)}, O_{R_1^{(2)}}) = 1. \tag{14}$$

Since $R_1^{(1)}$ and $R_1^{(2)}$ are fibers in the fiber space $V' \to E'_1$, all the one-dimensional cycles on $R_1^{(2)}$ are homologous to one-dimensional cycles on $R_1^{(1)}$, and conversely. Therefore any one-dimensional differential of first order on V vanishing on $R_1^{(1)}$ also vanishes on $R_1^{(2)}$, i.e. on $R_1^{(1)} + R_1^{(2)}$.

We denote the number of one-dimensional differentials of first order on V vanishing on $R_1^{(1)}$ by k. The number of connected components of the curve $R_1^{(1)}$ is clearly equal to one, and that of the curve $R_1^{(1)} + R_1^{(2)}$ to two, since $(R_1^{(1)} \cdot R_1^{(1)}) = 0$. From the result of Kodaira [25] cited above, it follows that

$$\begin{aligned} \dim H^1 (V, O\ [K + R_1^{(1)}]) &= 1 - 1 + k = k, \\ \dim H^1 (V, O\ [K + R_1^{(1)} + R_1^{(2)}]) &= 2 - 1 + k = 1 + k. \end{aligned} \tag{15}$$

From (13), (14), and (15) it follows that the mapping

$$H^1 (V, O\ [K + R_1^{(1)}]) \to H^1 (V, O\ [K + R_1^{(1)} + R_1^{(2)}])$$

is a monomorphism and the sequence (13) goes into

$$\begin{aligned} 0 \to H^0(V, O\ [K + R_1^{(1)}]) \to H^0(V, O\ [K + R_1^{(1)} + R_1^{(2)}]) \\ \to H^0 (R_1^{(2)}, O_{R_1^{(2)}} [K + R_1^{(1)} + R_1^{(2)}]_{R_1^{(2)}}) \to 0. \end{aligned} \tag{16}$$

Since V is a surface of fundamental type, the genus of the curve $p_{R_1^{(i)}} \geq 2$ for $i = 1, 2, 3$. Therefore

$$\dim H^0 (R_1^{(2)}, O_{R_1^{(2)}} [K + R_1^{(1)} + R_1^{(2)}]_{R_1^{(2)}})$$
$$= \dim H^0 (R_1^{(2)}, O_{R_1^{(2)}} [K_{R_1^{(2)}}]) \geqslant 2.$$

From this we have $\dim H^0(V, O[K + R_1^{(1)} + R_1^{(2)}]) \geq 2$, i.e. $|K + R_1^{(1)} + R_1^{(2)}|$ is an infinite linear system. Sequence (16) shows that $R_1^{(2)}$ is not a fixed curve for $|K + R_1^{(1)} + R_1^{(2)}|$.

It is known from the theory of algebraic curves that the canonical system on a curve does not have fixed points. Therefore sections from the group

$H^0(R_1^{(2)}, O_{R_1^{(2)}}[K + R_1^{(1)} + R_1^{(2)}]_{R_1^{(2)}})$ do not have a common zero on $R_1^{(2)}$, and (16) shows that the sections of the group $H^0(V, O[K + R_1^{(1)} + R_1^{(2)}])$ also do not have a common zero on $R_1^{(2)}$. This means that there are no base points of the system $|K + R_1^{(1)} + R_1^{(2)}|$ on $R_1^{(2)}$, and the fixed components of this system do not have points in common with $R_1^{(2)}$. It is clear that in all these arguments $R_1^{(1)}$ and $R_1^{(2)}$ could change roles and in the last statements $R_1^{(2)}$ could be replaced by $R_1^{(1)}$.

We will now show that $\dim H^0(V, O[K + R_1^{(i)}]) > 0$, $i = 1, 2, 3$. If the geometric genus $p_g > 0$, this is immediately clear. If $p_g = 0$, then, since V is a surface of fundamental type, Lemma 4 shows that the irregularity of V, $q = 0$. By the Riemann-Roch theorem for surfaces we have

$$\dim H^0(V, O\,[K + R_1^{(i)}]) \geqslant \frac{(K + R_1^{(i)})\,R_1^{(i)}}{2} + 1 - q + p_g = p_{R_1^{(i)}}.$$

From $p_{R_1^{(i)}} \geq 2$, we obtain $\dim H^0(V, O[K + R_1^{(i)}]) > 0$. Let $|M|$ be the nonfixed part of the system $|K + R_1^{(1)} + R_1^{(2)}|$. If we denote by Z the fixed part of the system $|K + R_1^{(1)} + R_1^{(2)}|$, then $|M| = |K + R_1^{(1)} + R_1^{(2)} - Z|$. Since $R_1^{(2)}$ is not contained in Z and $|K + R_1^{(1)}|$ is a nonnegative cycle, Z is contained in $|K + R_1^{(1)}|$.

Any curve of $|K + R_1^{(1)}|$ has the form $Z + C_1$, where C_1 is a nonnegative cycle. Then

$$Z + C_1 + R_1^{(2)} \in |K + R_1^{(1)} + R_1^{(2)}|$$

and

$$C_1 + R_1^{(2)} \in |M|.$$

We now apply enough σ-processes at the base points of $|M|$ so that the proper image of $|M|$ does not have base points. We denote the surface thus obtained by $\widetilde{V}$. The proper image on $\widetilde{V}$ of any linear system $|L|$ on V is denoted by $\overline{L}$, and its total image by $|\widetilde{L}|$. We have $\widetilde{L} = \overline{L} + \Sigma_L S^{(i)}$, where $\Sigma_L S^{(i)}$ is some collection of curves created by σ-processes.

The canonical class K' of the surface $\widetilde{V}$ is expressed in terms of $\widetilde{K}$ according to the formula $K' = \widetilde{K} + \Sigma S^{(i)}$, where $\Sigma S^{(i)}$ is some collection of curves created by σ-processes.

We have

$$\overline{M} + \sum_M S^{(i)} = \widetilde{M} = \widetilde{K} + \widetilde{R}_1^{(1)} + \widetilde{R}_1^{(2)} - \widetilde{Z} = K' - \sum S^{(i)}$$
$$+ \overline{R}_1^{(1)} + \sum_{R_1^{(1)}} S^{(i)} + \overline{R}_1^{(2)} + \sum_{R_1^{(2)}} S^{(i)} - \overline{Z} - \sum_Z S^{(i)}.$$

Hence

$$\overline{M} = K' + \overline{R}_1^{(1)} + \overline{R}_1^{(2)} - \overline{Z} + Y_S,$$

where Y_S is some cycle on V all of whose components are curves created under σ-processes.

On the other hand, $\overline{M} = \overline{C}_1 + \overline{R}_1^{(2)} + Y'_S$ where Y'_S is the nonnegative cycle consisting of the curves created under σ-processes.

We now consider $(\overline{M}^2)$.

$$(\overline{M}^2) = (\overline{M} \cdot \overline{M}) = ((\overline{C}_1 + \overline{R}_1^{(2)} + Y'_S) \cdot \overline{M}).$$

Since on $R_1^{(2)}$, there are no fixed points of the system $|M|$, $\overline{R}_1^{(2)}$ does not have points in common with either $\overline{Z}$ or Y_S, i.e. $(\overline{R}_1^{(2)} \cdot \overline{Z}) = (\overline{R}_1^{(2)} \cdot Y_S) = 0$. From the fact that $(R_1^{(1)} \cdot R_1^{(2)}) = 0$, it follows that $(\overline{R}_1^{(1)} \cdot R_1^{(2)}) = 0$. Using all this, we obtain

$$(\overline{R}_1^{(2)} \cdot \overline{M}) = (\overline{R}_1^{(2)} (K' + \overline{R}_1^{(2)})) + (\overline{R}_1^{(2)} \cdot \overline{R}_1^{(1)}) - (\overline{R}_1^{(2)} \overline{Z}) + (\overline{R}_1^{(2)} Y_S)$$
$$= ((K' + \overline{R}_1^{(2)}) \overline{R}_1^{(2)}) = 2p_{\overline{R}_1^{(2)}} - 2 \geqslant 2 \cdot 2 - 2 = 2.$$

Since $\overline{M}$ does not have fixed components,

$$(\overline{M} \cdot \overline{C}_1) \geqslant 0, \ (\overline{M} \cdot Y'_S) \geqslant 0.$$

Thus

$$(\overline{M} \cdot \overline{M}) = ((\overline{C}_1 + \overline{R}_1^{(2)} + Y'_S) \cdot \overline{M}) \geqslant 2. \tag{17}$$

Since $\overline{M}$ does not have fixed points, if it consisted of a pencil, that would imply that $(\overline{M}^2) = 0$, in contradiction to (17). Consequently, the system $|\overline{M}|$ is not composed of a pencil, and thus also the system $|M|$ on V is not composed of a pencil. With the aid of Bertini's theorem, we now obtain that $|M|$ is an irreducible linear system.

Since $|M|$ does not have fixed components, $(M \cdot S_1) \geq 0$. Clearly, $(R_1^{(3)} \cdot M) = (R_1^{(2)} \cdot M) = ((K + R_1^{(1)} + R_1^{(2)} - Z) \cdot R_1^{(2)})$.

Since $|K + R_1^{(1)} + R_1^{(2)}|$ does not have fixed points on $R_1^{(2)}$, $(Z \cdot R_1^{(2)}) = 0$. Thus $(R_1^{(3)} \cdot M) = ((K + R_1^{(2)}) \cdot R_1^{(2)}) = 2p_{R_1^{(2)}} - 2 \geq 2$.

We now assume that $(M \cdot S_1) > 0$. We write $R_1^{(3)} + S_1 = C$, and note that

$(C\cdot M)=(R_1^{(3)}\cdot M)+(S_1\cdot M)\geq 2+1=3.$

Since V does not contain a pencil of elliptic or rational curves, the geometric genus of a generic curve $M\in|M|$, $p_M\geq 2$. This circumstance and also equation (12) with $R_1^{(3)}$ instead of $R_1(t)$, and the fact that $\dim H^0(V, O[K+C])=\dim H^0(V, O[K+R_1^{(3)}+S_1])>0$, permit one to apply Lemma 2 to the systems $|K+M+C|$ and $|M|$. From $(C\cdot M)\geq 3$ we obtain that $|K+M+C|$ yields a birational mapping of the surface V into a projective space.

One says that a system $|D_1|$ is a part of a system $|D|$ (denoted $|D_1|\subseteq|D|$), if for a curve $D_1\in|D_1|$ there exist a curve $D\in|D|$ such that $D=D_1+D_2$ where $D_2\geq 0$.

It is clear that the field of functions $\mathcal{K}_{D_1}$ on the variety obtained under the mapping of V corresponding to the system $|D_1|$ is the subfield of the field $\mathbf{C}(V)$ (the field of functions on V) generated by the functions $f\in\mathbf{C}(V)$ for which $(f)+D_1\geq 0$. The field $\mathcal{K}_D$ corresponding to the system $|D|$ is generated by the functions $f\in\mathbf{C}(V)$ for which $(f)+D\geq 0$. It is clear that $\mathcal{K}_{D_1}\subseteq\mathcal{K}_D\subseteq\mathbf{C}(V)$. If $|D_1|$ yields a birational imbedding of V, then $\mathcal{K}_{D_1}=\mathbf{C}(V)$. But then $\mathcal{K}_D=\mathbf{C}(V)$, and $|D|$ also yields a birational imbedding of V. Clearly,

$$|K+M+C|=|K+M+R_1^{(3)}+S_1|$$

$$\subseteq|K+K+R_1^{(1)}+R_1^{(2)}+R_1^{(3)}+S_1|=\left|2K+\sum_{i=1}^{3}R_1^{(i)}+S_1\right|.$$

We obtain that $|2K+\Sigma_{i=1}^3R_1^{(i)}+S_1|$ yields a birational imbedding of the surface V into a projective space.

Now let $(S_1\cdot M)=0$. Since $|M|$ does not contain a fixed part, S_1 is a fundamental curve for $|M|$. This means that there is a curve of the form $B+n_1S_1$ in $|M|$, where B does not contain S_1. We must have $B\neq 0$, for otherwise we would have $M=n_1S_1$, and from $(M\cdot S_1)=0$, it would follow that $(M^2)=0$. From the connectedness of $B+n_1S_1$ (the irreducibility of $|M|$ and the principle of degeneracy) it follows that $(B\cdot S_1)>0$. From $(M\cdot S_1)=((B+n_1S_1)\cdot S_1)=0$ we obtain $(S_1^2)<0$. Since S_1 is an irreducible curve, $((K+S_1)\cdot S_1\geq -2$ and $(K\cdot S_1)\geq -2-(S_1^2)\geq -1$. But $(R_1^{(1)}\cdot S_1)=(R_1^{(2)}\cdot S_1)\geq 1$, and thus

$$((K+R_1^{(1)}+R_1^{(2)})\,S_1)\geqslant -1+1+1=1.$$

Comparing this with $(M\cdot S_1)=0$, we obtain that there are fixed components in $|K+R_1^{(1)}+R_1^{(2)}|$ intersecting S_1 (we note that S_1 is not contained in Z, since $(Z\cdot R_1^{(1)})=0$, while $(S_1\cdot R_1^{(1)})>0$). Let S' be the connected component of the carrier of Z that intersects with S_1.

Let us assume that $(S' \cdot M) > 0$. First let $R_1^{(3)}$ not be contained in S'. We will show that $H^1(V, O[K + R_1^{(3)} + S' + S_1]) = 0$. By the result of Kodaira [25] already cited

$$H^1(V, O[K + R_1^{(3)} + S' + S_1]) = m - 1 + k,$$

where m is the number of connected components of the curve $R_1^{(3)} + S' + S_1$, and k is the number of one-dimensional differentials of first order on V vanishing on $R_1^{(3)} + S_1 + S'$.

From $(R_1^{(3)} \cdot S_1) > 0$ and $(S' \cdot S_1) > 0$, it follows that $m = 1$, and from the fact that $R_1^{(3)}$ is parameterized by a point from E''_1, it follows that the number of one-dimensional differentials of first order on V vanishing on $R_1^{(3)} + S_1$ is equal to zero; but then also $k = 0$. Thus,

$$H^1(V, O[K + R_1^{(3)} + S' + S_1]) = 0.$$

Since $\dim H^0(V, O[K + R_1^{(3)} + S' + S_1]) > 0$ and $((R_1^{(3)} + S' + S_1) \cdot M) = (R_1^{(3)} \cdot M) + (S' \cdot M) \geq 2 + 1 = 3$, applying Lemma 2 to the systems $|K + M + (R_1^{(3)} + S' + S_1)|$ and $|M|$ shows that $|K + M + (R_1^{(3)} + S' + S_1)|$ yields a birational imbedding of V. But $|K + M + (R_1^{(3)} + S' + S_1)| \subseteq |K + M + Z + R_1^{(3)} + S_1| = |K + K + R_1^{(1)} + R_1^{(2)} + R_1^{(3)} + S_1| = |2K + \Sigma_{i=1}^{3} R_1^{(i)} + S_1|$, from which it follows that $|2K + \Sigma_{i=1}^{3} R_1^{(i)} + S_1|$ yields a birational imbedding.

Now let $R_1^{(3)}$ be included in S', i.e. also in Z. We consider the exact sequence

$$H^1(V, O[K + R_1^{(3)} + S_1]) \to H^1(V, O[K + 2R_1^{(3)} + S_1])$$
$$\to H^1(R_1^{(3)}, O_{R_1^{(3)}}[(K + R_1^{(3)}) R_1^{(3)} + R_1^{(3)} \cdot R_1^{(3)} + S_1 \cdot R_1^{(3)}]).$$

Since $R_1^{(3)}$ is a nonsingular curve, $(K + R_1^{(3)}) \cdot R_1^{(3)}$ is the canonical class on $R_1^{(3)}$, and $(S_1 \cdot R_1^{(3)}) > 0$, we have

$$H^1(R_1^{(3)}, O_{R_1^{(3)}}[(K + R_1^{(3)}) \cdot R_1^{(3)} + R_1^{(3)} \cdot R_1^{(3)} + S_1 \cdot R_1^{(3)}]) = 0.$$

But also $H^1(V, O[K + R_1^{(3)} + S]) = 0$, and consequently

$$H^1(V, O[K + 2R_1^{(3)} + S_1]) = 0.$$

Since $H^0(V, O[K + 2R_1^{(3)} + S_1]) > 0$, $((2R_1^{(3)} + S_1) \cdot M) = (2R_1^{(3)} \cdot M) \geq 4$, the application of Lemma 2 to the systems $|K + M + (2R_1^{(3)} + S_1)|$ and $|M|$ shows that

$|K + M + (2R_1^{(3)} + S_1)|$ yields a birational imbedding of V.

We note that

$$|K+M+(2R_1^{(3)}+S_1)| \subseteq |K+M+Z+R_1^{(3)}+S_1| = |K+K+R_1^{(1)}+R_1^{(2)} + R_1^{(3)}+S_1\Big| = \Big|2K+\sum_{i=1}^{3} R_1^{(i)}+S_1\Big|,$$

from which it follows that $|2K + \Sigma_{i=1}^{3} R_1^{(i)} + S_1|$ yields a birational imbedding of V.

We will now show that the suppositions $(S_1 \cdot M) = 0$, $(S' \cdot M) = 0$ lead to a contradiction. Thus, let $(S_1 \cdot M) = 0$, $(S' \cdot M) = 0$.

Let $S' = \Sigma_{j=1}^{N'} S_j'$ be a decomposition of S' into irreducible components, and let $S = \Sigma_{j=1}^{N'} m_j S_j'$ be the divisor in Z whose carrier is S'. Since $(M \cdot S_j') = 0$ for all $j = 1, \cdots, N'$,

$$(M \cdot S) = \sum_{j=1}^{N'} m_j (M \cdot S_j') = 0.$$

Since S' is the connected component in the carrier of Z, $((Z - S) \cdot S) = 0$. Thus

$$(M \cdot S) = ((K + R_1^{(1)} + R_1^{(2)} - Z) \cdot S) = ((K + R_1^{(1)} + R_1^{(2)}) \cdot S) - (S \cdot S) - ((Z - S) \cdot S) = ((K + R_1^{(1)} + R_1^{(2)}) \cdot S) - (S \cdot S).$$

From $(M \cdot S) = 0$ it follows that

$$(S \cdot S) = ((K + R_1^{(1)} + R_1^{(2)}) \cdot S). \tag{18}$$

Since $(S_j' \cdot M) = 0$, S_j' is a fundamental curve for $|M|$. Thus there is a connected curve of the form $B_j + n_{1j} S_j'$ in $|M|$, where $B_j \neq 0$ (from $B_j = 0$ and $(S_j' \cdot M) = 0$ it would follow that $(M^2) = 0$) and $(B_j \cdot B_j') > 0$.

From $(M \cdot S_j') = (B \cdot S_j') + n_{1j} (S_j^2) = 0$ we obtain $(S_j'^2) < 0$.

Since S_j' is an irreducible curve, $(K \cdot S_j') + (S_j'^2) \geq -2$, $(K \cdot S_j') \geq -2 - (S_j'^2) \geq -1$. The only time when $(K \cdot S_j') < 0$ is when $(K \cdot S_j) = -1$ and $(S_j'^2) = -1$. It is clear that these conditions characterize an exceptional curve of the first kind. From $(Z \cdot R) = 0$ it follows that $(S_j' \cdot R) = 0$. But it was remarked in the beginning of the proof of the theorem that it is possible to assume that there are no exceptional curves of the first kind on V whose index of intersection with R is equal to zero.

Thus $(K \cdot S_j') \geq 0$. Since $(R_1^{(1)} \cdot S_j') = (R_1^{(2)} \cdot S_j') = 0$, $((K + R_1^{(1)} + R_1^{(2)}) \cdot S_j') \geq 0$. Formula (18) now gives

$$(S \cdot S) \geq 0. \tag{19}$$

In order to arrive at a contradiction it remains to show

$$(S \cdot S) < 0. \tag{19a}$$

The proof we give practically repeats the proof of Mumford [35] of the negative definiteness of the matrix composed of the indices of intersection of the curves obtained under the resolution of a normal singular point on a surface.

We take a curve $M \in |M|$ passing through some point of the curve S_1. Since $(M \cdot S_1) = 0$, we must have

$$M = n_1 S_1 + \sum_k m'_k S'_k + B',$$

where $n_1 > 0$, all the $m'_k > 0$, $B' \geq 0$, and B' contains neither S_1 nor any of the S'_j. We will show that the k in $\Sigma_k m'_k S'_k$ runs through all the values from 1 to N'. If this were not so, it would follow from the connectedness of $S_1 + S'$ and of S' that there is an S'_l not included in $\Sigma_k m'_k S'_k$ but intersecting $S_1 + \Sigma_k m'_k S'_k$. This would mean that $(M \cdot S'_l) > 0$, in contradiction with $(M \cdot S'_l) = 0$.

Since $(M \cdot S'_i) = 0$ we have

$$\sum_{k=1}^{N'} m'_k (S'_k \cdot S'_i) = -((B' + n_1 S_1) \cdot S'_i) \leqslant 0,$$

or

$$\sum_{k=1}^{N'} (m'_k S'_k \cdot m'_i S'_i) \leqslant 0.$$

Moreover, it follows from the connectedness of $S_1 + S'$ that there exist values of i for which $\Sigma_{k=1} (m'_k S'_k \cdot m'_i S'_i) < 0$.

We set $(m'_k S'_k \cdot m'_i S'_i) = a_{ki}$. We obtain that $\Sigma_{k=1}^{N'} a_{ki} \leq 0$, where there exist values of i for which $\Sigma_{k=1}^{N'} a_{ki} < 0$. Since for $k \neq i$, $a_{ki} \geq 0$, the last inequality shows that there exists values of i for which $a_{ii} < 0$. If we now prove that the equation $\Sigma_{k=1}^{N'} \Sigma_{i=1}^{N'} a_{ki} \alpha_k \alpha_i = 0$ is possible only for $\alpha_1 = \cdots = \alpha_{N'} = 0$, then we obtain that the matrix $\|\alpha_{ki}\|$ is negative definite.

Let $\Sigma_{k=1}^{N'} \Sigma_{i=1}^{N'} a_{ki} \alpha_k \alpha_i = 0$. It is easy to verify that

$$\sum_{k=1}^{N'} \sum_{i=1}^{N'} a_{ki} \alpha_k \alpha_i = \sum_{i=1}^{N'} \Big(\sum_{k=1}^{N'} a_{ki} \Big) \alpha_i^2 - \sum_{1 \leqslant k < i \leqslant N'} a_{ki} (\alpha_i - \alpha_k)^2.$$

We obtain

$$\sum_{i=1}^{N'} \Big(\sum_{k=1}^{N'} a_{ki} \Big) \alpha_i^2 - \sum_{1 \leqslant k < i \leqslant N'} a_{ki} (\alpha_i - \alpha_k)^2 = 0,$$

and from the inequality $a_{ki} \geq 0$ for $k \neq i$, and $\Sigma_{k=1}^{N'} a_{ki} \leq 0$ it follows that

$$\sum_{i=1}^{N'}\left(\sum_{k=1}^{N'} a_{ki}\right)\alpha_i^2 = 0. \tag{20}$$

$$\sum_{1\leqslant k<i\leqslant N'} a_{ki}\,(\alpha_i - \alpha_k)^2 = 0. \tag{21}$$

Since there exists an i for which $\Sigma_{k=1}^{N'} a_{ki} < 0$, (20) shows that there exists an i for which $\alpha_i = 0$. It follows from connectedness of S' that from any i to any index k there can be found a sequence of indices $i_0 = i,\ i_1, i_2, \cdots, i_r = k$ such that for any adjacent members of the sequence $i_p,\ i_{p+1}$, $(S'_{i_p} \cdot S'_{i_{p+1}}) > 0$, i.e. $a_{i_p i_{p+1}} > 0$.

From (21) we obtain successively $\alpha_i = \alpha_{i_1} = \cdots = \alpha_k$, i.e. that all the α_k are zero.

We can write

$$(S\cdot S) = \sum_{k=1}^{N'}\sum_{i=1}^{N'} (m_k S'_k \cdot m_i S'_i) = \sum_{k=1}^{N'}\sum_{i=1}^{N'} (m'_k S'_k \cdot m'_i S'_i)\left(\frac{m_k}{m'_k}\right)\left(\frac{m_i}{m'_i}\right)$$
$$= \sum_{k=1}^{N'}\sum_{i=1}^{N'} a_{ki}\left(\frac{m_k}{m'_k}\right)\left(\frac{m_i}{m'_i}\right),$$

since the quantities m_k/m'_k are not zero.

Thus we have proved that $|2K + \Sigma_{i=1}^{3} R_1^{(i)} + S_1|$ yields a birational imbedding of the surface V.

We denote by m' the number of irreducible components into which a generic curve of the system $|R|$ is decomposed. Since $|R|$ does not contain fixed parts, there always exist curves $R \in |R|$ for which all the irreducible components are parameterized by points of E''_1. Let $m' \geq 3$. Then it is possible to assume that $R_1^{(1)}$, $R_1^{(2)}$, and $R_1^{(3)}$ are components of some curve $R \in |R|$, i.e. $|\Sigma_{i=1}^{3} R_1^{(i)}| \subseteq |R|$. We have

$$\left|2K + \sum_{i=1}^{3} R_1^{(i)} + S_1\right| \subseteq |\,2K + R + S_1\,| \subseteq |\,2K + nK\,| = |\,(n+2)\,K\,|,$$

i.e., for $m' \geq 3$ the system $|(n+2)K|$ yields a birational imbedding of the surface V.

For $m' = 2$, we can assume that $|2R| \supseteq |\Sigma_{i=1}^{3} R_1^{(i)}|$. Then $|2K + \Sigma_{i=1}^{3} R_1^{(i)} + S_1| \subseteq |2K + 2R + S_1| \subseteq |2K + 2nK| = |(2n+2)K|$, i.e. for $m' = 2$ the system $|(2n+2)K|$ yields a birational imbedding of V.

For $m' = 1$, we can assume that $|3R| \supseteq |\,\Sigma_{i=1}^{3} R_1^{(i)}|$, i.e.

$$\left|2K+\sum_{i=1}^{3} R_1^{(i)}+S_1\right| \subseteq |2K+3R+S_1| \subseteq |2K+3nK| = |(3n+2)K|,$$

and $|(3n+2)K|$ yields a birational imbedding of V. Since $|(n+2)K| \subseteq |(2n+2)K| \subseteq |(3n+2)K|$, it is possible to say that for $m' \geq 2$, the system $|(2n+2)K|$ and for any m' the system $|(3n+2)K|$ gives a birational imbedding of V.

For $P_n = 2$ the system $|nK|$ is necessarily composed of a pencil, and hence for the case $P_n = 2$ the statement of the theorem is proved in full.

We now take $P_n > 2$, and we will consider what kind of restriction the condition $m' \leq 2$ imposes on P_n. Thus, let us assume that $m' \leq 2$, $P_n > 2$.

We consider the regular mapping of V onto a curve X corresponding to the system $|R|$. We will denote by H a divisor of the hyperplane section of the curve X under the imbedding of V into a projective space corresponding to the mapping $f_R: V \to X$. We have already referred to the part of [25] where it is proved that one can take open subsets V' on V and X' on X such that to each point of X' there corresponds a nonsingular (possibly reducible) curve from V', where homeomorphic curves correspond to distinct points of X', i.e. in any case, curves having the same number of irreducible components. Let this number be h. Then if the degree of the divisor H is equal to $a(H)$, it is clear that the number m' of irreducible components is equal to $a(H) \cdot h$ for almost all of the curves of $|R|$. Thus, since $m' \leq 2$ we have $a(H) \cdot h \leq 2$ and $a(H) \leq 2$. It is clear, however, that $\dim H^0(X, O_X[H]) \geq \dim H^0(V, O[R])$.

Let $\widetilde{X}$ be a nonsingular model of the curve X, $[H]_{\widetilde{X}}$ be the fibering over $\widetilde{X}$ induced by the fibering $[H]$ over X, $\overline{O_{\widetilde{X}}[H]_{\widetilde{X}}}$ be the direct image of the sheaf $O_{\widetilde{X}}[H]_{\widetilde{X}}$ under the mapping $\widetilde{X} \to X$. It is clear that $O_X[H] \subseteq \overline{O_{\widetilde{X}}[H]_{\widetilde{X}}}$.

We obtain that $\dim H^0(X, O_X[H]_X) \leq \dim H^0(X, \overline{O_{\widetilde{X}}[H]_{\widetilde{X}}})$. But $\dim H^0(X, \overline{O_{\widetilde{X}}[H]_{\widetilde{X}}}) = \dim H^0(X, O_{\widetilde{X}}[H]_{\widetilde{X}})$. By the Riemann-Roch theorem for the curve $\widetilde{X}$ we have

$$\dim H^0(\widetilde{X}, O_{\widetilde{X}}[H]_{\widetilde{X}}) = a(H) + 1 - p_{\widetilde{X}} + \dim H^0(\widetilde{X}, O_{\widetilde{X}}[K_{\widetilde{X}} - (H_{\widetilde{X}})]).$$

Since $(H)_{\widetilde{X}}$ is a positive divisor on $\widetilde{X}$, from the absence of fixed points in the canonical system on the curve we obtain for $p_{\widetilde{X}} \neq 0$

$$\dim H^0(\widetilde{X}, O_{\widetilde{X}}[K_{\widetilde{X}} - (H)_{\widetilde{X}}]) \leqslant p_{\widetilde{X}} - 1.$$

It follows from this that for $p_{\widetilde{X}} \neq 0$

$$\dim H^0(\widetilde{X}, O_{\widetilde{X}}[H]_{\widetilde{X}}) \leqslant a(H) \leqslant 2.$$

Hence also $\dim H^0(V, O[R]) \leq 2$. Since $|R|$ is the fixed part of the system $|nK|$,

$$P_n = \dim H^0(V, O[nK]) = \dim H^0(V, O[R]) \leqslant 2.$$

We arrive at a contradiction with the assumption $P_n > 2$.

There remains the case $p_{\widetilde{X}} = 0$, i.e. $\widetilde{X}$ is a rational curve. Then $\dim H^0(\widetilde{X}, O_{\widetilde{X}}[H]) = a(H) + 1 \leq 3$. As above, one obtains from this $P_n \leq 3$. Together with $P_n > 2$, this gives $P_n = 3$. Then $\dim H^0(\widetilde{X}, O_{\widetilde{X}}[H]_{\widetilde{X}}) = 3$ and $a(H) = 2$. From $m' = a(H) \cdot h \leq 2$ we obtain $m' = 2$, $h = 1$, and $|R| = |R_1^{(1)} + R_1^{(2)}|$. The equation $h = 1$ means that irreducible curves of V correspond to points of X'. But these curves must be components of curves of $|R|$. Since X is a rational curve, we obtain that the curves $\{R_1^*\}$ are linearly equivalent.

We obtain that $R_1^{(1)}$ and $R_1^{(2)}$ vary in an infinite linear system. From results of [25] it now follows that $H^1(V, O[K + R_1^{(1)}]) = 0$. We proved earlier that $H^0(V, O[K + R_1^{(1)}]) > 0$. Since the genus of the curve $R_1^{(2)}$, $p_{R_1^{(2)}} > 2$ and $|R_1^{(2)}|$ is an infinite irreducible system, one can apply Lemma 2 to the systems $|K + R_1^{(2)} + R_1^{(1)}|$ and $|R_1^{(2)}|$. From this lemma we obtain that $|K + R_1^{(1)} + R_1^{(2)}|$ yields either a birational imbedding of V, or a mapping of V of degree two onto a rational surface. It is clear that

$$|K + R_1^{(1)} + R_1^{(2)}| \subseteq |K + nK| \subseteq |(2n + 1) K|.$$

Let f_1 be the rational mapping corresponding to the system $|K + R_1^{(1)} + R_1^{(2)}|$ and let f be the mapping corresponding to the system $|(2n + 1)K|$.

It is clear that we have the commutative diagram

$$\begin{array}{ccc} & \overset{f}{\nearrow} & f(V) \\ V & & \downarrow g \\ & \underset{f_1}{\searrow} & f_1(V), \end{array}$$

where g is a rational mapping of $f(V)$ onto $f_1(V)$. For the degrees of the mappings we have

$$d(f_1) = d(g)\, d(f).$$

Since $d(f_1) \leq 2$, $d(f) \leq 2$. When $d(f) = 1$, $|(2n + 1)K|$ yields a birational imbedding; when $d(f) = 2$, then $d(f_1) = 2$, $d(g) = 1$, i.e. $f(V)$ is birationally equivalent to $f_1(V)$, and $f(V)$ along with $f_1(V)$ is a rational surface.

We thus obtain that f for $m' \leq 2$ is either a birational imbedding, or a mapping of degree two on a rational surface. Let us assume that

$$\dim H^0(V, O[(n - 1) K]) > 0.$$

Then $|(n+2)K| \subseteq |(n+2)K+(n-1)K| = |(2n+1)K|$ and for $P_n = 3$, $m' > 2$, the system $|(2n+1)K|$ yields a birational imbedding of V. We obtain that the conditions that $P_n = 3$, that $\dim H^0(V, O[(n-1)K]) > 0$ and that $|nK|$ be composed of a pencil imply that $|(2n+1)K|$ yields a birational imbedding of V or a mapping of V of degree two onto a rational surface.

We showed above that $m' = 2$, $P_n = 3$ follow from the conditions $m' \leq 2$, $P_n > 2$.

It follows from this first of all that $P_n > 3$ implies that $m' \geq 3$, and thus if $P_n > 3$ and $|nK|$ is composed of a pencil, then $|(n+2)K|$ yields a birational imbedding.

If $P_n = 3$, then we obtain $m' \geq 2$, from which it follows that if $P_n = 3$ and $|nK|$ is composed of a pencil, then the system $|(2n+2)K|$ yields a birational imbedding of V. We see that in the case when $|nK|$ is composed of a pencil, all the assertions of the theorem have been proven.

Now let the system $|nK|$, and thus the system $|R|$, not be composed of a pencil. From Bertini's theorem it then follows that $|R|$ is an infinite irreducible system. Since V does not contain pencils of rational or elliptic curves, the genus of a generic curve $R \in |R|$, $p_R \geq 2$. We will show that $\dim H^0(V, O[K+R]) > 0$.

If $p_g > 0$, this is immediately clear. If $p_g = 0$, it follows from Lemma 4 that $q = 0$. The Riemann-Roch theorem gives us that $\dim H^0(V, O[K+R]) \geq ((K+R)\cdot R)/2 + 1 - q + p_g = ((K+R)\cdot R)/2 + 1$. Having noted that $((K+R)\cdot R)/2 + 1 \geq p_R \geq 2$, we obtain that $\dim H^0(V, O[K+R]) > 0$. Since R is a connected curve varying in an infinite linear system, it follows from the results of [25] that

$$H^1(V, O[K+R]) = 0.$$

It is now clearly possible to apply Lemma 2 to the systems $|K+R+R|$ and $|R|$, which shows us that in the case $(R\cdot R) \geq 3$, $|K+R+R|$ yields a birational imbedding, and for $(R\cdot R) \leq 2$, $|K+R+R|$ yields either a birational imbedding or a mapping of degree two onto a rational surface.

Let us consider the case $(R\cdot R) \leq 2$. We write down the exact sequence

$$0 \to H^0(V, O_V) \to H^0(V, O_V[R]) \to H^0(R, O_R[R\cdot R]). \tag{22}$$

Since $|R|$ does not have base points (see the beginning of the proof), by Bertini's theorem we can assume that R is a nonsingular curve.

Applying the Riemann-Roch theorem, we have

$$\dim H^0(R, O_R[R\cdot R]) = (R\cdot R) + 1 - p_R + \dim H^0(R, O_R[K_R - R\cdot R]).$$

Since $p_R \neq 0$ and $(R\cdot R) > 0$ we have $\dim H^0(R, O_R[K_R - R\cdot R]) \leq p_R - 1$ and thus $\dim H^0(R, O_R[R\cdot R]) \leq (R\cdot R) \leq 2$. Further, $\dim H^0(V, O_V) = 1$ and (22) shows that $\dim H^0(V, O_V[R]) \leq \dim H^0(V, O_V) + \dim H^0(R, O_R[R\cdot R]) \leq 1 + 2 = 3$. Thus $P_n = H^0(V, O[nK]) = H^0(V, O[R]) \leq 3$. Since $|nK|$ is not composed of a pencil, $P_n > 2$ and thus $P_n = 3$.

Since $|K + R + R| \subseteq |K + nK + nK| = |(2n+1)K|$, we obtain that the conditions that $P_n > 3$ and that $|nK|$ is not composed of a pencil imply that $|(2n+1)K|$ yields a birational imbedding of V, and the conditions that $P_n = 3$ and that $|nK|$ is not composed of a pencil imply that $|(2n+1)K|$ yields either a birational imbedding of V or a mapping of V of degree two onto a rational surface.

We now note that $(R\cdot R) \geq 2$. For the fact that $|R|$ is not composed of a pencil implies that $(R^2) > 0$, and it would follow from $(R^2) = 1$ that the image of V under the regular mapping f_R corresponding to R is a surface of first degree, i.e., a plane, and f_R has degree one. But then V would be a rational surface, which contradicts the assumption of the theorem.

From $(R\cdot R) \geq 2$, we obtain $2(R\cdot R) \geq 4 > 3$.

Applying Lemma 2 to the systems $|K + R + 2R|$ and $|R|$, we obtain that $|K + R + 2R| = |K + 3R|$ yields a birational imbedding of V.

Since $|K + 3R| \subseteq |K + 3nK| = |(3n+1)K|$; we obtain that from the conditions that $P_n = 3$ and that $|nK|$ is not composed of a pencil, it follows that $|(3n+1)K|$ yields a birational imbedding of V. All the assertions of the theorem are proved.

The most important consequence of Theorem 1 is the following theorem.

Theorem 2. *Let V be an algebraic surface of fundamental type. Then, if the geometric genus of V, $p_g > 3$, $|3K|$ yields a birational imbedding of V, while if $p_g = 3$, $|3K|$ yields either a birational imbedding of V or a mapping of V of degree two onto a rational surface.*

Proof. Since $P_1 = p_g \geq 3$, it is possible to take the number n of Theorem 1 to be one.

If one notes further that for $n = 1$ we have $P_{n-1} = P_0 = 1$ and $2n + 1 = n + 2 = 3$ then the assertions of Theorem 2 follow immediately from points 4) and 3) of Theorem 1. The theorem is proved.

The following theorem shows that for surfaces of basic types, when $p_g \leq 3$ small multiples of the canonical class will already yield a birational imbedding, and, in particular, $|9K|$ will yield a birational imbedding for any surface of fundamental type.

Theorem 3. *Let V be an algebraic surface of fundamental type. Then*

1) *for $p_g = 0$ the system $|7K|$ yields a birational imbedding of V;*

2) *for $p_g = 2$ the system $|5K|$ yields a birational imbedding of V;*

3) *for* $p_g = 3$ *the system* $|4K|$ *yields a birational imbedding of* V;

4) *for any* p_g *the system* $|9K|$ *yields a birational imbedding of* V.

Proof. It is clear that one can assume that V is a nonsingular surface without exceptional curves of the first kind. Then $(K^2) > 0$ (Lemma 5). Noether's formula gives us

$$p_a = \frac{1}{12}((K^2) + 2 - 4q + b_2),$$

or, since

$$p_a = 1 - q + p_g; \quad b_2 = h^{2,0} + h^{1,1} + h^{0,2} = 2p_g + h^{1,1},$$
$$12 - 12q + 12p_g = (K^2) + 2 - 4q + 2p_g + h^{1,1}.$$

From this we have

$$q = \frac{10(p_g + 1) - (K^2) - h^{1,1}}{8}. \tag{23}$$

1) Let $p_g = 0$. It then follows from Lemma 5 that $q = 0$.

The Riemann-Roch theorem gives us

$$P_n = \dim H^0(V, O[nK]) \geqslant \frac{(nK(nK - K))}{2} + 1 - q + p_g = \frac{n(n-1)}{2}(K^2) + 1. \tag{24}$$

For $n = 2$ and $n = 3$ we obtain from this

$$P_2 \geqslant \frac{2(2-1)}{2}(K^2) + 1 = (K^2) + 1 \geqslant 2 \tag{25}$$

$$P_3 \geqslant \frac{3(3-1)}{2}(K^2) + 1 = 3K^2 + 1 \geqslant 4. \tag{26}$$

The last inequality shows us that it is possible to apply point 4) of Theorem 1 for $n = 3$. Noting that for $n = 3$, $n + 2 = 5$, $2n + 1 = 7$, we obtain that the system $|5K|$ or the system $|7K|$ yields a birational imbedding.

It follows from (25) that $|2K|$ contains a positive cycle, and thus $|5K| \subseteq |7K| \subseteq |9K|$. This shows that for $p_g = 0$, $|7K|$ always yields a birational imbedding of V, and also $|9K|$ will always yield a birational imbedding of V.

2) Let $p_g = 2$. Then an application of point 1) of Theorem 1 with $n = 1$ shows ($3n + 2 = 5$ for $n = 1$) that $|5K|$ yields a birational imbedding of V. Since here $p_g > 0$ and $|K|$ thus contains a nonnegative cycle, $|5K| \subseteq |9K|$, i.e. $|9K|$ also yields a birational imbedding of V.

3) Let $p_g = 3$. Noting that $2n + 2 = 3n + 1 = 4$ for $n = 1$, we apply point 2) of Theorem 1 and obtain that $|4K|$ yields a birational imbedding of V. Since $|4K| \subseteq |9K|$, $|9K|$ also yields a birational imbedding of V.

4) To complete the proof it remains to consider the case $p_g = 1$. Formula (23) gives us that for $p_g = 1$

$$q = \frac{20-(K^2)-h^{1,1}}{8}.$$

It is evident from this that $q \leq 2$. By the Riemann-Roch theorem

$$P_n = \dim H^0(V, O, [nK]) \geqslant \frac{n(n-1)}{2}(K^2) + 1 - q + p_g \geqslant \frac{n(n-1)}{2}(K^2).$$

For $n = 3$ and $n = 4$ we have

$$P_3 \geqslant 3(K^2) \geqslant 3, \tag{27}$$

$$P_4 \geqslant \frac{4(4-1)}{2}(K^2) = 6(K^2) \geqslant 6. \tag{28}$$

One can see from this that it is possible to apply point 4) of Theorem 1 for $n = 4$. Noting that for $n = 4$, $n + 2 = 6$ and $2n + 1 = 9$, we obtain that the system $|6K|$ or the system $|9K|$ yields a birational imbedding of V.

It follows from (27) that $|3K|$ contains a positive cycle, and thus $|6K| \subseteq |9K|$. We obtain that $|9K|$ always yields a birational imbedding of V. The theorem is proved.

The results obtained can be somewhat improved if one assumes that V is a regular surface of fundamental type.

Theorem 4. *Let V be a regular algebraic surface of fundamental type and let n be a natural number satisfying the condition $P_n \geq 2$. Then:*

1) *for $p_g = 0$ the system $|(2n+1)K|$ yields either a birational imbedding of V or a mapping of V of degree two onto a rational surface;*

2) *for $p_g > 0$ the system $|(n+1)K|$ yields either a birational imbedding of V or a mapping of degree two onto a rational surface.*

Proof. Since $P_n \geq 2$, $|nK|$ is an infinite system. We can assume that V is a nonsingular surface and that the nonfixed part of the system $|nK|$ (we denote it by $|R|$) does not have base points.

Let us first assume that $|R|$ is composed of a pencil, and let $\{R_1^*\}$ be the irreducible algebraic system described in Theorem 1. Let $R_1^{(1)}$ and $R_1^{(2)}$ be any two curves of the system $\{R_1^*\}$. Since V is a regular surface, $R_1^{(1)}$ and $R_1^{(2)}$ are linearly equivalent. Let f be a function for which $R_1^{(1)}$ is a zero divisor and $R_1^{(2)}$ is a divisor of the poles.

We take any curve $R_1 \in \{R_1^*\}$ different from $R_1^{(2)}$. Since $(R_1^{(2)} \cdot R_1) = 0$, f is regular on R_1 and is thus constant on R_1. From the connectedness of R_1 (the irreducibility of $\{R_1^*\}$ and the principle of degeneracy) it follows that f takes the same constant value c on the whole curve R_1 and thus is a component of the zero divisor of the function $f - c$. Since this divisor must be homologous to the cycle $R_1^{(1)}$ and R_1 is homologous to $R_1^{(1)}$, R_1 coincides with the zero divisor of $f-c$. We obtain that all the curves $R_1 \in \{R_1^*\}$ are level curves of the function f, and thus that $\{R_1^*\}$ is a linear system. We denote it by $|R_1|$. It is clear that $|R_1| \subseteq |R| \subseteq |nK|$.

Now let $|R|$ not be composed of a pencil. Then by Bertini's theorem $|R|$ is an irreducible system. We denote it also by $|R_1|$. We obtain that in all cases there exists an infinite linear irreducible system $|R_1|$ that is a part of the system $|nK|$.

Since V is a surface of basic type, the geometric genus p_{R_1} of a generic curve $R_1 \in |R|$ is not less than two.

Let $p_g > 0$. Then $H^0(V, O_V[K]) > 0$. It follows from the regularity of V that $H^1(V, O_V[K]) = 0$. We can now apply Lemma 2 to the systems $|K + R_1|$ and $|R_1|$, from which we obtain that the system $|K + R_1|$ yields either a birational imbedding of V or a mapping of V of degree two onto a rational surface.

The assertion of the theorem for $p_g > 0$ now follows immediately from the fact that

$$|K + R_1| \subseteq |K + R| \subseteq |K + nK| = |(n+1)K|.$$

Let $p_g = 0$. By the Riemann-Roch theorem we have

$$\dim H^0(V, O[K+R_1]) \geqslant \frac{((K+R_1)R_1)}{2} + 1 - q + p_g = \frac{((K+R_1)R_1)}{2} + 1 = p_{R_1} \geqslant 2.$$

Further, $H^1(V, O[K + R_1]) = 0$, since R_1 is a connected curve and there are no one-dimensional differentials of first order on V. We now apply Lemma 2 to the systems $|K + R_1 + R_1|$ and $|R_1|$. We obtain that the system $|K + 2R_1|$ yields either a birational imbedding of V or a mapping of V of degree two onto a rational surface. It remains to note that $|K + 2R_1| \subseteq |K + 2nK| = |(2n+1)K|$. The theorem is proved.

Theorem 5. *Let V be a regular algebraic surface of fundamental type. Then*

1) *for $p_g \geq 2$ the system $|2K|$ yields either a birational imbedding of V or a mapping of V of degree two onto a rational surface;*

2) *for $p_g = 1$ the system $|3K|$ yields either a birational imbedding or a mapping of degree two onto a rational surface, and the system $|7K|$ yields a birational imbedding of V;*

3) *for $p_g = 0$ the system $|5K|$ yields either a birational imbedding of V or a mapping of V of degree two onto a rational surface;*

4) *for any regular surface of fundamental type the system $|7K|$ yields a birational imbedding.*

Proof.

1) Let $p_g \geq 2$. Then the assertion of the theorem follows immediately from point 2) of Theorem 4 for $n = 1$.

2) Let $p_g = 1$. The Riemann-Roch theorem gives us

$$\dim H^0(V, O[2K]) \geqslant \frac{2(2-1)}{2}(K^2) + 1 - q + p_g = (K^2) + 2 \geqslant 3$$

Applying point 2) of Theorem 4, with $n = 2$, we obtain that $|3K|$ yields either a birational imbedding or a mapping of degree two onto a rational surface. We also apply points 2) and 4) of Theorem 1 for $n = 2$. Noting that for $n = 2$, $n + 2 = 4$, $2n + 1 = 5$, $2n + 2 = 6$, $3n + 1 = 7$ and that for $p_g > 0$, $|4K| \subseteq |5K| \subseteq |6K| \subseteq |7K|$, we obtain that $|7K|$ yields a birational imbedding of the surface V.

3) Let $p_g = 0$. By the Riemann-Roch theorem we have

$$\dim H^0(V, O[2K]) \geqslant (K^2) + 1 - q + p_g = (K^2) + 1 \geqslant 2.$$

Applying point 1) of Theorem 4 with $n = 2$ ($2n + 1 = 2 \cdot 2 + 1 = 5$), we obtain that the system $|5K|$ yields either a birational imbedding of V or a mapping of V of degree two onto a rational surface.

The assertion of 4) follows easily from 2) and from the results of Theorems 2 and 3. The theorem is proved.

The results of Theorems 2, 3, and 5 may be easily given in a table. We note that by Lemma 4 any surface of fundamental type with $p_g = 0$ is regular.

§3. Regular surfaces of fundamental type with $p_g = 3$ for which $|3K|$ does not yield a birational imbedding

Theorem 6. *In order for an algebraic surface V to be a regular surface of fundamental type with $p_g = 3$ for which $|3K|$ does not give a birational imbedding, it is necessary and sufficient that there exist in the class of surfaces birationally equivalent to V a surface V' given in three-dimensional affine space by the equation $z^2 = F_8(x, y)$, where $F_8(x, y)$ is a polynomial of eighth degree of x and y that does not have multiple factors and the curve C on the plane (x, y) with the equation $F_8(x, y) = 0$ possess the following properties:*

1) *C is not tangent to the infinite line at any point;*

2) *C does not have singular points at points of intersection with the infinite line;*

3) *any singular point of C is either of quadratic singularity or is a singular point of third order which, after one σ-process on the plane at the point, goes into a point of multiplicity not greater than two on the proper image of the curve C.*

Table of the results of Theorems 2, 3, and 5
(only for surfaces of fundamental type)

Geometric genus	Irregularity	Multiple of the canonical class yielding a birational imbedding	Multiple of the canonical class yielding either a birational imbedding or a mapping of degree two onto a rational surface
$p_g > 3$	arbitrary	$3K$	
$p_g \geqslant 2$	$q = 0$		$2K$
arbitrary	arbitrary	$9K$	
	$q = 0$	$7K$	
$p_g = 0$		$7K$	$5K$
$p_g = 1$	arbitrary	$9K$	
	$q = 0$	$7K$	$3K$
$p_g = 2$	arbitrary	$5K$	
$p_g = 3$	arbitrary	$4K$	$3K$

Proof. Let V be an arbitrary regular surface of fundamental type with $p_g = 3$ for which $|3K|$ does not yield a birational imbedding. We can assume that V is nonsingular.

Let $|\overline{K}|$ be the nonfixed part of the canonical system $|K|$, and let S be its fixed part. Let us assume that $|\overline{K}|$ is composed of a pencil.

We perform enough σ-processes on V so that the system $|\overline{K}|$ does not have any fixed points. Let $\{R_1\}$ be the one-dimensional irreducible algebraic system on V of which $|\overline{K}|$ is composed. Let E be a curve parametrizing the system $\{R_1\}$, and let E' be an open subset on this curve corresponding to the nonsingular members of $\{R_1\}$.

If there is among the exceptional curves of the first kind on V one whose restriction leaves elements of $\{R_1\}$ parametrizing points of E' nonsingular, then we eliminate it. We thus obtain from V a surface V_1. Again we eliminate on it an exceptional curve of the first kind whose restriction leaves elements of $\{R_1\}$ parametrizing points of E' nonsingular (if such a curve of first order exists), and we continue this process. Since the second Betti number is reduced at each step, this process must terminate. This means that we can assume that there are no exceptional curves of the first kind on V whose restriction leaves elements of $\{R_1\}$ parametrizing points of E' nonsingular.

The last statement can also be formulated in the following way: there is no exceptional curve of the first kind T_1 on V such that $(R_1 \cdot T_1) \leq 1$. Since V is a regular surface, all the elements of the system $\{R_1\}$ are linearly equivalent. Therefore if $\{R_1\}$ is not a linear pencil, $\{R_1\}$ may be made part of an irreducible linear system whose dimension is greater than one. But this would contradict the assumption that $|\overline{K}|$ is composed of a pencil.

Thus $\{R_1\}$ is a linear pencil. We will denote it in the future by $|R_1|$. We note that V is birationally equivalent to a surface $\widetilde{V}$ (with the aid of σ-processes) such that $|R_1|$ does not have base points i.e. such that the elements of $|R_1|$ are fibers of the mapping $\widetilde{V} \to E$.

Let $\overline{K}$ be homologous to mR_1. It is easy to show that the functions of $L_{\widetilde{V}}(\overline{K})$ ($f \in L_{\widetilde{V}}(\overline{K})$ if $((f) + \overline{K}) > 0$) are induced by functions of $C(E)$ under the mapping $\widetilde{V} \to E$, and more precisely by the functions of $L_E(mP)$, where P is an arbitrary point on E. Since E is a line, $\dim L_E(mP) = m + 1$. We have $m + 1 = \dim L_C(mP) = \dim L_{\widetilde{V}}(\overline{K}) = p_g(V) = 3$. Thus $m = 2$. Returning to V, we obtain that $|\overline{K}| = |2R_1|$. We consider first the case $(R_1^2) > 0$. Then, noting that

$$\dim H^0(V, O[K + 3R_1]) > 0, \quad H^1(V, O[K + 3R_1]) = 0$$

(the latter follows from the connectedness of the curve $R_1^{(1)} + R_1^{(2)} + R_1^{(3)}$, where $R_1^{(1)}$, $R_1^{(2)}$ and $R_1^{(3)}$ are arbitrary distinct irreducible elements of $|R_1|$), $(3R_1 \cdot R_1) \geq 3$, and applying Lemma 2 to the systems $|M| = |R_1|$ and $|L| = |K + R_1 + 3R_1|$, we obtain that the system $|K + 4R_1|$ yields a birational imbedding of V into the projective space. Since $|K + 4R_1| = |K + 2\overline{K}| \subseteq |3K|$, $|3K|$ also yields a birational imbedding of V. This contradicts our assumption with respect to the surface V.

Thus, we can assume that $(R_1^2) = 0$ on V. But then we obtain

$$2 \leqslant ((K + R_1) R_1) = (K \cdot R_1) = ((\overline{K} + S) \cdot R_1) = ((2R_1 + S) \cdot R_1) = (S \cdot R_1),$$

since $((K + R_1) \cdot R_1)/2 + 1 \geq 2$. This means that there can be found in S an irreducible curve S_1 such that $(R_1 \cdot S_1) > 0$.

We consider the exact sequence

$$0 \to H^0(V, O[K]) \to H^0(V, O[K + R_1]) \to H^0(R_1, O[K + R_1]_{R_1}) \to 0, \quad (29)$$

where R_1 is an arbitrary nonsingular element of $|R_1|$. $(K + R_1)_{R_1}$ is the canonical class on R_1, and from the fact that the canonical system on R_1 does not have fixed points, we obtain that the sections of $H^0(R_1, O[K+R_1]_{R_1})$ do not have a common zero. Formula (29) then shows that the sections of $H^0(V, O[K+R_1])$ do not have a common zero on R_1, and this means that the system $|K+R_1|$ does not have fixed points of R_1, i.e., in particular, its fixed part does not intersect with R_1. Let Z be the fixed part of the system $|K+R_1|$ and D its nonfixed part. Since $\dim|K+R_1| \geq |\dim K| = 2$, D is an infinite system; hence its generic curve does not have multiple components, does not have the component S_1, and the condition $(D \cdot S_1) > 0$ is sufficient for the connectedness of the curve $D + S_1$.

Let us assume that $(D \cdot S_1) = 0$. We will show that in this case

$$(Z \cdot S_1) > 0. \tag{30}$$

Since $(S_1 \cdot R_1) > 0$ and $(Z \cdot R_1) = 0$, S_1 cannot be contained in Z and $(Z \cdot S_1) \geq 0$. If it were true that $(Z \cdot S_1) = 0$, then

$$((K+R_1)\cdot S_1) = (D \cdot S_1) + (Z \cdot S_1) = 0,$$

i.e. $(R_1 \cdot S_1) = -(K \cdot S_1)$, and we would have $(K \cdot S_1) < 0$. Since $|K|$ contains effective cycles ($p_g = 3$), it follows from $(K \cdot S_1) < 0$ that $(S_1^2) < 0$. From $((K+S_1)\cdot S_1) \geq -2$ we obtain

$$(K \cdot S_1) = -1, \ S_1^2 = -1, \ \frac{((K+S_1)S_1)}{2} + 1 = 0,$$

$$(R_1 \cdot S_1) = -(K \cdot S_1) = 1.$$

But these conditions characterize an exceptional curve of the first kind on V for which $(R_1 \cdot S_1) = 1$. This contradicts our choice of the surface V. This proves (30).

Let S' be the connected component of the carrier of Z intersecting S_1.

Let us assume first that $(S' \cdot D) > 0$. From $(S' \cdot S_1) > 0$ and $(S' \cdot D) > 0$ it follows that there exists in the system $|D + S' + S_1|$ a connected curve without multiple components, and thus

$$H^1(V, O[K+D+S'+S_1]) = 0.$$

Further,

$$((D+S'+S_1)\cdot R_1) = ((K+R_1-Z+S'+S_1)R_1) = ((K+R_1)R_1) + (S_1 \cdot R_1) \geqslant 2p(R_1) - 2 + (S_1 \cdot R_1) \geqslant 2+1 = 3.$$

Now, noting that $H^0(V, O[K+D+S'+S_1]) \neq 0$, and applying Lemma 2 to the systems $|M| = |R_1|$ and $|L| = |K+R_1+D+S'+S_1|$, we obtain that $|K+R_1+D+S'+S_1|$ yields a birational imbedding of V. Since

$$|K+R_1+D+S'+S_1|\subseteq|K+R_1+D+Z+S_1|$$
$$=|K+R_1+K+R_1+S_1|=|2K+\bar{K}+S|\subseteq|3K|,$$

$|3K|$ yields a birational imbedding of V. This contradicts our assumption with respect to V.

Now let $(S'\cdot D)=0$. We note that $|D|$ is not composed of a pencil. Indeed, it follows from (29) that there is a function f_1 in $L_V(D)$ that is nonconstant on R_1, and from $|R_1|\subseteq|D|$ it follows that there is a nonconstant function f_2 in $L_V(D)$ that is constant on R_1. Since f_1 and f_2 are algebraically independent, $|D|$ yields a mapping of V onto an algebraic surface, i.e. it is not composed of a pencil. $(S'\cdot D)=0$ means that S' is a fundamental curve of the system D. Repeating the argument given in the proof of formula (19a), we obtain that $(S_2^2)<0$, where S_2 is the divisor in Z whose carrier is S', and that $(S_1'^2)<0$, where S_1' is an arbitrary component of S_2. However,

$$(S_2^2)=((D+S_2)\cdot S_2)=((D+S_2+Z-S_2)\cdot S_2)$$
$$=((D+Z)\cdot S_2)=((K+R_1)\cdot S_2)=(K\cdot S_2).$$

From $(K\cdot S_2)<0$ we obtain that there exists in S_2 an irreducible component S_1' such that $(K\cdot S_1')<0$. Uniting this with $(S_1'^2)<0$ and $((K+S_1')\cdot S_1')\geq -2$, we have $(S_1'^2)=-1$, $(K\cdot S_1')=-1$; $((K+S_1')\cdot S_1')/2+1=0$. This means that S_1' is an exceptional curve of the first kind with $(S_1'\cdot R_1)=0$. This contradicts our choice of V.

Thus the assumption $(D\cdot S_1)=0$ leads to a contradiction in every case. Then $(D\cdot S_1)>0$. But then $D+S_1$ is a connected curve, $H^1(V, O[K+D+S_1])=0$, $H^0(V, O[K+D+S_1])\neq 0$, $((D+S_1)\cdot R_1)=((K+R_1)\cdot R_1)+(S_1R_1)\geq 2+1=3$ and the application of Lemma 2 to the systems $|M|=|R_1|$ and $|L|=|K+R_1+D+S_1|$ shows that $|K+R_1+D+S_1|$ yields a birational imbedding of V. But then, since

$$|K+R_1+D+S_1|\subseteq|2K+2R_1+S_1|=|2K+K+S_1|\subseteq|3K|,$$

$|3K|$ also yields a birational imbedding of V. We finally obtain that for a regular surface of basic type with $p_g=3$ possessing the property that $|3K|$ does not yield a birational imbedding, the unfixed part of the canonical system cannot be composed of a pencil.

We will further assume that V is a minimal model. We will show that $|K|$ does not have a fixed part. Let us assume the contrary, i.e. that $S\neq 0$, and we will show that one can find in S an irreducible component S_1 for which $(\bar{K}\cdot S_1)>0$.

If this were not so, we would have that $(\bar{K}\cdot S)=0$. Let $\bar{S}$ be the carrier of S. From $(\overline{K\cdot S})=0$ it follows that $\bar{S}$ is a fundamental curve of the system $\bar{K}$. As in the proof of formula (19a) it is possible to establish here that $(S^2)<0$ and that for any irreducible component S_i of the divisor S, $(S_i^2) < 0$. Since $(S^2) < 0$, one can find in S an irreducible component S_i such that $(S \cdot S_i) < 0$. From $(\overline{K} \cdot S) = 0$ we obtain

$$(\overline{K}, S_i) = 0,\ ((\overline{K} + S)\, S_i) < 0,\ (K \cdot S_i) < 0,\ (S_i^2) < 0.$$

Combining the inequalities $(K \cdot S_i) < 0$ and $(S_i^2) < 0$ with the inequality $((K + S_i) \cdot S_i) \geq -2$, we obtain

$$(K \cdot S_i) = -1,\ (S_i^2) = -1,\ \frac{((K+S_i)\, S_i)}{2} + 1 = 0,$$

i.e. S_i is an exceptional curve of the first kind on V. This contradicts the minimality of V.

Thus, $(\overline{K} \cdot S) > 0$, and let S_1 be an irreducible component in S such that $(\overline{K} \cdot S_1) > 0$. Let V' be the surface onto which $|\overline{K}|$ maps V, d be the degree of the mapping $f\colon V \longrightarrow V'$ given by $|\overline{K}|$, and m be the degree of V' in the projective imbedding corresponding to the mapping f. It is easy to see that $(\overline{K}^2) \geq d \cdot m$. If it were true that $d \cdot m = 1$, then $d = 1$ and $m = 1$, and V would be a rational surface. Consequently, $d \cdot m \geq 2$ and $(\overline{K}^2) \geq 2$. Since $|\overline{K}|$ is not composed of a pencil, by the theorem of Bertini it is an irreducible system. It follows from $(\overline{K} \cdot S_1) > 0$ that there exists in $|\overline{K} + S_1|$ a connected curve without multiple components and thus

$$H^1(V, O\,[K + \overline{K} + S_1]) = 0.$$

Noting that $H^0(V, O[K + \overline{K} + S_1]) \neq 0$ and $((\overline{K} + S_1) \cdot \overline{K}) = (\overline{K}^2) + (S_1 \cdot \overline{K}) \geq 2 + 1 = 3$, and applying Lemma 2 to the systems $|M| = |\overline{K}|$ and $|L| = |K + \overline{K} + \overline{K} + S_1|$ we obtain that $|K + \overline{K} + \overline{K} + S_1|$ yields a birational imbedding of V. Since $|K + \overline{K} + \overline{K} + S_1| \subseteq |3K|$, it follows that $|3K|$ also gives a birational imbedding of V.

The assumption $S \neq 0$ led us to a contradiction, so that the system $|K|$ does not have a fixed part. If it were true that $(K^2) \geq 3$, then by applying Lemma 2 to the systems $|M| = |K|$ and $|L| = |K + K + K|$, we would obtain that $|3K|$ yields a birational imbedding of V.

Consequently $(K^2) \leq 2$ and the argument above with $f\colon V \longrightarrow V'$ now shows that $(K^2) = 2$. Let us assume that $|K|$ has base points. We then apply enough σ-processes on V so that the proper image of the system $|K|$, a system $|K'|$, does not have base points. Then the mapping f is regular, and $(K'^2) = m \cdot d$. But it is clear that $(K'^2) < 2$, and we arrive at a contradiction with $d \cdot m \geq 2$. Thus $|K|$ is an irreducible linear system without a fixed part and without base points and such that $(K^2) = 2$.

It follows from Bertini's theorem that a generic curve of the system $|K|$ is nonsingular. From $(K^2) = d \cdot m = 2$ and $d > 1$, we obtain $d = 2$, $m = 1$, i.e. $|K|$ effects a regular mapping of degree two of the surface V onto a projective plane $\mathbf{P}^2$. The images of the curves of the system $|K|$ under this mapping are lines of the plane $\mathbf{P}^2$. There exists on $\mathbf{P}^2$ a set A, consisting of a finite number of points such that for each point $P \in \mathbf{P}^2 - A$, $f^{-1}(P)$ consists of a finite number of points (one or two). A branch curve of the mapping f is a curve $C \subset \mathbf{P}^2$ such that for any point $P \in \mathbf{P}^2 - (C \cup A)$, $f^{-1}(P)$ consists of exactly two points, and for a point $P' \in C - (C \cap A)$, $f^{-1}(P')$ consists of one point. Let l be a line on $\mathbf{P}^2$ such that $l \cap A = \emptyset$, l is not a component of C, l does not pass through singular points of C and is not tangent to it anywhere, and $f^{-1}(l)$ is a nonsingular element K_l of the system $|K|$. For any point $P \in l - l \cap C$, $f^{-1}(P')$ consists of two points, while for any point $P' \in l \cap C$, $f^{-1}(P')$ consists of one point. We obtain that $f|_{K_l}$: $K_l \to l$ is a regular mapping of the curve K_l onto the line l, that has a degree of two and a set of branch points that coincide with the set $C \cap l$.

Since $f|_{K_l}$ is a covering of second degree, all the branch points have an order equal to one. From $(K^2) = 2$ it follows that the geometric genus of the curve K_l

$$p(K_l) = \frac{(K_l(K_l + K))}{2} + 1 = 3.$$

For the Euler characteristic of the curve K_l we have $2p(K_l) - 2 = 2 \cdot (-2) + \Delta$, where Δ is the number of branch points of the mapping $f|_{K_l}$. From this we have

$$\Delta = 2p(K_l) - 2 - 2 \cdot (-2) = 2 \cdot 3 - 2 + 4 = 8.$$

Since l does not pass through singular points of C and is nowhere tangent to it, $C \cdot l = 8$, i.e. C is a curve of eighth degree on $\mathbf{P}^2$. It follows from Theorem 3 (this is also easily proved directly) that for a sufficiently large n the system $|nK|$ yields a birational imbedding of V. It easily follows from this that there exists a function z on V such that a pole of z is concentrated on K_l and z is a primitive element of the field $\mathbf{C}(V)$ over $\mathbf{C}(x, y)$, where (x, y) are affine coordinates on $S^2 = \mathbf{P}^2 - l$.

Let W' be an affine variety with a generic point (x, y, z). Since x, y, and z are regular functions on $V' = V - K_l$, there exists a regular mapping g': $V' \to W'$. For any point $(c_1, c_2, c_3) \in W'$ an extension of the specialization $(x \to c_1,\ y \to c_2,\ z \to c_3)$ into the field $\mathbf{C}(V)$ can have as a center only a point of V', for at any point of K_l one of the functions $1/x$ or $1/y$ is regular and equal to zero. It follows from this that g' is a mapping of V' onto all of W'. The function z can have only a finite specialization over each point $x = c_1$, $y = c_2$ (a pole of z is located on K_l), and therefore z must satisfy the following equation over $\mathbf{C}(x, y)$:

$$z^2 + \mathcal{P}(x, y)\, z + \mathcal{Q}(x, y) = 0,$$

where $\mathcal{P}(x, y)$ and $\mathcal{Q}(x, y)$ are polynomials. Replacing z by $z + \mathcal{P}(x, y)/2$, it is possible to assume that z satisfies the equation

$$z^2 = G(x, y):$$

where $G(x, y)$ is a polynomial.

Let $G(x, y) = \Pi_{j=1}^{m}[G_j(x, y)]^{n_j}$ be a decomposition of $G(x, y)$ into irreducible factors, $n_j = 2k_j + \epsilon_j$, where ϵ is zero or one depending on the evenness or oddness of n_j. We consider the function

$$z_1 = \frac{z}{\left[\prod_{j=1}^{m} G_j(x, y)\right]^{k_j}}.$$

z_1 satisfies the equation

$$z_1^2 = \prod_{j=1}^{m} [G_j(x, y)]^{\epsilon_j}. \tag{31}$$

Let W'' be the variety in the space $S^3 = S^3(x, y, z)$ given by this equation. From the form of equation (31) it follows that z_1 can have only a finite specialization over any point $x = c_1$, $y = c_2$ of S^2. This means that z_1 is a regular function on V', a pole of which is concentrated on K_l. Moreover, it is clear that z_1 is a primitive element of the field $\mathbf{C}(V)$ over $\mathbf{C}(x, y)$.

As above for g', we find that there exists a regular birational mapping g'' of V' onto W''. If we denote by e the projection of W'' onto $S^2 = S^2(x, y)$, then $f' = e \circ g''$, where f' is the restriction of f on V'.

For any point $P = (c_1, c_2) \in S^2$ satisfying the condition $\Pi_{j=1}^{m}[G_j(c_1, c_2)]^{\epsilon_j} \neq 0$, $e^{-1}(P)$ consists of two points, and thus $f'^{-1}(P)$ also consists of at least two points, i.e. $P \notin (C - (A \cap C))$. On the other hand, if $P \notin (C - (A \cap C))$, then f' is locally biregular at P, and thus e must also be locally biregular over P, i.e. P does not belong to the curve C_e with the equation $\Pi_{j=1}^{m}[G_j(x, y)]^{\epsilon_j} = 0$. We obtain that $C = C_e$, and since the polynomial $\Pi_{j=1}^{m} G_j(x, y)$ does not have multiple factors, it must be a polynomial of eighth degree. We denote it by $F_8(x, y)$. By the choice of l we obtain that the curve $F_8(x, y) = 0$ coinciding with the curve C, is nowhere tangent to the infinite line and does not have singular points on that line. It is thus proved that V is birationally equivalent to a variety W'' with equation $z^2 = F_8(x, y)$, where $F_8(x, y)$ does not have multiple factors and determines a curve on the plane that is nowhere tangent to the infinite line and does not have singular points on it. To complete the proof of Theorem 6 it remains to show that in order that the surface

The subsets in $\widetilde{U}_0$ and $\widetilde{U}_1$ determined respectively by the conditions $\eta_1 \neq 0$ and $\eta \neq 0$ (we denote them by $\widetilde{U}'_0$ and $\widetilde{U}'_1$) must be identical according to the rule: if $(Q_1(\xi_0, \xi_1))$ is a point of $\widetilde{U}'_0(Q \in A_0)$, then the point

$$(Q\ (\xi'_0, \xi'_1)) \in \widetilde{U}'_1, \quad \text{where} \quad \frac{\xi'_1}{\xi'_0} = v_Q^2 \frac{\xi_1}{\xi_0},$$

is put in correspondence with it, where v_Q designates the value of $v = \eta_1/\eta_0$ at the point Q. We denote by $\overline{U}_1$ the variety which is obtained from $\overline{U}$ as a result of a σ-process along the curve $u = 0$, $y_1 = 0$, by $\overline{X}_1$ the proper image of $\overline{X}$ on $\overline{U}_1$, and by S_1 the surface which is joined into $\overline{U}$. We introduce four open sets A_{01}, A_{02}, A_{11}, A_{12}. Here A_{01} and A_{02} are the subsets of $\widetilde{U}_0$ determined respectively by the conditions $\xi_0 \neq 0$ and $\xi_1 \neq 0$, and A_{11} and A_{12} are the subsets of $\widetilde{U}_1$ determined respectively by the conditions $\xi'_0 \neq 0$ and $\xi'_1 \neq 0$. We set

$$\frac{\xi_1}{\xi_0} = t_1, \quad \frac{\xi_0}{\xi_1} = t'_1, \quad \frac{\xi'_1}{\xi'_0} = s_1, \quad \frac{\xi'_0}{\xi'_1} = s'_1, \quad v' = \frac{1}{v}.$$

Then the local coordinates are (u, v, t_1) in A_{01}, (y_1, v, t'_1) in A_{02}, (u, v', s_1) in A_{11}, and (y_2, v', s'_1) in A_{12}.

We will find the local equation of $\overline{X}_1$ in A_{01}. For this we set $y_1 = ut_1$ in (35):

$$u^6 - y_1^2\Phi\ (u, 1, v) = u^6 - u^2 t_1^2 \Phi\ (u, 1, v) = 0.$$

Cancelling u^2, we obtain the needed equation:

$$u^4 - t_1^2\Phi\ (u, 1, v) = 0. \tag{37}$$

We now find the intersection $\overline{X}_1 \cap A_{01} \cap S_1$. For this we set $u = 0$ in (37), obtaining $t_1^2\Phi(0, 1, v) = 0$. From this we have: 1) $t_1 = 0$; this is the curve $u = 0$, $t_1 = 0$; 2) $\Phi(0, 1, v) = 0$. Let $v_1, \cdots, v_8$ be the roots of the polynomial $\Phi(0, 1, v)$ (all of them are distinct). We obtain eight curves S_{1k} determined by the conditions $u = 0$, $v = v_k$, $k = 1, \cdots, 8$. Singular points of $\overline{X}_1 \cap A_{01}$ can lie only on $\overline{X}_1 \cap A_{01} \cap S_1$. It is immediately evident that each point of the curve $u = 0$, $t_1 = 0$ is singular for $\overline{X}_1$. Now let P be a point on the curve S_{1k}, $t_1 \neq 0$ at P. Since v_k is a nonmultiple root of $\Phi(0, 1, v)$ ($\Phi(0, 1, v)$ does not have multiple roots), $\partial\Phi(0, 1, v_k)/\partial v \neq 0$. Moreover,

$$\frac{\partial}{\partial v}(u^4 - t_1^2\Phi\ (u, 1, v)) = -\ t_1^2 \frac{\partial\Phi}{\partial v}(u, 1, v).$$

For $u = 0$, $t_1 = 0$, $v = v_k$, this expression is not equal to zero. Consequently, P is a nonsingular point on $\overline{X}_1$.

We now consider $\overline{X}_1$ in A_{02}. Setting $u = t'_1 y_1$ in (35), we obtain

$$u^6 - y_1^2 \Phi(u, 1, v) = t_1'^6 y_1^6 - y_1^2 \Phi(t_1' y_1, 1, v) = 0.$$

Cancelling y_1^2, we obtain the equation of $\overline{X}_1$ in A_{02}:

$$t_1'^6 y_1^4 - \Phi(t_1' y_1, 1, v) = 0. \tag{38}$$

We obtain the intersection $\overline{X}_1 \cap S_1 \cap A_{02}$ by setting $y_1 = 0$ in (38), i.e. $\Phi(0, 1, v) = 0$.

We obtain the eight curves S_{1k} already found earlier.

If there are also singularities of $\overline{X}_1$ in $\overline{X}_1 \cap A_{02}$, they can only be at the points $y_1 = 0$, $t'_1 = 0$, $v = v_k$. But

$$\frac{\partial}{\partial v}(t_1'^6 y_1^4 - \Phi(t_1' y_1, 1, v)) = -\frac{\partial \Phi}{\partial v}(t_1' y_1, 1, v).$$

For $t'_1 = 0$, $y_1 = 0$, $v = v_k$, this expression is not equal to zero. Consequently there are no singular points of $\overline{X}_1$ in $A_{02} \cap \overline{X}_1$.

Consideration of the sets A_{11} and A_{12} does not add anything new, and we obtain that $\overline{X}_1$ has a curve of singularities l_1 whose equation in A_{01} is $u = 0$, $t_1 = 0$, and in A_{11} is $u' = 0$, $s_1 = 0$.

We now perform a σ-process on $\overline{U}_1$ along the curve l_1. For this we consider the two products $A_{01} \times \mathbf{P}^1$ and $A_{11} \times \mathbf{P}^1$ (the homogeneous coordinates in $\mathbf{P}^1$ are (ζ_0, ζ_1) or (ζ'_0, ζ'_1)), and then in the first product the subset V_0 determined by the equation $u\zeta_1 = t_1\zeta_0$, and in the second product the subset V_1 determined by the equation

$$u'\zeta'_1 = s_1\zeta'_0.$$

The subsets in V_0 and V_1 determined respectively by the conditions $v \neq 0$ and $v' \neq 0$ (we denote them by V'_0 and V'_1), must be identified according to the rule: if $(Q(\zeta_1, \zeta_0))$ is a point of $V'_0 (Q \in A_{01})$, then the point $(Q(\zeta'_1, \zeta'_0)) \in V'_1$ is put in correspondence with it, where

$$\frac{\zeta'_1}{\zeta'_0} = v_Q^2 \frac{\zeta_1}{\zeta_0}.$$

Let $\overline{U}_2$ be the variety obtained from $\overline{U}_1$ as a result of the σ-process, let $\overline{X}_2$ be the proper image of $\overline{X}_1$ on $\overline{U}_2$, and let the surface joined in to $\overline{U}_1$ be S_2.

We consider four open sets in $\overline{U}_1$: $A_{011}, A_{012}, A_{111}, A_{112}$. Here A_{011} and A_{012} are the subsets in V_0 determined respectively by the conditions $\zeta_0 \neq 0$ and $\zeta_1 \neq 0$, and A_{111} and A_{112} are the subsets in V_1 determined respectively by the conditions $\zeta' \neq 0$, $\zeta'_1 \neq 0$. We set $\zeta_1/\zeta_0 = t$, $\zeta_0/\zeta_1 = t'$. The local coordinates are (u, v, t) in A_{011} and (t_1, v, t') in A_{012}. We now find the equation of $\overline{X}_2$ in A_{011}. For this we set $t_1 = u\zeta_1/\zeta_0 = ut$ in (37): $u^4 - t_1^2\Phi(u, 1, v) = u^4 - u^2t^2\Phi(u, 1, v) = 0$. Cancelling u^2, we obtain the desired equation:

$$u^2 - t^2\Phi(u, 1, v) = 0. \tag{39}$$

The singular points of $\overline{X}_2$ can be only on $\overline{X}_2 \cap S_2$. We now find $\overline{X}_2 \cap S_2 \cap A_{011}$. We set $u = 0$ in (39), obtaining

$$t^2\Phi(0, 1, v) = 0.$$

From this we have 1) $t = 0$, i.e. the curve $u = 0$, $t = 0$; 2) $\Phi(0, 1, v) = 0$ gives us eight curves S_{2k} determined by the conditions $u = 0$, $v = v_k$, $k = 1, \cdots, 8$.

From the absence of multiple roots for $\Phi(0, 1, v)$ it follows that the singular points of $\overline{X}_2$ satisfy the curve $u = 0$, $t = 0$, and that none of them are outside it.

We now perform a σ-process on $\overline{U}_2$ along this curve.

Let $\overline{U}_3$ be what is obtained from $\overline{U}_2$ as a result of this σ-process, let $\overline{X}_3$ be the proper image of $\overline{X}_2$ on $\overline{U}_3$, let S_3 be the variety joined in, and let $\overline{A}_{011}$ be what is obtained from A_{011} under the σ-process. Then it is possible to cover $\overline{A}_{011}$ with two open sets A_{0111} and A_{0112}, where the local coordinates are s, t, v in A_{0111} and s', u, v in A_{0112}; here $s' = 1/s$, $u = st$. The equations for $\overline{X}_3$ in A_{0111} and A_{0112} respectively have the form

$$s^2 - \Phi(st, 1, v) = 0,$$

$$1 - s'^2\Phi(u, 1, v) = 0.$$

It is easy to verify that $\overline{X}_3$ is a nonsingular variety.

We now turn to the consideration of regular differentials of different degree on X.

Any regular differential of mth degree on $X - \phi^{(m)}$ must have the form

$$\varphi^{(m)} = \mathcal{P}(y_1, y_2, y_3)\left[\frac{dy_1 \wedge dy_3}{\frac{\partial}{\partial y_2}(y_3^6 - \Phi(y_3, y_1, y_2))}\right]^m, \tag{40}$$

where $\mathcal{P}(y_1, y_2, y_3)$ is a polynomial of y_1, y_2, and y_3 of degree not greater than $m(8-4) = 4m$ that satisfies some additional conditions. In order to discover these conditions, we look at the differential $\phi^{(m)}$ on $\overline{X}_3$ and require its regularity there. It is easy to see that a question about the regularity of $\phi^{(m)}$ on $\overline{X}_3$ arises only in the preimage of the point $P_0(y_1 = y_2 = y_3 = 0)$ under the mapping $\overline{X}_3 \to X \cap U$. This preimage consists of the curve $s^2 - \Phi(0, 1, v) = 0$, $t = 0$ on $\overline{X}_3$ and of the curves $\overline{S}_{2k}$, $\overline{S}_{1k}$, $k = 1, \cdots, 8$, where $\overline{S}_{ik}$ denotes the proper image of the curve S_{ik} on $\overline{V}_3$.

It is easy to verify that it is sufficient to consider only A_{0111}. We note that

$$y_1 = ut_1 = u\cdot ut = u^2t = s^2t^3,$$
$$y_2 = \frac{\eta_1}{\eta_0} y_1 = vy_1 = vs^2t^3,$$
$$y_3 = \frac{\eta_2}{\eta_0} y_1 = uy_1 = s^3t^4.$$

Thus

$$\begin{aligned} y_1 &= s^2t^3; \\ y_2 &= vs^2t^3; \\ y_3 &= s^3t^4. \end{aligned} \tag{41}$$

Hence

$$dy_1 = 2st^3 ds + 3s^2t^2\, dt;$$
$$dy_3 = 3s^2t^4\, ds + 4s^3t^3 dt;$$
$$dy_1 \wedge dy_3 = -\, s^4t^6\, ds \wedge dt;$$
$$\frac{\partial}{\partial y_2}(y_3^6 - \Phi(y_3, y_1, y_2)) = -\frac{\partial \Phi}{\partial y_2}(s^3t^4,\ s^2t^3,\ vs^2t^3) = -(s^2t^3)^7 \frac{\partial \Phi}{\partial y_2}(st,\ 1,\ v)$$
$$= -\, s^{14}t^{21} \frac{\partial \Phi}{\partial v}(st,\ 1,\ v),$$

$$\frac{dy_1 \wedge dy_3}{\frac{\partial}{\partial y_2}(y_3^6 - \Phi(y_3, y_1, y_2))} = \frac{ds \wedge dt}{s^{10}t^{15} \frac{\partial \Phi}{\partial v}(st,\ 1,\ v)}.$$

We consider the eight points P_k, $k = 1, \cdots, 8$ on $\overline{X}_3$ with the coordinates $t = 0$, $s = 0$, $v = v_k$. It is easy to verify that at each point P_k the functions s and t are local coordinates on $\overline{X}_3$; this follows from the fact that

$$\frac{\partial}{\partial v}(s^2 - \Phi(st,\ 1,\ v))_{\substack{s=0,\\ t=0,\\ v=v_k}} = -\frac{\partial \Phi}{\partial v}(0,\ 1,\ v_k) \neq 0.$$

Let $m = 1$. We represent $P(y_1, y_2, y_3)$ in the form

$$P(y_1, y_2, y_3) = \sum_{j=0}^{4} \sum_{r=0}^{4-j} B_{j,r}(y_1, y_2)\ y_3^j,$$

where $B_{j,r}(y_1, y_2)$ is a homogeneous polynomial of y_1 and y_2 of degree r.

Expressing y_1, y_2, and y_3 in terms of v, s, and t, we obtain in a neighborhood of the point P_k an expression of the form

$$P(y_1, y_2, y_3) = \sum_{k_1=0}^{12} s^{k_1} \sum_{\max(0,\, k_1-8) \leqslant j \leqslant \frac{k_1}{3}} t^{\frac{3k_1-j}{2}} \cdot B_{j,\, \frac{k_1-3j}{2}}(1,\ v)^{1)}$$

1) Here it is assumed that $B_{j,(k_1-3j)/2} \equiv 0$ when $k_1 - 3j$ is an uneven number.

and for the differential (40) the expression

$$\varphi^{(1)} = \frac{\sum_{k_1=0}^{12} s^{k_1} \sum_{\max(0,\, k_1-8)\leqslant j\leqslant \frac{k_1}{3}} t^{\frac{3k_1-j}{2}} \cdot B_{j,\, \frac{k_1-3j}{2}}(1, v)}{s^{10}t^{15}\, \frac{\partial\Phi}{\partial v}(st, 1, v)}\, ds \wedge dt.$$

Since at the point P_k, $\partial\Phi(st, 1, v)/\partial v = \partial\Phi(0, 1, v_k)/\partial v \neq 0$, the regularity of $\phi^{(1)}$ requires the regularity at P_k of the function

$$\mathcal{H}(s, t) = \frac{\sum_{k_1=0}^{12} s^{k_1} \sum_{\max(0,\, k_1-8)\leqslant j\leqslant \frac{k_1}{3}} t^{\frac{3k_1-j}{2}} B_{j,\, \frac{k_1-3j}{2}}(1, v)}{s^{10}t^{15}}.$$

Thus, let $\mathcal{H}(s, t)$ be a regular function at P_k, $k = 1, \cdots, 8$.

We will show by induction on k_1 that for $0 \le k_1 \le 10$, $B_{j,\,(k_1-3j)/2}(1, v) \equiv 0$ for j such that $\max(0, k_1 - 8) \le j \le k_1/3$.

1) Let $k_1 = 0$. It is necessary to show that $B_{0,0}(1, v) \equiv 0$. From the regularity of $\mathcal{H}(s, t)$ at P_k it follows that at P_k the numerator of the expression for $\mathcal{H}(s, t)$ is equal to zero. But for $s = 0$, $t = 0$, $v = v_k$, this numerator is equal to $B_{0,0}(1, v_k)$, from which we have $B_{0,0}(1, v_k) = 0$, and, since $B_{0,0}(1, v)$ is a polynomial of zero degree, $B_{0,0}(1, v) \equiv 0$.

2) Let us assume that $0 < k'_1 \le 10$ and that our assertion has been proved for $k_1 < k'_1$. But this means that

$$\mathcal{H}(s, t) = \frac{\sum_{k_1=k'_1}^{12} s^{k_1} \sum_{\max(0,\, k_1-8)\leqslant j\leqslant \frac{k_1}{3}} t^{\frac{3k_1-j}{2}} B_{j,\, \frac{k_1-3j}{2}}(1, v)}{s^{10}t^{15}}$$

$$= \frac{\sum_{k_1=k'_1}^{12} s^{k_1-k'_1} \sum_{\max(0,\, k_1-8)\leqslant j\leqslant \frac{k_1}{3}} t^{\frac{3k_1-j}{2}} B_{j,\, \frac{k_1-3j}{2}}(1, v)}{s^{10-k'_1}t^{15}}.$$

We consider the numerator of this expression on the curve $s = 0$ in a neighborhood of the point P_k on $\overline{X}_3$. From the equation for $\overline{X}_3$: $s^2 - \Phi(st, 1, v) = 0$, it follows for $s = 0$ that $\Phi(0, 1, v) = 0$, also in a neighborhood of the point P_k on the curve $s = 0$, $v = v_k$. Hence for $s = 0$ the numerator takes the form

$$\sum_{\max\left(0,k_1'-8\right)\leqslant j\leqslant \frac{k_1'}{3}} t^{\frac{3k_1'-j}{2}} B_{j,\frac{k_1'-3j}{2}}(1, v_k). \tag{42}$$

First let $k'_1 < 10$. Then (42) must be zero on the curve $s = 0$, and thus all the $B_{j,(k'_1-3j)/2}(1, v_k)$ with $\max(0, k'_1 - 8) \leq j \leq k'_1/3$ are zero. Since $r \leq 4$, $B_{j,r}$ is a polynomial of degree not greater than four, and therefore since $B_{j,r}(1, v_k) = 0$ for all $k = 1, \cdots, 8$ it follows that $B_{j,r}(1, v) = 0$.

Now let $k'_1 = 10$. Then (42) can be divided by t^{15}. For j we have

$$10-8 \leqslant j \leqslant \frac{10}{3}, \quad \text{i.e.} \quad 2 \leqslant j \leqslant 3.$$

If $j = 3$, then $(k'_1 - 3j)/2 = 1/2$ is not an integer and $B_{j,(k_1-3j)/2} \equiv 0$ by definition.

We obtain in (42) the single member

$$t^{14} B_{2,2}(1, v_k).$$

From this we have $B_{2,2}(1, v_k) = 0$ and $B_{2,2}(1, v) \equiv 0$. We have thus proved the assertion made above.

We thus have at our disposal only $k_1 = 11$ and $k_1 = 12$. For $k_1 = 11$ we have for j that $3 \leq j \leq 11/3$ i.e. $j = 3$ and $(k_1 - 3j)/2 = 1$; for $k_1 = 12$ one has $j = 4$ and $(k_1 - 3j)/2 = 0$. We obtain that the regularity of $\phi^{(1)}$ on $\overline{X}_3$ requires that only $B_{3,1}(y_1, y_2)$ and $B_{4,0}(y_1, y_2)$ need be different from zero among the $B_{j,r}(y_1, y_2)$.

For $P(y_1, y_2, y_3)$ we obtain the following necessary condition:

$$P(y_1, y_2, y_3) = B_{3,1}(y_1, y_2)\, y_3^3 + B_{4,0}(y_1, y_2)\, y_3^4 = y_3^3(a_1y_1 + a_2y_2 + a_3y_3).$$

A direct computation shows that the expression

$$\varphi^{(m)} = \frac{y_3^{3m} Q_m(y_1, y_2, y_3)\,(dy_1 \wedge dy_3)^m}{\left(\frac{\partial}{\partial y_2}\left(y_3^6 - \Phi(y_3, y_1, y_2)\right)\right)^m} \tag{43}$$

($Q_m(y_1, y_2, y_3)$ is a homogeneous polynomial of degree m of y_1, y_2, and y_3) is a regular double differential of degree m on $\overline{X}_3$.

In order for $p_g = 3$ to be true, we must require that X have only singularities for $x_0 \neq 0$ which do not impose new restrictions on

$$\varphi^{(1)} = \frac{y_3^3 Q_1(y_1, y_2, y_3)}{\frac{\partial}{\partial y_2}\left(y_3^6 - \Phi(y_3, y_1, y_2)\right)}.$$

We turn to the equation of X in $S^3 = S^3(x, y, z)$:

$$z^2 - F_8(x, y) = 0.$$

We call a surface X for which the equation in $S^3(x, y, z)$ has the form

$$z^2 - F(x, y) = 0,$$

a double plane of canonical form, and if moreover $F(x, y)$ does not have multiple factors, a double plane of normal form.

Let Y be a double plane of normal form for which the point $Q_0(x = y = z = 0)$ is a singular point. Singular points can lie only on the curve C_Y having the equation $F(x, y) = 0$, and must be singular points of this curve. Let Q_0 be a singular point of multiplicity s on the curve C_Y. We perform a σ-process at the point Q_0 of the space $S^3(x, y, z)$. We denote the direct image of Y by $\overline{Y}$, the direct image of the plane $S^2(x, y)$ $(z = 0)$ by $\overline{S}^2$, and what is obtained from S^3 by $\overline{S}^3$. Then it is possible to cover $\overline{S}^3$ with three open sets B_1, B_2, and B_3, where it is possible to introduce local coordinates x, u_1, v_1 in B_1, y, u_2, v_2 in B_2, and z, u_3, v_3 in B_3 such that

$$y = u_1 x, \quad z = v_1 x;$$
$$x = u_2 y, \quad z = v_2 y;$$
$$x = u_3 z, \quad y = v_3 z.$$

The equation of $\overline{Y}$ in B_1 has the form

$$v_1^2 - x^{s-2} F_1(x, u_1) = 0, \tag{44}$$

where $F_1(x, u_1) = 0$ is the equation of the direct image of the curve C_Y in $\overline{S}^2 \cap B_1$ (x, u_1 are local coordinates in $\overline{S}^2$). We see from (44) that $\overline{Y} \cap B_1$ is a double plane of canonical form in B_1.

We consider the differential

$$\frac{dx \wedge dz}{\frac{\partial}{\partial y}(z^2 - F(x,y))}.$$

We have $dz = x dv_1 + v_1 dx$, $dx \wedge dz = x dx \wedge dv_1$,

$$\frac{\partial}{\partial y}(z^2 - F(x, y)) = -\frac{\partial F(x, y)}{\partial y} = -x^{s-1}\frac{\partial F_1(x, u_1)}{\partial u_1}$$
$$= x\left(-x^{s-2}\frac{\partial F_1(x, u_1)}{\partial u_1}\right) = x\frac{\partial}{\partial u_1}(v_1^2 - x^{s-2}F_1(x, u_1)),$$
$$\frac{dx \wedge dz}{\frac{\partial}{\partial y}(z^2 - F(x, y))} = \frac{dx \wedge dv_1}{\frac{\partial}{\partial u_1}(v_1^2 - x^{s-2}F_1(x, u_1))}. \tag{45}$$

The consideration of $\overline{Y}$ in B_2 does not add anything new. As for $\overline{Y} \cap B_3$, there occurs there only one point of the preimage of Q_0 under the σ-process that is not contained in $(\bar{Y} \cap B_1) \cup (\bar{Y} \cap B_2)$, the point $u_3 = v_3 = 0$. It is easy to see, however, that this point does not lie on $\bar{Y}$. Therefore we will not consider $\bar{Y} \cap B_3$. Equation (45) shows that if during the resolution of the point Q_0 by consecutive σ-processes we obtain only double planes of normal form, then Q_0 does not impose any new restrictions on the double differential. If, however, at some step we do not obtain a double surface of normal form, this means that the surface acquires a curve of singularities, which a fortiori lays some restriction on the double differential.

It is clear from equation (44) that in order for $\bar{Y}$ to be a double plane of normal form, it is necessary and sufficient that $s \leq 3$, i.e. $s = 2$ or $s = 3$. If $s = 2$, then (44) takes the form

$$v_1^2 - F_1(x, u_1) = 0.$$

Since a σ-process does not raise the multiplicity of singular points, we obtain that the curve $F_1(x, u_1) = 0$ on $\bar{S}^2 \cap B_1$ can have in the preimage of the point Q_0 singularities of at most second order, i.e. we have again arrived at the consideration of a double plane of normal form with a double point on a branch curve. This shows that the last σ-processes repeat this situation and that we can never meet a double plane that is not of normal form, i.e. for $s = 2$ the point Q_0 does not impose any restrictions on the double differential.

Now let $s = 3$. Equation (44) takes the form

$$v_1^2 - xF_1(x, u_1) = 0.$$

If the curve $F_1(x, u_1) = 0$ has singularities of third order among the preimages of the point Q_0, then, since in these preimages $x = 0$, $v_1^2 - xF(x, u_1) = 0$ will be a double plane with a singular point whose multiplicity on the branch curve is equal to four. And this, as we showed above, implies that during the preceding σ-process a double plane of normal form was not obtained.

Now let any preimage Q_{0i} of the point Q_0 on the curve $F_1(x, u_1) = 0$ have a multiplicity not exceeding two. If this multiplicity is equal to one, we obtain for $\bar{Y} \cap B_1$ a singular point whose multiplicity on the branch curve is equal to two. As was mentioned above, restrictions on the double differential cannot arise from such singularities.

Now let Q_{0i} be a point of multiplicity two on $F_1(x, u_1) = 0$. Let $F(x, y) = f_3(x, y) + f_4(x, y) + \cdots$ be the representation of $F(x, y)$ in the form of a sum of homogeneous polynomials of third, fourth, fifth, etc. degrees. Then

$$F_1(x, u_1) = f_3(1, u_1) + xf_4(1, u_1) + x^2 f_5(1, u_1) + \ldots$$

Let the point Q_{0i} have the coordinates $x = 0$, $u = c_i$. We have at this point

$$F_1 = f_3(1, c_i) = 0;$$
$$\frac{\partial F_1}{\partial x} = f_4(1, c_i) = 0;$$
$$\frac{\partial F_1}{\partial u_1} = \frac{\partial f_3}{\partial u_1}(1, c_i) = 0;$$

and since $f_3(1, u_1)$ is a polynomial of not greater than third degree

$$F_1(x, u_1) = \frac{\partial^2 f_3}{\partial u_1^2}(1, c_i)\frac{(u_1 - c_i)^2}{2} + \frac{\partial^3 f_3}{\partial u_1^3}(1, c_i)\frac{(u_1 - c_i)^3}{6}$$
$$+ x\frac{\partial f_4}{\partial u_1}(1, c_i)(u_1 - c_i) + x^2 f_5(1, c_i) + H,$$

where H contains the members of degree greater than two with respect to x and $u_1 - c_i$.

Since the multiplicity of Q_{0i} is equal to two, at least one of the quantities

$$\frac{\partial^2 f_3}{\partial u_1^2}(1, c_i),\quad \frac{\partial f_4}{\partial u_1}(1, c_i),\quad f_5(1, c_i)$$

must be different from zero.

We perform a σ-process at the point Q_{0i} on $\overline{S}^3$. Here it is sufficient to consider the space B'_1 with the coordinates (u, u', v'), where $u = u_1 - c_i$, $u' = x/u$, $v' = v_1/u$, and in it the surface given by the equation

$$v'^2 - uu'\Big[\frac{1}{2}\frac{\partial^2 f_3}{\partial u_1^2}(1, c_i) + \frac{\partial f_4}{\partial u_1}(1, c_i)u'$$
$$+\frac{1}{6}\frac{\partial^3 f_3}{\partial u_1^3}(1, c_i)u + f_5(1, c_i)u'^2 + H_1\Big] = 0, \tag{46}$$

where H_1 consists of the members of degree greater than one with respect to u and u', and necessarily containing u. We are interested in the singular points of the surface (46) for $u = 0$. These singular points can only be for values of u' satisfying the equation

$$u'\left(\frac{1}{2}\frac{\partial^2 f_3}{\partial u_1^2}(1, c_i) + \frac{\partial f_4}{\partial u_1}(1, c_i)u' + f_5(1, c_i)u'^2\right) = 0. \tag{47}$$

Since

$$\frac{1}{2}\frac{\partial^2 f_3}{\partial u_1^2}(1, c_i) + \frac{\partial f_4}{\partial u_1}(1, c_i)u' + \frac{1}{6}\frac{\partial^3 f_3}{\partial u_1^3}(1, c_i)u + f_5(1, c_i)u'^2 + H_1 = 0 \tag{48}$$

is the equation of the direct image of the curve $F_1(x, u_1) = 0$ after the σ-process, and the σ-process does not raise multiplicities, the roots of equation (47) can give only singular points of second order on the curve (48). To a nonzero root of equation (47) there will correspond a singular point on the surface (46) whose multiplicity on the branch curve does not exceed three. It is easy to verify that a cubic singularity that arises has a lowered multiplicity after a σ-process.

We now consider the root $u'=0$. If $\partial^2 f_3(1, c_i)/\partial u_1^2 \neq 0$, then the curve (48) does not pass through the point $u=u'=0$ and on the surface (46) there corresponds to this root a singularity whose multiplicity on the branch curve is two.

Now let $\partial^2 f_3(1, c_i)/\partial u_1^2 = 0$. Since $f_3(1, c_i) = \partial f_3(1, c_i)/\partial u_1 = 0$, it must be true that $\partial^3 f_3(1, c_i)/\partial u_1^3 \neq 0$. But this shows that the point $u=u'=0$ is simple on the curve (48), and thus the surface has a singular point at it whose multiplicity on a branch curve is equal to three. It is easy to verify, however, that this cubic singularity is such that after a σ-process on the curve

$$uu'\left(\frac{\partial f_4}{\partial u_1}(1, c_i)\, u' + \frac{1}{6}\frac{\partial^3 f_3}{\partial u_1^3}(1, c_i) u + f_5(1, c_i)\, u'^2 + H_1\right) = 0$$

it has a smaller multiplicity. This proves that if a double plane of normal form Y

$$z^2 - F(x, y) = 0$$

is such that a singular point Q_0 on it is either a quadratic singularity on a branch curve, or a cubic singularity that decreases in multiplicity after a σ-process, then after a σ-process on Y at the point Q_0 we again meet a double plane of normal form on which the singular points corresponding to Q_0, considered as singularities of a branch curve, will again possess the same property.

This shows that a necessary and sufficient condition for a singularity on X for $x_0 \neq 0$ not to impose new restrictions on the differential is the satisfaction of condition 3) of Theorem 6.

Consequently, this condition 3) is necessary and sufficient in order that $p_g(X) = 3$. This finishes the proof of the necessity in Theorem 6.

We can now assume that condition 3) is satisfied for X and that consequently all the double differentials of first order on X have the form

$$\varphi^{(1)} = \frac{y_3^3 (a_1 y_1 + a_2 y_2 + a_3 y_3)\, dy_1 \wedge dy_3}{\frac{\partial}{\partial y_2}(y_3^6 - \Phi(y_3, y_1, y_2))} = \frac{(a_1 x + a_2 y + a_3)\, dx \wedge dz}{\frac{\partial}{\partial y}(z^2 - F(x, y))}.$$

Let X_1 be a nonsingular model of X which coincides with $\overline{X}_3$ over $U \cap X$. A simple analysis then shows that the canonical system $|K|$ on X_1 consists of a nonfixed part $|\overline{K}|$ without base points, a generic element of which is the preimage of the line $a_1 x + a_2 y + a_3$ on $S^2 = S^2(x, y)$ under the projection of X_1 onto the plane $S^2(x, y)$, and of a fixed part

$$S = 3\sum_{k=1}^{8} \overline{S}_{0k} + 2\sum_{k=1}^{8} \overline{S}_{1k} + \sum_{k=1}^{8} \overline{S}_{2k},$$

CHAPTER VII

SURFACES WITH A PENCIL OF ELLIPTIC CURVES

This chapter studies surfaces with a pencil of elliptic curves. A description is given of the connection between this class of surfaces with other classes of algebraic surfaces (surfaces with $(K^2) = 0$, surfaces with $\kappa = 1$). Finally, there is presented a classification of surfaces with a pencil of elliptic curves, or, more precisely, a classification of the pencils themselves. This last question was considered in the works [25, 42, 57, 40]. We present the results of these papers without proofs, proving only those assertions which are not contained in them.

The base field k is assumed to be the field of complex numbers. The majority of the arguments remain valid when k is an algebraically closed field of characteristic 0 (or with even weaker restrictions on the characteristic). We shall note the places where the assumption $k = \mathbf{C}$ is essential in an argument.

§1. Basic concepts

Definition. *A surface with a pencil of elliptic curves is a triple* (V, B, π) *consisting of a nonsingular surface* V, *a nonsingular curve* B, *and a regular mapping* $\pi: V \longrightarrow B$ *such that a generic fiber of the fibering* π *is a nonsingular curve of genus* 1.

Two surfaces with a pencil of elliptic curves (V, B, π) *and* (V', B, π') *are said to be biregularly (respectively, birationally) equivalent if there exists a biregular (respectively, birational) mapping* $f: V \longrightarrow V'$ *such that* $\pi' f = \pi$.

A surface with an elliptic pencil (V, B, π) *is said to be a minimal model, if the fibers of the fibering* π *do not contain exceptional curves of the first kind, i.e. (by the theorem of Castelnuovo, Chapter* II, §4) *it does not contain nonsingular rational curves* C *with* $(C^2) = -1$.

Every surface with an elliptic pencil is birationally equivalent to a minimal model — a transformation contracting an exceptional curve of the first kind contained in a fiber into a point clearly commutes with a projection.

Remark. A surface (V, B, π) with a pencil of elliptic curves can be a minimal model in the sense of the definition of this section, while at the same time the surface V is not a minimal model in the sense of Chapter II. As an example we consider two plane nonsingular cubic curves $G = 0$ and $H = 0$ that have nine distinct points of intersection $P_1, \cdots, P_9$. We denote by V the surface obtained by an application of a σ-process to the points $P_1, \cdots, P_9$ of the plane. The total preimages on V of the plane curves $\lambda G + \mu H = 0$ determine on V a pencil of elliptic curves. The surface obtained is a minimal model in the sense of the definition of this section. This can be easily verified directly if one assumes that no three of the P_i lie on the same line, and that no six of them lie on the same second order curve; one may also deduce it from the description of the possible types of degenerate fibers given in §6. The surface, however, is of course not a minimal model in the sense of the definition of Chapter II.

The concept of a minimal model of a surface with an elliptic pencil is analogous to the concept of a relatively minimal model of an arbitrary surface. It also plays the role of an absolute minimal model, however, as the following result shows.

Theorem 1. *If (V, B, π) and (V', B, π') are two surfaces with a pencil of elliptic curves, $f\colon V' \to V$ is a birational mapping of V' onto V, and if V is a minimal model, then the mapping f is regular.*

Proof. Let F_β and F'_β be generic fibers of the fiberings π and π'. We denote by ξ and ξ' generic points of the curves F_β and F'_β, and by o_ξ, $o_{\xi'}$, O_ξ, and $O_{\xi'}$, their local rings on the curves F_β and $F_{\beta'}$ and on the surfaces V and V' respectively. It is clear that the mapping f induces a biregular isomorphism between F_β and F'_β. Thus

$$f(\xi') \in o_\xi, \quad f(0') \in o_\xi \otimes k(\beta).$$

It follows easily from this that exceptional curves of the mappings f and f^{-1} are contained in fibers of the fiberings π and π'. If f were not regular, then there would exist on V exceptional curves of the first or second kind. Exceptional curves of the first kind cannot exist on V by definition of a minimal model. We shall show that there also do not exist any exceptional curves of the second kind.

For this we note that if there exists an exceptional curve of the second kind, then there also exists an irreducible exceptional curve of the second kind. The proof of this fact given in Chapter II (Theorem 1, §5) remains valid in the present case (we cannot directly apply this theorem because of the different sense of the term "minimal model").

If there were on V an irreducible exceptional curve of the second kind C contained in a fiber F_0, then, by Theorem 1, Chapter II, §1, we would have

$$(C^2) \geq 0.$$

Let

$$F_0 = nC + \sum n_i C_i, \quad n > 0.$$

It is clear that $(F_0 \cdot C) = 0$, since in the calculation of $(F_0 \cdot C)$ we can replace F_0 by any other fiber. Therefore

$$(C^2) = -\frac{1}{n} \sum n_i (C_i \cdot C) \leqslant 0.$$

Consequently it must be true that $(C^2) = 0$, which is possible only if all the $(C_i \cdot C) = 0$. This means that C coincides with a connected component of F_0, which, because of the connectedness of F_0, is possible only for $F_0 = nC$. But then we have $p_a(C) = 1$, at the same time that $p_a(C) = 0$ (by Theorem 1 of §1, Chapter II). The theorem is proved.

Corollary. *Two birationally equivalent minimal models are biregularly equivalent. A birational automorphism of a minimal model is biregular.*

We consider an example of the application of Theorem 1. Let (V, B, π) be the surface with a pencil described in the remark. It is clear that $B = \mathbf{P}^1$ (a point on B is a ratio $(\lambda : \mu)$). The lines $L_i = \sigma(P_i)$, $i = 1, \cdots, 9$ are mapped by the projection π biregularly onto B. Thus it is possible to define biregular mappings $s_i : B \to L_i$ such that $\pi s_i = 1$. Taking the point $F_\beta \cdot L_1$ for 0, we can define on the generic fiber F_β the structure of a one-dimensional abelian variety. The mappings ϕ_i, $i = 1, \cdots, 9$, defined by

$$\varphi_i(v) = v + s_i \pi(v)$$

are biregular transformations, if $v \in F_\beta$ are contractions onto the points $s_i(\beta)$. They are thus birational, and by the corollary are also biregular automorphisms of V. The group formed by these automorphisms is, as it is easy to show, the free abelian group with eight generators $\phi_2, \cdots, \phi_9$. We thus obtain an example of a surface that has an infinite group of automorphisms, but does not have, as is easily verified, an algebraic group (or, for $k = \mathbf{C}$, a Lie group) of automorphisms.

On the other hand, there exist on V curves with negative squares (for example, L_i). Using the automorphisms constructed, we can obtain an infinite number of such curves. Thus, we obtain an example of a surface containing an infinite number of curves with a negative square (and even exceptional curves of the first kind).

§2. The structure of fibers

Lemma 1. *Let V be a surface, π its regular mapping onto a curve B, F_0 one of the fibers of the fibering π, and $F_0 = \Sigma n_i C_i$, $n_i > 0$, where the C_i are irreducible curves. If $D = \Sigma m_i C_i$, then $(D^2) \leq 0$.*

Proof. Assume $(D^2) > 0$. It then follows from the Riemann-Roch theorem that for any E and for a sufficiently large n, $l(nD - E) > 0$. We choose for E a hyperplane section of the surface V and set

$$nD - E \sim D_1 > 0.$$

Then, if F is any fiber of the fibering π,

$$n(D \cdot F) = (E \cdot F) + (D_1 \cdot F). \tag{1}$$

Since all the components of the cycle D are contained in F_0, $(D \cdot F) = 0$. On the other hand, $(E \cdot F) > 0$ and $(D_1 \cdot F) \geq 0$, since we can choose an irreducible fiber for F. Thus we have a contradiction and the lemma is proved.

The assertion of Lemma 1 can be expressed differently. For this we consider the space X (over the field of rational numbers) a basis of which is formed by the irreducible components $C_1, \cdots, C_k$ of the fiber F_0. The index of intersection $(C \cdot D)$ determines in X a scalar product and the quadratic form

$$\psi(C) = (C^2). \tag{2}$$

The lemma asserts that this form is not positive.

Theorem 2 (cf. Zariski [22]). *If, in the notation of Lemma* 1, *the fiber* F_0 *is connected and* $(D^2) = 0$, *then* $D = rF_0$, *where* r *is a rational number.*

Since $(F_0 \cdot C_i) = 0$ for all the $C_1, \cdots, C_k$, the quadratic form ψ defined by (2) has rank $\leq k - 1$. The assertion of Theorem 2 is equivalent to saying that this rank is equal to $k - 1$. Assume that the rank of ψ is equal to $l \leq k - 2$. In the space X we denote by Y the subspace consisting of all D for which $(D \cdot C) = 0$ for all $C \in X$. It is clear that in the factor space X/Y the form ψ is negative definite. We can choose $C_1, \cdots, C_l$ such that they form a basis in X/Y, and then

$$(x_1 C_1 + \cdots + x_l C_l)^2 < 0 \tag{3}$$

for all $x_1, \cdots, x_l$ not simultaneously equal to zero.

We set

$$C_{l+1} = L + D,$$

where $D \in Y$ and L is a linear combination of $C_1, \cdots, C_l$. Let

$$L = L_1 - L_2,$$

where $L_1 > 0$, $L_2 > 0$, and L_1 and L_2 do not have common components. Then, on the one hand, $(C_{l+1} \cdot L_1) \geq 0$, since C_{l+1} and L_1 do not have common components, and on the other hand,

$$(C_{l+1} \cdot L_1) = (L_1^2) - (L_1 \cdot L_2) \leq 0.$$

Hence $(L_1^2) = 0$ and $(L_1 \cdot L_2) = 0$. But from (3) it follows that $L_1 = 0$,

$$C_{l+1} + L_2 \in Y,$$

and, in particular, $((C_{l+1} + L_2) \cdot C_i) = 0$. This means that the cycles C_i that are components of $C_{l+1} + L_2$ do not intersect with other cycles, and this, in turn, means that two sets of cycles constitute two connected components of the fiber F_0, which contradicts the assumption.

§3. The canonical class

Lemma 2. *On a surface with a pencil of elliptic curves, the canonical class contains a divisor consisting of components of fibers.*

Proof. Let F_β be a generic fiber of the fibering $\pi: V \longrightarrow B$. Since the canonical class of the curve F_β is equal to 0, $K \cdot F_\beta \sim 0$. If $D \in K$ and $D \cdot F_\beta = (f)$, then $\overline{D} = D - (f) \in K$ and $(\overline{D} \cdot F_\beta) = 0$. As we saw in Chapter IV, §7, it follows from this that D consists of components of fibers.

Remark. It follows from the proof of the lemma that any effective divisor consists of components of fibers.

Theorem 3. *On a minimal model of a surface with a pencil of elliptic curves* $(K^2) = 0$.

We have to show the impossibility of a) $(K^2) > 0$, b) $(K^2) < 0$.

a) It follows from the Riemann-Roch theorem that for any E and for some $n > 0$, $l(nK - E) > 0$. We take for E a hyperplane section of the surface V and let

$$nK - E \sim D > 0.$$

Since a fiber F is an elliptic curve and $(F^2) = 0$, we have $(K \cdot F) = 0$. Thus

$$0 = n\,(K \cdot F) = (E \cdot F) + (D \cdot F).$$

But $(E \cdot F) > 0$, and $(D \cdot F) \geq 0$, which leads to a contradiction.

b) According to Lemma 2, there exists in the canonical class a representative K of the form

$$K = \sum_0^m K_i,$$

where the K_i consist of components of fibers F_i.

Since $(K_i \cdot K_j) = 0$ for $i \neq j$,

$$(K^2) = \sum (K_i^2)$$

and it is sufficient for us to show that all the $(K_i^2) \geq 0$.

Let $(K_0^2) < 0$, and let C_0 be an irreducible component of the fiber F_0 contained in K_0. If $F_0 = nC_0$, then $(C_0^2) = 0$, and thus $(K_0^2) = 0$. If F_0 contains still other components $F_0 = nC_0 + \Sigma_1^k n_i C_i$, then

$$0 = (F_0 \cdot C_0) = n\,(C_0^2) + \sum n_i\,(C_i \cdot C_0).$$

Here $(C_i \cdot C_0) \geq 0$ and for at least one i, $(C_i \cdot C_0) > 0$; otherwise C_0 would be a connected component of the fiber F_0, which contradicts its connectedness. Therefore, $(C_0^2) < 0$. For at least one component C_0 of the cycle F_0 we have $(C_0 \cdot K_0) < 0$. For, if $(C_i \cdot K_0) \geq 0$ for $i = 0, \cdots, k$, then $(C_i \cdot K_0) = 0$ for all i – otherwise for some C_j it would be true that $(C_j \cdot K_0) > 0$, and since for all j, $(C_j \cdot K_0) \geq 0$, then $(F_0 \cdot K_0) > 0$ at the same time as $(F_0 \cdot K_0) = 0$. This proves that if for all i, $(C_i \cdot K_0) \geq 0$, then $(C_i \cdot K_0) = 0$. But then $(K_0^2) = 0$, which contradicts our assumption.

We have proved the existence of an irreducible component C_0 of the divisor K_0 for which $(C_0 \cdot K_0) < 0$. But then the inequality already established

$$(C_0 \cdot K) = (C_0 \cdot K_0) < 0, \quad (C_0^2) < 0$$

shows that

$$p_a(C_0) = \frac{(C_0 \cdot K) + (C_0^2)}{2} + 1 \leqslant 0,$$

which is possible only for $p_a(C_0) = 0$, $(C_0^2) = -1$. This means that C_0 is an exceptional curve of the first kind that is a component of a fiber, and this contradicts the minimality of V.

Theorem 4. *On a minimal model of a surface with a pencil of elliptic curves, the canonical class contains a divisor that is a rational combination of fibers.*

Proof. Let K be a divisor of the canonical class that consists of components of fibers, $K = \Sigma K_i$. Since by Theorem 3 $(K_i^2) = 0$, it follows from Theorem 1 that $K_i = r_i F_i$, where r_i is some rational number. This is the assertion of the theorem.

Remark. It follows from the remark after Lemma 2 that any effective divisor of the canonical class is a rational combination of fibers.

Definition. A fiber F_0 is said to be multiple if

$$F_0 = \sum n_i C_i, \quad n_i > 1.$$

Corollary to Theorem 4. *If the fibering $\pi: V \longrightarrow B$ does not contain multiple fibers, the canonical class contains a divisor that is an integral linear combination of fibers.*

§4. Surfaces with an elliptic pencil and surfaces with $(K^2) = 0$

Theorem 5. *Assume that the surface V is neither rational nor ruled, but is a minimal model and for it $(K^2) = 0$. Then either $12K = 0$ or there exists an m such that, for all sufficiently large n, the linear system $|mnK|$ has neither fixed curves nor base points and is composed of a pencil of elliptic curves. The property given* *uniquely determines this pencil.*

Proof. We consider two cases: A) All the $P_n \leq 1$, and B) some $P_n \geq 2$.

A) We consider separately the cases $p = 0$ and $p = 1$. If $p = 0$ and $q = 0$, then, for $P_2 = 0$, by the theorem of Castelnuovo (Chapter III, §2), the surface is rational, and for $P_2 = 1$, by the remark at the end of §1, Chapter VIII, we have an Enriques surface, for which $2K = 0$, and thus $12K = 0$. If $p = 0$ and $q > 1$, then by Theorem 5 (§3, Chapter IV) V is a ruled surface. Finally, for $p = 0$, $q = 1$, by Theorems 11 and 12 (§8, Chapter IV) V can be represented in the form $(B \times C)/G$, where B is an elliptic curve, C is a curve of arbitrary genus g, and G is the finite group of automorphisms without fixed points of the surface $B \times C$. If $g = 0$, the surface V is ruled. If $g > 1$, then by the remark after Lemma 14 (Chapter IV), $P_n(V)$ takes values as large as desired, and we have case B). It remains to consider the case when $g = 1$, i.e., when B and C are elliptic curves.

By the theorem of Enriques (Theorem 13, Chapter IV), if V is not a ruled surface, then $P_{12}(V) > 0$, i.e., $12K \sim D > 0$. We shall show that $D = 0$. Thus, if $f: B \times C \longrightarrow V$ is a projection, then the canonical class $\overline{K}$ of the surface $B \times C$ has the form $f^*(K)$. Therefore, if $D > 0$, $D \neq 0$, then

$$12\overline{K} = f^*(12K) = f^*(D)$$

and it is clear that $f^*(D) > 0$, $f^*(D) \neq 0$. This, however, contradicts the fact that $B \times C$ is an abelian variety, so $\overline{K} = 0$ and $12\overline{K} = 0$.

We now consider the case $p = 1$ and thus $P_2 = 1$. Then

$$p_a(V) = 1 - q + p = 2 - q = \frac{\chi}{12} = \frac{2 - 4q + 2p + h^{1,1}}{12} = \frac{4 - 4q + h^{1,1}}{12}.$$

Since $h^{1,1} > 0$, it follows from this that $q \leq 2$. If $q = 0$, then by the Riemann-Roch theorem

$$l(-K) + l(2K) \geq 2,$$

i.e., $l(-K) \geq 1$. Since $l(K) \geq 1$, it follows from this that $K = 0$.

The case $q = 1$ is impossible by Theorem 1 (Chapter VIII). Finally, for $q = 2$, V is an abelian variety according to Theorem 3 (Chapter VIII), and thus again $K = 0$.

B) Let $P_\nu \geq 2$ for some ν, i.e. $l(\nu K) \geq 2$.

By Bertini's theorem the system $|\nu K|$ is composed of a pencil C_λ. If D is the fixed part of this system, then

$$\nu K \sim D + \Sigma C_i, \tag{4}$$

where the C_i are curves of the pencil C_λ and $(C_\lambda^2) \geq 0$. On the other hand, $(C_i \cdot K) \geq 0$ and $(D \cdot K) \geq 0$, for otherwise, by the lemma of Chapter II, §4, the surface V would be ruled or would not be a minimal model. Since $(K^2) = 0$, it follows from (4) by the multiplication of both the parts by K that $(C_i \cdot K) = 0$. Now multiplying both parts of (4) by C_i, we obtain from this (and from the fact that $(C_i^2) \geq 0$) that $(C_i^2) = 0$ and $(C_i \cdot C_j) = 0$. Thus, $p_a(C_i) = 1$, and distinct curves C_i do not intersect. It follows from Bertini's theorem that C_λ is a pencil of elliptic curves, where, since $(C_\lambda^2) = 0$, it does not have fundamental points. It determines a regular mapping of V onto some curve B.

It remains to prove the assertion about the fixed components of the system $|nK|$. By the remark after Theorem 4, the divisor $D + \Sigma C_i$ is a rational linear combination of fibers. Let ν' be the common denominator of the coefficients of this combination. We set $m = \nu \cdot \nu'$. Then $|mK|$ contains an integral linear combination of fibers:

$$mK \sim \Sigma n_i F_{b_i}, \quad n_i > 0.$$

Since for a sufficiently large n the class $n(\Sigma n_i b_i)$ on the curve B does not contain fixed components, the class mnK on V also does not contain fixed components.

Corollary. *If a surface V is a minimal model and is not ruled, then $\kappa = 1$ for it if and only if $(K^2) = 0$, $12K \neq 0$.*

§5. The Jacobian fibering

In this and the following sections we shall present the results of the works [25, 42, 40, 57] on the classification of surfaces with a pencil of elliptic curves. We shall not give proofs of the majority of the results set forth. The reader can find them in the works indicated.

Let $\pi: V \longrightarrow B$ be a fibering, whose fibers determine a pencil of elliptic curves on the surface V. If β is a generic point of the curve B, then the fiber F_β is a curve of genus 1 defined over the field $k(B)$. We can thus apply to the analysis of the surfaces V the theory of curves of genus 1. This is the point of view of the works [42, 56, 57].

It is in general impossible to introduce on the curve F_β the structure of a one-dimensional abelian variety over the field $k(B)$; for this it is necessary that it have a rational point over this field. The existence of a rational point on the curve F_β over the field $k(B)$ is equivalent to the existence of a rational (and thus regular) mapping $\sigma: B \longrightarrow V$ such that $\pi\sigma = 1$. The image $\sigma B = C$ is characterized by the fact that it is an irreducible curve on V and $(C \cdot F) = 1$, where F is any fiber of the fibering π. We shall call such a curve C a section of the fibering π. With each fibering π one may associate another fibering having the same base and already possessing a section. For this it is necessary to consider the Jacobian curve A_β of the curve F_β. The curve A_β possesses the property that there exists a birational mapping ϕ of the curve F_β onto A_β, defined over some finite extension of the field $k(B)$ and establishing an isomorphism between the group of the classes of divisors of degree zero of the curve F_β defined over some field $K \supset k(B)$ and the group of points on the curve A_β defined over the same field. It will be convenient later to assume that the curves F_β (and the corresponding surfaces V with a pencil of elliptic curves) are distinct if the mappings ϕ are distinct, i.e., do not differ by an automorphism of the one-dimensional abelian variety A_β. A generic point of the curve A_β determines some algebraic surface, which, in view of the inclusion $k(B) \subset k(A_\beta)$, has a pencil of elliptic curves with the base B.

We denote by J a nonsingular minimal model of this surface with a pencil of elliptic curves. We shall call J a Jacobian fibering of the fibering $\pi: V \longrightarrow B$. Since the curve A_β has a rational point over the field $k(B)$, the zero point, the fibering J has a section σ, which we shall call the zero section. For each fibering $\pi: V \longrightarrow B$ there exists a covering $C \longrightarrow B$ such that the fibering $V \times_B C$ over C has a section. Every fibering with a section is isomorphic to its Jacobian fibering.

The method of classification to be used is to classify first all the fiberings having a section, and then all the fiberings having a given Jacobian fibering. An arbitrary fibering is associated with its Jacobian fibering in the following way. Let $C \longrightarrow B$ be some normal covering with a Galois group G and let $\widetilde{J}_C$ be a nonsingular minimal model of the fibering $J \times_B C$ over C. The group G, naturally, operates on $\widetilde{J}_C$. The sections $\sigma: C \longrightarrow \widetilde{J}_C$ form a group, which is clearly a G-operator group. We denote this group by $\mathfrak{A}_J(C)$. Then all the fiberings $\pi: V \longrightarrow B$ for which the fibering $V \times_B C$ over C has a section are in one-to-one correspondence with the elements of the group $H^1(G, \mathfrak{A}_J(C))$.

The group structure can be carried over with the help of this correspondence onto the set of all the fiberings of V having J as a Jacobian fibering and for which $V \times_B C$ has a section. For any two fiberings V_1 and V_2 over B with the same Jacobian fibering it is possible to find a covering $C \longrightarrow B$ such that $V_1 \times_B C$ and $V_2 \times_B C$ have a section. Using this, it is possible to introduce a group operation into the whole set of fiberings on elliptic curves $\pi: V \longrightarrow B$ having a given Jacobian fibering J. The group obtained is denoted by $\mathfrak{H}(B, J)$. It is a torsion group.

Let $\pi: V \longrightarrow B$ be a fibering on elliptic curves having the Jacobian fibering J. Let $C \longrightarrow B$ be a covering such that $V \times_B C$ has a section over C and

$u \in H^1(G, \mathfrak{A}_J(C))$ is the element corresponding to V. Thus u is a one-dimensional cocycle, i.e., has the form $u_g \in \mathfrak{A}_J(C)$, $g \in G$. Since u_g is a section of $\widetilde{J}_C$ over C, the transformation $x \to g(x) + u_g(\pi(x))$ is (since $\widetilde{J}_C$ is a minimal model) a biregular automorphism of $\widetilde{J}_C$. We thus have a mapping $\phi\colon G \to \operatorname{Aut}(\widetilde{J}_C)$ which is a a monomorphism. Now V is defined in terms of $\widetilde{J}_C$ and u_g:

$$V \simeq \widetilde{J}_C/\phi(G). \tag{5}$$

§6. Fibers of a Jacobian fibering

The works [25] and [40] describe all the types of degenerate fibers that can be met in Jacobian fiberings. In order to present this description, we recall that a generic fiber F_β is birationally equivalent over $k(B)$ to the curve given by the equation

$$y^2 = x^3 + p\cdot x + q, \quad p, q \in k(B).$$

Let $b \in B$, let t be a local parameter at the point b, and let $\nu(f)$ be the index of the function $f \in k(B)$ at the point b. One may assume that in the above equation $\nu(p) \geq 0$, $\nu(q) \geq 0$, and $\min(\nu(p) - 4, \nu(q) - 6) < 0$. We set

$$\Delta = 4p^3 + 27q^2.$$

The fiber F_b is degenerate if and only if $\nu(\Delta) > 0$. We consider two cases.

A. For $k = 0$ or $k = 1$, one has the representation

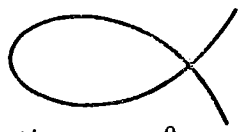
$$p = t^{2k}a, \quad q = t^{3k}b, \quad \nu(a) = \nu(b) = 0, \quad n = \nu(4a^3 + 27b^2) > 0.$$

B. Such a representation does not exist.

We shall denote case A for $k = 0$ by A'_n depending on n, and for $k = 1$, by A''_n. The number n can take any positive integral value.

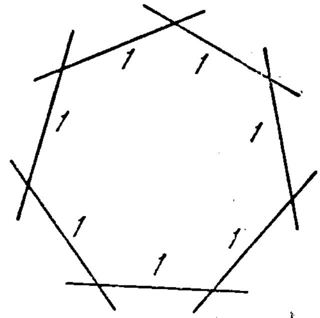
In case B the number $\nu(\Delta)$ can take the values 2, 3, 4, 6, 8, 9, 10. We shall denote these cases by B_n, $n = 2, 3, 4, 6, 8, 9, 10$.

The description of the degenerate fibers in all these cases is given below. Here the Θ_i are rational curves without singular points, except for the case A'_1, when Θ has one double point with different tangents, and for the case B_2, when Θ has one double point with a double tangent. The intersections of the curves Θ_i are shown in Figures 1–4. All the curves are transversal at the points of intersection, except for case B_3, when Θ_1 and Θ_2 have a tangent point of first order.



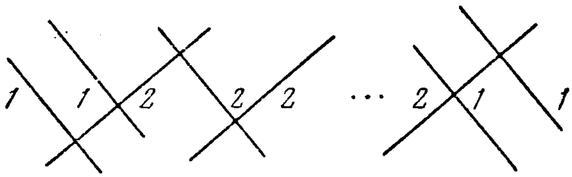
$A'_1;\ A_b = A_b^0$

Figure 1



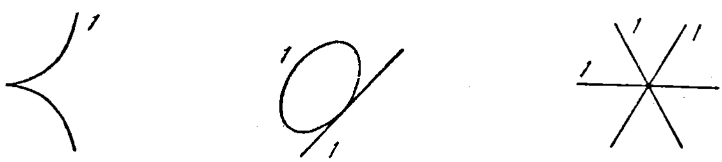
$A'_n,\ n > 1;\ A_b/A_b^0 = \mathbf{Z}_n$

Figure 2

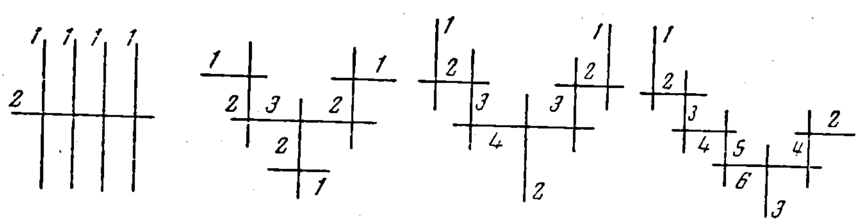


A''_n; for $n \equiv 0\,(2)$, $A_b/A_b^0 = \mathbf{Z}_2 \times \mathbf{Z}$; for $n \not\equiv 0\,(2)$, $A_b/A_b^0 = \mathbf{Z}_4$

Figure 3



$B_2;\ A_b = A_b^0$ $B_3;\ A_b/A_b^0 = \mathbf{Z}_2$ $B_4;\ A_b/A_b^0 = \mathbf{Z}_3$



$B_6;\ A_b/A_b^0 = \mathbf{Z}_2 \times \mathbf{Z}_2$ $B_8;\ A_b/A_b^0 = \mathbf{Z}_3$ $B_9;\ A_b/A_b^0 = \mathbf{Z}_2$ $B_{10};\ A_b = A_b^0$

Figure 4

The fibers have the following structure (Figures 1–4):

$$A_1': \quad F_b = \Theta,$$

$$A_n',\ n > 1: F_b = \Theta_1 + \ldots + \Theta_n,$$

$$A_n'': F_b = \Theta_0 + \Theta_1 + 2\Theta_2 + \ldots + 2\Theta_{n+2} + \Theta_{n+3} + \Theta_{n+4},$$

$$B_2: F_b = \Theta,$$

$$B_3: F_b = \Theta_1 + \Theta_2,$$

$$B_4: F_b = \Theta_1 + \Theta_2 + \Theta_3,$$

$$B_6: F_b = 2\Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4,$$

$$B_8: F_b = \Theta_1 + \Theta_2 + \Theta_3 + 2\Theta_4 + 2\Theta_5 + 2\Theta_6 + 3\Theta_7,$$

$$B_9: F_b = \Theta_1 + \Theta_2 + 2\Theta_3 + 2\Theta_4 + 2\Theta_5 + 3\Theta_6 + 7\Theta_7 + 4\Theta_8,$$

$$B_{10}: F_b = \Theta_1 + 2\Theta_2 + 2\Theta_3 + 3\Theta_4 + 3\Theta_5 + 4\Theta_6 + 4\Theta_7 + 5\Theta_8 + 6\Theta_9.$$

It is evident from the consideration of the separate cases that it is always true that $\nu(\Delta) = \chi(F_b)$.

More detailed properties of the degenerate fibers come from a consideration of the group structure on them. The group operation on a generic fiber F_β of the fibering J determines a regular mapping $F_\beta \times F_\beta \rightarrow F_\beta$. This mapping can be considered as a rational mapping $J \times_B J \rightarrow J$. It turns out that if J is a minimal model, this mapping is regular and defines the structure of an algebraic group on the set of nonsingular points of each fiber F_b. We denote this group by A_b. In particular, the component that intersects the zero section forms a subgroup A_b^0, a connected component of the unit of the group A_b. The group $A_b = A_b^0$ is an elliptic curve if the fiber F_b is nonsingular, is a multiplicative group in the case of A_n', and an additive group in all the remaining cases. The group A_b/A_b^0 is shown in Figures 1–4.

Finally, we shall indicate the form of Jacobian fiberings that do not have degenerate fibers.

Definition. Let A be an elliptic curve (with a fixed group structure) over the field k, let B be an arbitrary curve, let $\overline{B} \rightarrow B$ be a normal nonramified covering whose Galois group is isomorphic to some subgroup of the group of automorphisms of the curve A (as an abelian variety), and let $\phi: G \rightarrow \operatorname{Aut} A$ be the corresponding automorphism. We define the operation of G on $\overline{B} \times A$ according to the rule

$$\sigma(\bar{b} \times a) = \sigma\bar{b} \times a\varphi(\sigma)^{-1}.$$

The fibering $(B \times A)/G \rightarrow B$ does not have degenerate fibers. Such a fibering is said to be a *fiber bundle*. We have

Theorem 6. *Every Jacobian fibering without degenerate fibers is a fiber bundle.*

Corollary. *If B is a rational curve, then a Jacobian fibering without degenerate fibers over B has the form $B \times F$, where F is an elliptic curve.*

§7. Local classification

In order for a fibering of V over B by elliptic curves to be isomorphic to its Jacobian fibering, it is necessary and sufficient that it have a section.

Every section s over B gives a local section at any point $b \in B$. By this is meant the mapping $s_b: B \to V$ given by a formal power series in the powers of the local parameter t at the point b, and such that $\pi s_b = 1$. In other words, a local section is a rational point on the curve F_β over the field of power series $k\{t\}$. An important necessary condition for the existence of a section is the existence of a local section at some point $b \in B$. In connection with this we introduce also the concept of a local isomorphism of fiberings, i.e. an isomorphism given by a formal power series in the powers of the local parameter t at the point b. In other words, the fiberings $\pi: V \to B$ and $\pi': V' \to B$ are locally isomorphic at the point $b \in B$ if the curves F_β and F'_β are isomorphic over the field of power series $k\{t\}$.

We now give the classification of a fibering up to a local isomorphism at a given point $b \in B$. For any fibering $V \to B$ there exists a covering $C \to B$ having one branch point over the point $b \in B$ such that $V \times_B C$ is formally isomorphic to $J \times_B C$, where J is the Jacobian fibering of V.

We denote by $\widetilde{V} \to C$ a minimal nonsingular model of the fibering $V \times_B C$. Then the fiberings $\widetilde{V} \to C$ and $\widetilde{J} \to C$ are isomorphic. Since, moreover, there is a unique projection $\widetilde{J} \to J$, we obtain a mapping $\widetilde{V} \to J$. Let U_b and $\widetilde{F}_c$ be fibers of the fiberings $\widetilde{J}$ and $\widetilde{V}$ lying over the point b and its preimage $c \in C$ respectively. The mapping $\widetilde{V} \to J$ determines a mapping $\widetilde{F}_c \to U_b$, which, as can be easily seen, is an unramified cyclic covering. The Galois group H of this covering, naturally, is isomorphic to the subgroup of the Galois group G of the covering $C \to B$ at the point b. The group G has a distinguished character ψ_0. In fact, the corresponding extension is obtained by the addition of the element $\sqrt[n]{t}$ to $k\{t\}$. The character ψ_0 is determined by $\psi_0(\sigma) = \sqrt[n]{t^{1-\sigma}}$. A restriction of the character ψ_0 determines some character on the Galois group H of the covering $\widetilde{F}_c \to U_b$. Thus we have some unramified covering $\widetilde{F}_c \to U_b$ and some character of the Galois group H of this covering. Since the group H (when $k = \mathbf{C}$) is a factor group of the group $H_1(U_b)$, we thus have some character ψ of the group $H_1(U_b)$:

$$\psi \in \operatorname{Char} H_1(U_b),$$

where Char denotes the group of characters of finite order, and $H_1(U_b)$ is the one-dimensional Betti group with integer coefficients.

The character ψ is a basic invariant of a fibering from the point of view of a formal isomorphism. It is convenient to replace it with another invariant, however. For this we consider the open set $A_b^0 \subset U_b$ and the natural mapping $j: H_1(U_b) \to \overline{H}_1(A_b^0)$, where $\overline{H}_1$ is the homology group with arbitrary carriers (if the fiber U_b is singular, A_b^0 is not compact).

It is easy to verify for all types of fibers that j is an isomorphism, and consequently determines an isomorphism j^* of the groups $\mathrm{Char}\, H_1(U_b)$ and $\mathrm{Char}\, \overline{H}_1(A_b^0)$.

Since A_b^0 is a variety, the group $\mathrm{Char}\, \overline{H}_1(A_b^0)$ is isomorphic to $H_1(A_b^0, \mathbf{Q}/\mathbf{Z})$, where $\mathbf{Q}$ is an additive group of rational numbers. As a result we have an isomorphism of the groups $\mathrm{Char}\, H_1(U_b)$ and $H_1(A_b^0, \mathbf{Q}/\mathbf{Z})$. We denote by $h_b(V)$ the element of the group $H_1(A_b^0, \mathbf{Q}/\mathbf{Z})$ corresponding under this isomorphism to the character ψ which we associated with the fibering of V.

The following is a basic result of the local classification.

Theorem 7. *For $k = \mathbf{C}$ the fiberings $\pi: V \to B$ and $\pi': V' \to B$ are locally isomorphic at the point $b \in B$ if and only if $h_b(V) = h_b(V')$.*

For a somewhat more general class of fields k an analogous result is obtained in the works [42] and [57].

Corollary 1. *The invariant $h_b(V)$ can be different from zero only if the fiber U_b is nonsingular or has the type A_n'.*

In fact, in the remaining cases A_b^0 is an additive group which is simply connected.

Corollary 2. *For an arbitrary fibering $\pi: V \to B$ a singular fiber F_b is either isomorphic to the corresponding fiber U_b of the Jacobian fibering or (when U_b is nonsingular or of type A_n') is multiple and is obtained from U_b by multiplication by some integer.*

If the fiber U_b is singular, but not of type A_n', the assertion follows from Corollary 1. The remaining cases are easily verified by a direct construction of the fiber F_b according to formula (5).

Corollary 3. *If the fibering $V \to B$ has no degenerate fibers other than multiple nonsingular fibers, there exists a covering $C \to B$ such that the fibering $V \times_B C$ over C is isomorphic to $C \times F$, where F is an elliptic curve.*

For the proof it is sufficient to choose C such that $V \times_B C \simeq J \times_B C$ and to apply Corollary 2 and Theorem 6.

Corollary 4. *The fibering* $V \longrightarrow B$ *has a local section at a point* $b \in B$ *if and only if the fiber* F_b *is not multiple.*

The proof is obvious.

Corollary 5. *The fibering* $V \longrightarrow B$ *has a local section at a point* $b \in B$ *if and only if it has a differentiable section in some neighborhood of* b.

In fact, if V does not have a local section, then the fiber F_b has the form mD, $m > 1$. If the fibering had a differentiable section s, then we would have $(s \cdot F_b) = 1$ at the same time that $(s \cdot F_b) = m(s \cdot D) > 1$.

§8. Classification of fiberings

For a fibering $\pi: V \longrightarrow B$ and any point $b \in B$ we have defined the invariant $h_b(V) \in H^1(A_b^0, \mathbf{Q}/\mathbf{Z})$. It is easy to see that $h_b(V) = 0$ if F_b is a nondegenerate fiber of the fibering π. Thus for a given V, $h_b(V) \neq 0$ only for a finite number of points $b \in B$. Therefore the correspondence

$$V \longrightarrow \{h_b(\nu), \quad b \in B\}$$

determines the homomorphism

$$\varphi: \quad \mathfrak{H}(B, J) \rightarrow \sum_{b \in B} H_1(A_b^0, \mathbf{Q}/\mathbf{Z}).$$

Our first goal is the description of the kernel and cokernel of the homomorphism ϕ. We begin with the description of the cokernel.

Theorem 8. *If* J *is not a direct product* $B \times A$, *where* A *is an elliptic curve, then the homomorphism* ϕ *is epimorphic. If* $J \simeq B \times A$, *then the cokernel of the homomorphism* ϕ *is isomorphic to the group of points of finite order of the variety* A.

The kernel of the homomorphism ϕ consists of those fiberings which, by Theorem 7, have a local section at any point $b \in B$. In other words, they are locally isomorphic to J at any point of the base. We shall call such fiberings locally trivial, and we shall denote by $\mathfrak{H}(B, J)$ the group consisting of all such fiberings.

Now let us assume that $k = \mathbf{C}$. The analysis of the group $\mathfrak{H}(B, J)$ can be conducted according to the classical example of the analysis of one-dimensional vector fiberings. Namely, it is based on the comparison of the algebraic and differential structure of the fiberings $V \in \mathfrak{H}(B, J)$.

We denote by $\mathfrak{H}_0(B, J)$ the subgroup of the group $\mathfrak{H}(B, J)$ consisting of those fiberings which are isomorphic to J as differentiable fiberings. These fiberings are also characterized by the fact that they possess a differentiable section.

Theorem 9. *The group* $\mathfrak{D}(B, J) = \mathfrak{H}(B, J)/\mathfrak{H}_0(B, J)$ *is finite.*

We first describe the group $\mathfrak{H}_0$ and then the group $\mathfrak{D}$ and we shall then show that $\mathfrak{H}$ is their direct sum.

Let $V \in \mathfrak{H}_0(B, J)$ and let $u_\sigma \in H^1(G, \mathfrak{A}_J(C))$ be a cocycle. It easily follows from the condition $V \in \mathfrak{H}_0(B, J)$ that

$$u_\sigma = s - \sigma s, \quad \sigma \in G, \tag{6}$$

where $s\colon C \to \widetilde{J}$ is a differentiable section of the fibering $\widetilde{J}$, and the section σs is defined by the formula

$$(\sigma s)(c) = s(\sigma^{-1} c).$$

The cocycle u_σ, as well as s and σs, are two-dimensional cycles of the variety $\widetilde{J}$. We denote by $\hat{u}_\sigma$, $\hat{s}$ and $\hat{\sigma s}$ the two-dimensional cycles corresponding to them – elements of the group $H^2(\widetilde{J}, \mathbf{Z})$. Let $P^{2,0}$ be the operator associating with an element of $H^2(\widetilde{J}, \mathbf{Z})$ its component of type $(2, 0)$. Since u_σ is an algebraic cycle, by a theorem of Lefschetz, $P^{2,0}\hat{u}_\sigma = 0$. Therefore, we obtain from (6) that

$$P^{2,0}\hat{s} = \sigma P^{2,0}\hat{s}. \tag{7}$$

The mapping $f\colon \widetilde{J} \to J$ determines the imbedding

$$f^*\colon\ H^2(J, \mathbf{C}) \to H^2(\widetilde{J}, \mathbf{C}).$$

As is known (and easily verified), here elements of $H^{2,0}(J, \mathbf{C})$ are mapped into elements of $H^{2,0}(\widetilde{J}, \mathbf{C})$ that are invariant with respect to the operation of the group G. Equation (7) shows then that there exists an element $x \in H^{2,0}(J, \mathbf{C})$ such that

$$f^* x = P^{2,0}\hat{s}.$$

Direct verification shows that the element x is determined by a given fibering of V uniquely up to a term of the type $P^{2,0}J$, $y \in H^2(J, \mathbf{Z})$.

The coset in the group $H^{2,0}(J, \mathbf{C})/P^{2,0}H^2(I, \mathbf{Z})$ determined by the element x is denoted by $\gamma(V)$.

Theorem 10. *The mapping*

$$\gamma\colon\ \mathfrak{H}_0(B, J) \to H^{2,0}(J, \mathbf{C})/P^{2,0}H^2(J, \mathbf{Z})$$

determines an isomorphism of the group $\mathfrak{H}_0(B, J)$ *and a torsion part of the group* $H^{2,0}(J, \mathbf{C})/P^{2,0}H^2(J, \mathbf{Z})$.

The homomorphism

$$P^{2,0}\colon\ H^2(J, \mathbf{Z}) \to H^{2,0}(J, \mathbf{C})$$

has a kernel, according to a theorem of Lefschetz, the group $H_a^2(J)$ consisting of algebraic cycles. The factor group

$$H_t^2(J) = H^2(J, \mathbf{Z})/H_a^2(J)$$

is said to be a group of transcendental cycles. This group is imbedded in the group $H^{2,0}(J, \mathbf{C})$ with the aid of the homomorphism $P^{2,0}$. On the other hand, if Y is a subgroup of an infinitely divisible group X, then the torsion part of the group X/Y is isomorphic, as can easily be seen, to the group $Y \otimes \mathbf{Q}/\mathbf{Z}$. In conjunction with Theorem 9 this gives us the following result.

Theorem 11. *The group* $\mathfrak{H}_0(B, J)$ *is isomorphic to* $H_t^2(J) \otimes \mathbf{Q}/\mathbf{Z}$.

It remains for us to describe the structure of the group $\mathfrak{D}(B, J)$.

Theorem 12. *If the fibering* J *has any degenerate fibers, or if it has the form* $B \times A$, *where* A *is an elliptic curve, then* $\mathfrak{D}(B, J) = 0$. *If* J *does not have degenerate fibers and* $J \not\simeq B \times A$, *then the group* $\mathfrak{D}(B, J)$ *is isomorphic, depending on the type of the group of automorphisms of a fiber, to a group of order* 1, 2, 3, *or* 4 (*in the last case to* $\mathbf{Z}_2 \oplus \mathbf{Z}_2$).

The proof is based on results found in [57]. Let us assume that J has a degenerate fiber. It follows from the above that the group $\mathfrak{D}(B, J)$ is isomorphic to the factor group of the group $\mathfrak{H}(B, J)$ over the subgroup of the infinitely divisible elements. According to [57] this factor group is isomorphic to the group of those sections s of the fibering J that have finite order in the group $\mathfrak{A}_J(B)$ of all sections and for any $b \in B$ intersect the fiber F_b in some point belonging to the subgroup A_b^0. We have to show that such a section s is equal to the zero section o.

We note first that if $s \neq o$, then s does not intersect the zero section. For, in the terminology of [57], s determines for each $b \in B$ a nonzero point of finite order on the curve F_β that is rational over the field $k\{t\}$ of power series in the powers of the local parameter t at the point b. This point belongs to a connected component α_b^0 of the group of all points α_b of the curve F_β that are rational over $k\{t\}$. On the other hand, under the specialization $\beta \longrightarrow b$ the points of finite order in the group α_b^0 are mapped isomorphically onto the points of finite order of the group A_b^0. This means that $s \cdot A_b^0 \neq 0$, and since this is true for any $b \in B$, it follows from this that s does not intersect the zero section.

Let $ms = o$. Then the divisor $ms \cdot F_\beta - mo \cdot F_\beta$ is equivalent to 0 on F_β and, consequently, determines some function f on F_β and thus on J. We want to apply Lemma 8 of Chapter IV to this function. This cannot be done directly, since the lemma applies to fiberings without degenerate fibers, while J has degenerate fibers. In order to be able to apply the result of Lemma 8, we remove from J in each degenerate fiber F_b all the components different from A_b^0. We denote the open set left by J^0. It is easy to verify that the proof of Lemma 8 remains valid without any changes for the surface J^0 if one considers in the formulation of the lemma only divisors C_i on J that intersect with singular fibers F_b only at points of the sets A_b^0. s and o are such divisors. The divisor $ms - mo$, as we have seen, is not ramified, and we can now apply Lemma 8. It gives us, in particular, that all the fibers of the fibering J^0 are isomorphic, which is possible only if the fibering J does not have degenerate fibers. Thus, if J has a degenerate fiber, then $s = o$.

The case $J = B \times A$ is analyzed on the basis of the description of the group $\mathfrak{D}(B, J)$ given in [25]. In this case $\mathfrak{D}(B, J)$ is a periodic subgroup of the group $\mathbf{Z} \oplus \mathbf{Z}$, and is consequently equal to zero.

Now let J not have degenerate fibers, and consequently be a fiber bundle, but let it not be a direct product. Then J has the form $(C \times A)/G$, where $f: C \to B$ is an unramified covering with a Galois group G, A is an elliptic curve, ϕ: $G \to \operatorname{Aut} A$ is a homomorphism, and G operates on $C \times A$ according to the rule

$$\sigma\,(c \times a) = \sigma\,(c) \times a\varphi\,(\sigma)^{-1}. \tag{8}$$

According to [57], the group $\mathfrak{D}(B, J)$ is isomorphic to the group of sections of finite order of the fibering J. We now indicate what these sections are like. Let s: $B \to J$ be a periodic section. It determines the section

$$s^*\colon\quad C \to J \times_B C,\quad s^*(c) = (sf)\,(c) \times c.$$

The section s^* possesses the property

$$\sigma s^*(c) = s^*(\sigma c),$$

where σ operates on $J \times_B C$ according to the rule

$$\sigma\,(x \times c) = x \times \sigma\,(c). \tag{9}$$

In our case, $J \times_B C \simeq A \times C$, and under this isomorphism the operation (8) of the automorphism σ goes into (9).

In view of this s^* can be written in the form

$$s^*(c) = u\,(c) \times c, \tag{10}$$

where u: $C \to A$ is a regular mapping. Rule (10) gives

$$u\,(\sigma\,(c)) = u\,(c)\,\varphi\,(\sigma)^{-1}. \tag{11}$$

Since s is a section of finite order, for some $m > 1$

$$ms^*(c) = 0, \quad mu\,(c) = 0.$$

Thus $u\,(c)$ is an element of period m in A. The number of such elements is finite, and then it follows from the regularity of the mapping u that this mapping is constant:

$$u\,(c) = u_0 \in A.$$

The condition (11) shows that

$$\phi(\sigma) u_0 = u_0. \tag{12}$$

As is known, σ can be an automorphism of order 2, 3, 4, or 6. It is easy to verify that the solutions of equation (12) form in these cases groups respectively of orders 4 (isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$), 3, 2, and 1.

§9. One particular case

We apply the obtained classification of fiberings on elliptic curves to the study of one type of surfaces which we have already met earlier (Chapter IV, §7). Namely, we consider the surfaces V with the invariants $p = 0$, $q = 1$ for which the Albanese mapping $\pi: V \to B$ has as fibers elliptic curves. We saw in Chapter IV, §7 that in this case the fibers can be only nonsingular or multiples of nonsingular curves. It follows from Corollary 2 of Theorem 7 that the Jacobian fibering J of a fibering of V does not have degenerate fibers and is, consequently, a fiber bundle

$$J \simeq (C \times A)/G.$$

We first establish that V has the invariants we need if and only if J is not a direct product (i.e., $\phi(G) \neq 1$). For this we have to give an argument very close to that which is contained in Chapter IV, §8. We note that by formula (5) of §5 the fibering of V is a factor of the fibering $\widetilde{J} \simeq J \times_B \overline{C}$ for some covering $\overline{C} \to B$ with a Galois group $\overline{G}$. We can choose $\overline{C}$ such that the covering $C \to B$ will be a factor of it. This gives a homomorphism $\psi: \overline{G} \to G$. Then, as it is easy to see, $\widetilde{J} \simeq \overline{C} \times A$. It is not difficult to interpret the operation of the group $\overline{G}$ on $\widetilde{J}$. A simple calculation shows that

$$\overline{\sigma}\,(\overline{c} \times a) = \overline{\sigma}\,(\overline{c}) \times a \cdot u\,(\overline{\sigma})^{-1},$$

where $u(\overline{\sigma})$ is an automorphism of the curve A (but not of the corresponding abelian variety), and $u: \overline{G} \to \operatorname{Aut} A$ is a homomorphism of the group $\overline{G}$ into the group of automorphisms on the curve A. Here $u(\overline{\sigma})$ is a combination of an automorphism and a translation of the abelian variety A

$$u\,(\overline{\sigma})\,(a) = \varphi\psi\,(\overline{\sigma})\,(a) + v\,(\overline{\sigma}),$$

where $v(\overline{\sigma})$ is a point of A.

Repeating the argument given in the proof of Lemma 13, §8, Chapter IV, we see that

$$\Omega^i\,(V) \simeq \Omega^i\,(\overline{C} \times A)^{\overline{G}}, \quad i = 1, 2.$$

It follows from this that

$$\Omega^1\,(V) \simeq \Omega^1\,(B) \oplus \Omega^1\,(A)^{\varphi(G)}$$

and thus $q = 1$ if and only if $\phi(G) \neq 1$, i.e. when J is not a direct product.

Analogously,

$$\Omega^2(V) \simeq (\Omega^1(\overline{C}) \oplus \Omega^1(A))^{\overline{G}}.$$

Let H be the kernel of the homomorphism $\psi\colon \overline{G} \to G$. Since

$$(\Omega^1(\overline{C}) \otimes \Omega^1(A))^{\overline{G}} = ((\Omega^1(\overline{C}) \otimes \Omega^1(A))^H)^G,$$

$$\Omega^1(\overline{C})^H \simeq \Omega^1(C),\ \Omega^1(A)^H = \Omega^1(A),\ \Omega^1(C)^G \simeq \Omega^1(B),$$

we have

$$\Omega^2(V) \simeq \Omega^1(B) \otimes \Omega^1(A)^{\varphi(G)},$$

from which it follows that $p = 0$ if and only if $\phi(G) \neq 1$.

Now applying the classification developed in the previous sections, we arrive at the following result.

Theorem 13. *The surfaces V with the invariants $p = 0$, $q = 1$, for which the fibers of the Albanese fibering $\pi\colon V \to B$ have genus 1 are classified in the following way. The Jacobian fibering of the fibering of V has the form*

$$J \simeq (C \times A)/G,$$

where $f\colon C \to B$ is an unramified covering with Galois group G that is not the identity mapping, A is an elliptic curve, $\phi\colon G \to \operatorname{Aut} A$ is an imbedding of G into the group of automorphisms of A (as an abelian variety), and G operates on $C \times A$ according to the rule $\sigma(c \times a) = \sigma(c) \times a\phi(\sigma)^{-1}$. The structure of the group $\mathfrak{H}(B, J)$ of all the fiberings of V with a given Jacobian fibering is determined from the exact sequence (F_b is a fiber of J over the point $b \in B$):

$$0 \to \mathfrak{h}(B, J) \to \mathfrak{H}(B, J) \to \sum_{b \in B} H_1(F_b, \mathbf{Q}/\mathbf{Z}) \to 0$$

and the relationship

$$\mathfrak{h}(B, J) = \mathfrak{D}(B, J), \tag{13}$$

where $\mathfrak{D}(B, J)$ has order 1, 2, 3, or 4 depending on whether the group $\phi(G)$ has order 6, 4, 3, or 2.

We need to verify only the relationship (13). It follows from the fact that $\mathfrak{D}(B, J) = \mathfrak{h}(B, J)/\mathfrak{h}_0(B, J)$ and from $\mathfrak{h}_0(B, J) = 0$. The last assertion follows from the fact that $H_t^2(J) = 0$. For, from the fact that $\chi = 0$, $q = 1$ it follows that $b_2 = 2$. On the other hand, we have immediately two nonhomologous algebraic cycles in J, for example a fiber and a section. Thus the rank of the group $H_a^2(J)$ is also equal to two. Since, according to the criterion of Lefschetz, the group $H_t^2(J) = H^2(J, \mathbf{Z})/H_a^2(J)$ is torsion-free, it must be equal to zero.

Remark. In his consideration of our class of surfaces, Enriques ([59], Chapter X, §11) asserts that V has a pencil of elliptic curves that are transversal to the fibers of the fibering $\pi: V \to B$. It is easy to see, however, that there is an elliptic curve on V distinct from the fibers of the fibering π only if V does not have multiple fibers. In fact, let L be such a curve. By assumption, $(L \cdot F_b) = r > 0$. Since the base B is, like L, an elliptic curve, the projection π determines on L the structure of an unramified covering of B. This means that $L \cdot F_b$ consists of r distinct points for any $b \in B$. But if F_b is a multiple fiber, $F_b = m \cdot U$, $m > 1$, then it is clear that $L \cdot F_b$ consists of the points of $L \cdot U$ taken m times.

Thus the classification of Enriques evidently only applies to fiberings without multiple fibers. As we saw, these are everywhere locally trivial. They form a finite subgroup $\mathfrak{D}(B, J) = \mathfrak{h}(B, J)$ of the group $\mathfrak{H}(B, J)$. The fiberings corresponding to the elements of the infinite factor group $\Sigma H_1(F_b, Q/Z)$ of this group are clearly omitted in Enriques' classification.

CHAPTER VIII

ALGEBRAIC SURFACES WITH $\kappa = 0$

This chapter studies Kähler (sometimes only algebraic) surfaces with $\kappa = 0$.

The existence of a multicanonical model of a surface F means that for some natural number n_0 the number $P_{n_0} = \dim H^0(F, \Omega(n_0 K))$ is greater than zero, where K denotes the canonical line bundle; zero-dimensionality in such a model is equivalent to the requirement $P_n \leq 1$ for all $n > 0$.

§1 discusses the possible values of the integral invariants of surfaces of the indicated type.

These surfaces are classified in §§2, 3, and 4. We will assume from the beginning that all surfaces considered are minimal models, i.e. do not contain exceptional curves of the first kind.

§1. Values of invariants

We begin this section with several lemmas and will terminate it with a table of the possible values of the integral invariants of the surfaces with $\kappa = 0$.

Lemma 1. *Let F be a nonsingular Kähler surface with $\kappa = 0$. Then $(K^2) \geq 0$. Moreover, if θ is an irreducible component of an effective divisor $D \in |nK|$ (n is any positive integer), then $(D \cdot \theta) \geq 0$.*

Proof. Since $\kappa = 0$, there exists a number n such that the system $|nK|$ contains an effective divisor D. Let $D = \Sigma \rho_i \theta_i$, where $\rho_i > 0$ and θ_i is an irreducible curve. By the adjunction formula, $(D \cdot \theta_i) = n(2p_a(\theta_i) - 2 - (\theta_i^2))$. Let us assume that $(D \cdot \theta_i) < 0$. Then $2p_a(\theta_i) - 2 - (\theta_i^2) < 0$. This is possible only when $(\theta_i^2) \geq 0$ (the values $(\theta_i^2) = -1$, $p_a(\theta_i) = 0$ are excluded by the assumption of the minimality of the surface F). Since obviously $(\theta_i \cdot \theta_j) \geq 0$ for $i \neq j$, we arrive at a contradiction. The lemma is proved.

Corollary. *Let F be a nonsingular Kähler surface with $\kappa = 0$. Then $p_a(F) \geq 0$.*

Indeed $12p_a = (K^2) + \chi \geq \chi \geq 4p_a - 2 - 2p \geq 4p_a - 4$.

Lemma 2. *Let F be a nonsingular Kähler surface with $\kappa \geq 0$, $(K^2) > 0$. Then $\kappa > 0$.*

Proof. By the Riemann-Roch theorem

$$\dim H^0(F, \Omega(nK)) + \dim H^0(F, \Omega((1-n)K)) \geqslant \frac{n(n-1)}{2}(K^2) + p_a.$$

Since $\kappa \geq 0$, $\dim H^0(F, \Omega((1-n)K)) \leq 1$ for $n > 1$.

Corollary. *Let F be a nonsingular surface with $\kappa = 0$. Then $(K^2) = 0$. Moreover, if θ is an irreducible component of an effective divisor $D \in |nK|$ (n is any positive integer), then $(D \cdot \theta) = 0$.*

This assertion follows trivially from Lemmas 1 and 2.

Lemma 3. *Let F be a nonsingular Kähler surface, $P_n = 1$, $P_m = 1$, $m \neq n$. Then either $\kappa > 0$ or $P_d = 1$, where $d = (m, n)$.*

Proof. We denote by D_0 and D_1 effective divisors of the systems $|nK|$ and $|mK|$ respectively, and we consider the effective divisors $(m/d) D_0$, $(n/d) D_1$. These divisors belong to the system $|(mn/d) K|$. If they are distinct, then $P_{mn/d} > 1$ and $\kappa > 0$. If they coincide, one of the divisors $D_0 - D_1$, $D_1 - D_0$ is effective (depending on the relative value of the numbers m, n). This divisor (we denote it by D_2) belongs to the system $|(m-n)K|$ (or $|(n-m)K|$). The assertion of the lemma now follows from the existence of the Euclid algorithm for finding the greatest common divisor.

Lemma 4. *Let F be a nonsingular Kähler surface with $\kappa = 0$. Then either $p_g = 1$, or $2K \sim 0$, or $p_g = q - 1 = 0$.*

Proof. Let us assume that $p_g = 0$, $q \neq 1$. Then, since $\kappa = 0$, we have $p_a > 0$, $q = 0$, and $P_2 \neq 0$. By Lemma 3, $P_3 = 0$. By the Riemann-Roch theorem, $\dim H^0(F, \Omega(3K)) + \dim H^0(F, \Omega(-2K)) \geq p_a > 0$. Consequently, $\dim H^0(F, \Omega(-2K)) > 0$, which, together with $P_2 \neq 0$, gives $2K \sim 0$.

The above results permit us to determine those values of the invariants p_a, p_g, q, (K^2) which can correspond to surfaces with $\kappa = 0$.

Let $p_g = 0$. Then by the corollary to Lemma 1, $q = 0$ or $q = 1$.

Let $p_g = 1$. Then $q = 0$, 1, or 2.

Surfaces with $p_g = 0$, $q = 1$ were studied in Chapter VII.

Surfaces with $p_g = 0$, $q = 0$ (Enriques surfaces) will be studied in Chapter X.

Remark. From the results of this section it follows that Enriques surfaces are defined by the following values of the invariants: $p_g = q = P_3 = 0$, $P_2 = 1$.

In §3, we will describe the projective models of the surfaces with $p_g = 1$, $q = 0$. The question of the "number of moduli" of such surfaces is studied in Chapter IX.

In §4, we prove that Kähler surfaces with $p_g = 1$, $q = 2$ are complex tori.

§2. Surfaces with $\kappa = 0$, $p_g = q = 1$

Theorem 1. *There do not exist algebraic surfaces with $\kappa = 0$, $p_g = 1$, $q = 1$.*

The proof of this theorem is preceded by a simple lemma.

Lemma 1. *Let* $(K^2) = 0$ *and* $P_n > 0$. *Then all the irreducible components of an effective divisor* $D \in |nK|$ *have an arithmetic genus of* 0 *or* 1.

Proof of the lemma. By the adjunction formula,

$$p_a(\theta) = \frac{(\theta(K+\theta))}{2} + 1$$

for every irreducible curve θ. According to the corollary of Lemma 2, $(K \cdot \theta) = 0$ if θ is a component of D. Consequently, $(\theta^2) \leq 0$. Now we have

$$0 \leqslant p_a(\theta) = 1 + \frac{(\theta^2)}{2} \leqslant 1.$$

Proof of Theorem 1. We denote by $A(F)$ the Albanese variety of the surface F with $\kappa = 0$, $p_g = q = 1$, that is a nonsingular curve of genus 1, and by α the natural mapping $F \longrightarrow A(F)$. We denote by D an effective divisor of the system $|K|$. We consider two cases.

1) The image of every irreducible component of the divisor D is a point (or the empty set, if $D = 0$). We denote by Γ_C the "fiber" of the mapping over a point $C \in A(F)$. As is known, for a generic point $C \in A(F)$, Γ_C is a nonsingular curve. Clearly, $(\Gamma_C^2) = (\Gamma_C \cdot K) = 0$. By the Riemann-Roch theorem,

$$\begin{aligned} \dim H^0(F, \Omega(\Gamma_C - K)) &+ \dim H^0(F, \Omega(2K - \Gamma_C)) \\ &= 1 + \dim H'(F, \Omega(\Gamma_C - K)). \end{aligned} \qquad (*)$$

Since $\kappa = 0$, $\dim H^0(F, \Omega(2K - \Gamma_C)) = 0$. Hence it follows from the equation $(*)$ that $\dim H^0(F, \Omega(\Gamma_C)) \geq 2$ (if $D > 0$, this is clear; if $D = K = 0$, then

$$\dim H'(F, \Omega(\Gamma_C - K)) = \dim H'(F, \Omega(\Gamma_C + K)) \geqslant 1$$

by a theorem of Kodaira [25], Theorem 2.3).

We consider the linear system $|\Gamma_C|$. We denote its fixed part by θ, its nonfixed part by $|H_\nu|$: $|\Gamma_C| = \theta + |H_\nu|$. Since $(\theta) \subset \Gamma_C$, $(H_{\nu_0}) \subset (\Gamma_C)$ (by (M) we denote the support of the divisor M) for a suitable index ν_0, we have $(\theta \cdot \Gamma_C) = (H_\nu \cdot \Gamma_C) = (H_\nu \cdot K) = 0$. Since, moreover, $(H_\nu^2) = 0$, the system $|H_\nu|$ is composed of a pencil L, a generic curve of which may be assumed to be irreducible. Since $(L \cdot \Gamma_C) = 0$, all the curves of the pencil L belong to fibers of the mapping α.

As above, it is possible to prove that the linear system $|L_0|$ has a positive dimension, where L_0 is an irreducible curve of the pencil L, and that the nonfixed part L_1 of this system (which is a linear pencil) has a zero index of intersection with curves of the pencil L. From the irreducibility of the pencils L and L_1 it now follows that they coincide. This means, however, that there exists a regular mapping of the curve $\mathbf{P}^1$ of genus 0 parametrizing the pencil L onto a curve of $A(F)$ of genus 1. We have a contradiction.

2) There exists at least one irreducible component of the divisor D whose image under the mapping α is a curve of $A(F)$.

We denote by θ the irreducible component of the divisor D whose existence we have just required. It follows from Lemma 1 that θ is a nonsingular curve of genus 1. Let $D = s\theta + \widetilde{D}$, where θ is not a component of the divisor $\widetilde{D}$. It is clear that the divisors θ and $\widetilde{D}$ do not have common points. It easily follows from this that the canonical line bundle over the curve θ coincides with the line bundle $(s+1)\theta|_\theta$, where $\theta|_\theta$ is the line bundle over the curve induced by the imbedding $\theta \subset F$ of the line bundle on F determined by the divisor θ. Consequently, $(s+1)\theta|_\theta \sim 0$, where we denote by 0 the trivial line bundle on θ.

By the Riemann-Roch theorem

$$\dim H^0(F, \Omega(D-(2s+2)\theta)) + \dim H^0(F, \Omega((2s+2)\theta)) = \frac{((D-(2s+2)\theta)(2s+2)\theta)}{2} + 1 + \dim H'(F, \Omega(D-(2s+2)\theta)).$$

Since $\kappa = 0$, we have $\dim H^0(F, \Omega(D-(2s+2)\theta)) = 0$, $\dim H^0(F, \Omega((2s+2)\theta)) = 1$. Since the index of intersection on the right-hand side is equal to zero, $\dim H^1(F, \Omega(D-(2s+2)\theta)) = 0$. We now write the exact cohomology sequence associated with the exact sequence of sheaves:

$$0 \to \Omega(D-(2s+2)\theta) \to \Omega(D-(2s+1)\theta) \to \Omega_\theta(D-(2s+1)\theta|_\theta) \to 0;$$

$$0 \to H^0(F, \Omega(D-(2s+2)\theta)) \to H^0(F, \Omega(D-(2s+1)\theta)) \to H^0(\theta, \Omega_\theta(D-(2s+1)\theta|_\theta)) \to 0.$$

Since $D-(2s+1)\theta|_\theta \sim \widetilde{D} - (s+1)\theta|_\theta \sim -(s+1)\theta|_\theta \sim 0$, we have $\dim H^0(\theta, \Omega_\theta(D-(2s+1)\theta|_\theta)) = 1$, and consequently $\dim H^0(F, \Omega(D-(2s+1)\theta)) > 0$. We obtain a contradiction with the assumption that $\kappa = 0$. Theorem 1 is proved.

§3. Surfaces with $\kappa = q = 0$, $p_g = 1$

We indicate in this section the projective models of algebraic surfaces with $\kappa = q = 0$, $p_g = 1$. We begin with three essential lemmas.

Lemma 1. $K(F) \sim 0$.

Proof. By the Riemann-Roch theorem

$$\dim H^0(F, \Omega(-K)) + \dim H^0(F, \Omega(2K)) \geqslant p_a(F) = 2.$$

Since $\kappa = 0$, $\dim H^0(F, \Omega(2K)) \leq 1$, and thus $\dim H^0(F, \Omega(-K)) \geq 1$, from which our assertion follows.

Corollary. *If D is an irreducible divisor on a surface F, then $H^1(F, \Omega(D)) = 0$.*

This assertion follows directly from a theorem of Kodaira [25], Theorem 2.5.

Remark. It follows from the proof of the lemma that the surfaces of the type under consideration are defined by the following values of the invariants: $q = 0$,

$p_g = P_2 = 1$.

Lemma 2. *If D is an irreducible divisor on a surface F and $(D^2) > 0$, then a generic curve of the complete linear system $|D|$ is nonsingular.*

Proof. By Bertini's theorem, it is sufficient to show that the system $|D|$ does not have base points. For this, in turn, it is sufficient to prove that the system $|D|$ cuts out on the curve D a complete linear system without fixed points.

According to Kodaira ([25], § 1), we have the exact sequence

$$0 \to H^0(\tilde{D}, \Omega(D|_{\tilde{D}} - \mathfrak{c})) \to H^0(D, \Omega_D(D|_D)) \to H^0(M) \to H^1(\tilde{D}, \Omega(D|_{\tilde{D}} - \mathfrak{c})) \to H^1(D, \Omega_D(D|_D)) \to 0, \quad (1)$$

where $\tilde{D}$ denotes a nonsingular model of the curve D, $\mathfrak{c}$ is a divisor on $\tilde{D}$ depending on the singularities of the curve D, $D|_{\tilde{D}}$ denotes the line bundle over the curve $\tilde{D}$ which is induced from the line bundle corresponding to the divisor D by means of the natural mapping $\tilde{D} \to F$, and M is a sheaf concentrated at the preimages of the singular points of the curve D. As is known, the canonical class $K(\tilde{D})$ is given by the formula $K(\tilde{D}) \sim K(F)|_{\tilde{D}} + D|_{\tilde{D}} - \mathfrak{c}$, and since $K(F) \sim 0$, $K(\tilde{D}) \sim D|_{\tilde{D}} - \mathfrak{c}$. We will now calculate the dimension of the group $H^1(D, \Omega_D(D|_D))$. We consider for this the exact cohomology sequence which corresponds to the sequence of sheaves

$$0 \to \Omega(0) \to \Omega(D) \to \Omega_D(D|_D) \to 0; \quad (2)$$

$$H^1(F, \Omega(D)) \to H^1(D, \Omega_D(D|_D)) \to H^2(F, \Omega(0)) \to H^2(F, \Omega(D)).$$

corollary to lemma 1

By ~~Theorem~~ 1 and the duality theorem, the groups at the ends of this exact sequence are trivial, and $\dim H^2(F, \Omega(0)) = \dim H^0(F, \Omega(K)) = 1$. Thus, $\dim H^1(D, \Omega_D(D|_D)) = 1$. Now the exact sequence (1) takes the form

$$0 \to H^0(\tilde{D}, \Omega(D|_{\tilde{D}} - \mathfrak{c})) \xrightarrow{i} H^0(D, \Omega_D(D|_D)) \xrightarrow{j} H^0(M) \to 0.$$

Since the mapping i is a monomorphism, the sections of $H^0(D, \Omega_D(D|_D))$ do not have common zeros outside the points of the divisor $\mathfrak{c}$, and since the mapping j is an epimorphism, they also do not have common zeros at points of the divisor $\mathfrak{c}$.

From the exact cohomology sequence

$$H^0(F, \Omega(D)) \xrightarrow{r} H^0(D, \Omega_D(D|_D)) \to H^1(F, \Omega(0)) \approx 0,$$

corresponding to the exact sequence of sheaves (2), we conclude that the restriction mapping r is an epimorphism, and thus the sections of the group $H^0(F, \Omega(D))$ do not have common zeros on D. Thus the system $|D|$ does not have base points. The lemma is proved.

Lemma 3. *If a generic curve of an irreducible linear system $|D|$, $(D^2) > 0$, is not hyperelliptic, then this system yields a birational mapping of the surface F into a projective space, while if this curve is hyperelliptic, but has a genus greater then two, then $|2D|$ yields such a mapping.*

Proof. By Lemma 2, a generic curve of the system $|D|$ is nonsingular. Depending on the genus and hyperellipticness of this curve, it can be imbedded in the projective space of the system $|K(D)| = |D|_D|$ or $|2K(D)| = |2D|_D|$. Let $|sK(D)|$ yield a birational imbedding of the curve D into a projective space. We will show that then the system $|sD|$ yields a birational mapping of the surface F. In fact, from the exact sequence

$$H^0(F, \Omega(sD)) \xrightarrow{r} H^0(D, \Omega_D(sD/D)) \to H^1(F, \Omega((s-1)D))$$

(corresponding to the exact sequence of sheaves

$$0 \to \Omega((s-1)D) \to \Omega(sD) \to \Omega_D(sD|_D) \to 0)$$

we conclude by the corollary of Lemma 1 and the regularity of F that the mapping r is an epimorphism. Therefore the restriction $|sD|_D|$ of the system $|sD|$ on the curve D coincides with $|sK(D)|$. On the other hand, if x_0 is any point of some curve $D \in |D|$, and x_1 is a point of the surface F joined with x_0 under the mapping corresponding to the system $|sD|$, then the point x_1 also lies on the curve D. Therefore the mapping $|sD|$ is one-to-one at almost all the points of almost all the curves of the system $|D|$, i.e. this mapping is birational. The lemma is proved.

We denote by π the minimal possible value of the arithmetic genus of the irreducible curves D on the surface F that satisfy the condition $(D^2) > 0$. We have

$$\pi = \frac{(D^2)}{2} + 1.$$

Moreover, we conclude from the Riemann-Roch theorem and the corollary of Theorem 1 that $\dim |D| = \pi$.

Theorem 2. *If $\pi = 2$, the irreducible system $|D|$ maps the surface F onto the projective space $\mathbf{P}^2$ with the nonhomogeneous coordinates x, y. Since $(D^2) = 2$, the field of functions $k(F)$ on the surface F is an extension of second degree of the field of rational functions $k(x, y)$ and the equation of the surface F has the form $z^2 = F^n(x, y)$. Since every divisor $\{ax + by + c = 0\}$ belongs to the system $|D|$, it has a genus of two, and thus $n = 6$.*

If $\pi > 2$, and a generic curve of the system $|D|$ is not hyperelliptic, this system maps the surface F into a space $\mathbf{P}^\pi$ where the degree of the image is equal to $(D^2) = 2\pi - 2$, and the mapping is birational.

If $\pi > 2$ and a generic curve of the system $|D|$ is hyperelliptic, then the *system* $|2D|$ *birationally maps the surface* F *into the space* $\mathbf{P}^{4\pi-3}$, *and the image has a degree of* $8\pi - 8$. *The system* $|D|$ *in this case yields a two-sheeted mapping of our surface onto a rational surface of degree* $\pi - 1$ *in* $\mathbf{P}^{\pi}$.*

We note that, as follows from results of G.N. Tjurina (Chapter IX), the mappings described are defined for a generic surface of the considered type with uniqueness up to a projective equivalence.

§4. Surfaces with $\kappa = 0$, $p_g = 1$, $q = 2$

It will be proved in this section that every Kähler surface with $\kappa = 0$, $p_g = 1$, $q = 2$ without exceptional curves of the first kind is biregularly equivalent to an algebraic torus. The proof reduces to the study of the canonical class of a surface and of the surface's Albanese mapping.

Theorem 3. *Let* F *be an algebraic surface without exceptional curves and with* $\kappa = 0$, $p_g = 1$, $q = 2$. *Then* F *is an abelian variety.*

The proof consists in the analysis of the three a priori possible cases: 1) the Albanese mapping α of the surface F is a mapping onto its Albanese variety $A(F) = T^2$, and the canonical class $K(F)$ is equivalent to zero; 2) $\alpha(F) = A(F)$ and $K \not\sim 0$; 3) $\alpha(F)$ is a curve in $A(F)$.

1) We consider the first case. We denote by ω_1 and ω_2 two linearly independent holomorphic differentials of first degree on the surface F. Since $\alpha(F) = T^2$, $\omega_1 \wedge \omega_2 \neq 0$. Since $K \sim 0$, the differential of second degree $\omega_1 \wedge \omega_2$ does not generally have zeros, i.e. the mapping α is locally biholomorphic. This means that the surface F is an unramified covering over its Albanese variety, the torus T^2, i.e. is itself a two-dimensional abelian variety (and the mapping α is one-to-one).

The two remaining cases lead to a contradiction.

2) We recall that, by Lemma 2 of §1 and Lemma 1 of §2, $(K^2) = 0$ and K (an effective divisor, since $p_g = 1$) consists of rational and elliptic components.

We will show that the canonical class does not contain irreducible components whose geometric genus is zero. Thus, let H be such a component. Then $\omega_1|_H = \omega_2|_H = 0$, and thus the curve H is contracted under the mapping α. If we denote by $\theta_1, \cdots, \theta_k$ the irreducible components of the divisor K that have a geometric genus of 1, and by $s_1, \cdots, s_k$ the multiplicities of these components, then the effective divisor $K - s_1\theta_1 - \cdots - s_k\theta_k$ will consist of components of geometric

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* *Added in proof*: Such surfaces exist for $\pi = 3$; they will be studied in my article: B. G. Averbuh, *On special types of Kummer and Enriques surfaces*, Izv. Akad. Nauk SSSR 29 (1965), 1095–1118. For $\pi > 3$ apparently nothing is known. *Translator's note*: This article has been included as the appendix to the present translation.

genus zero, will be contracted into a certain number of points under the mapping α, and will have a zero index of self-intersection. This, however, contradicts the theorem of Mumford [35] about the negative definiteness of the matrix of the indices of intersection arising under the resolution of a normal singular point.

Now let θ be a nonsingular elliptic component of K. Since the curve θ is elliptic, there exists a nonzero holomorphic differential ω of first degree on the surface F whose restriction to the curve θ is equal to zero. This means that the image $\alpha(\theta)$, nontrivial by the same theorem of Mumford, lies completely in some coset of the group T^2 over the subgroup that is the elliptic curve T^1. Moreover, the differential ω induces a holomorphic mapping $\phi: F \longrightarrow T^1$.

We consider on the surface F the algebraic system $\{L\}$ of the fibers of the mapping ϕ. Clearly, $(L^2) = (L \cdot \theta) = 0$. If H is an arbitrary irreducible component of a curve of the system $\{L\}$ containing θ, then $(H \cdot \theta) = 0$, since $(L \cdot \theta) = (\theta^2) = 0$. The curve θ is thus a connected component of its fiber under the mapping ϕ.

Considering now a normalization of the curve T^1 in the field $k(F)$, we find a nonsingular algebraic curve Γ and a regular mapping $F \longrightarrow \Gamma$ such that the curve θ is a topological, and the divisor $s\theta$, an algebraic fiber over some point of Γ (we denote by $s - 1$ the order of the zero of the differential ω on the curve θ). Since the genus of the curve Γ does not exceed one, the divisor $2s\theta$ on the surface F must change in the linear system. Since $2s\theta \leq 4K$, this contradicts the assumption that $\kappa = 0$.

3) If C is a point of $\alpha(F)$, we denote by Γ_C an (algebraic) "fiber" over this point ($(\Gamma_C^2) = 0$). Let θ be a rational curve on the surface F. Since $\omega_1|_\theta = \omega_2|_\theta = 0$, the curve θ lies in some fiber of the mapping α.

If θ is a component of an effective divisor of K that does not lie in any fiber, then we have $(\theta^2) = 0$, $(\theta \cdot \Gamma_C) > 0$. Since $((\theta + \Gamma_C)^2) > 0$, for all except possibly a finite number of irreducible curves D on the surface F, we have $(\theta + \Gamma_C) \cdot D > 0$. If in particular D is a component of some fiber, then $(\theta \cdot D) > 0$, and the curves θ and D have a common point. Since $p_g(\theta) = 1$, there exists a nonzero holomorphic differential ω of first degree on the surface F that is equal to zero on θ. It can easily be noted, however, that integration of this differential along a path lying completely in fibers of the mapping α and on the curve θ gives zero. Since, on the other hand, such a path must connect, by what has been proved, almost every pair of points of the surface F, we arrive at a contradiction, from which we conclude that all the components of an effective divisor of K lie in fibers of the mapping α and $(K \cdot \Gamma_C) = 0$.

By the Riemann-Roch theorem and the theorem of Kodaira mentioned in §2, we can conclude for a generic fiber Γ_{C_0} that

$$\dim H^0(F, \Omega(K + \Gamma_{C_0})) = \dim H^1(F, \Omega(K + \Gamma_{C_0})) = m - 1 + 2 \geqslant 2,$$

where we denote by m the number of connected components of the fiber Γ_{C_0}. If we denote by θ the fixed and by $|D_\nu|$ the nonfixed part of the system $|\Gamma_{C_0} + K|$, then it is easy to see that $(\theta \cdot \Gamma_C) = (D_\nu \cdot \Gamma_C) = (D_\nu \cdot K) = 0$, and thus $(D_\nu^2) = 0$. The system $|D_\nu|$, consequently, is composed of curves of an irreducible pencil L. Since $(L \cdot \Gamma) = 0$, the curves of the pencil L lie in fibers, hence $\dim H^1(F, \Omega(K + L_0)) = 2$ (the same theorem of Kodaira). Since $(L \cdot K) = 0$ we thus obtain by the Riemann-Roch theorem

$$\dim H^0(F, \Omega(K + L_0)) = \dim H^1(F, \Omega(K + L_0)) = 2,$$

where L_0 denotes an irreducible curve of the pencil L. We denote by θ_1 and $|\overline{D}_\nu|$ the fixed and nonfixed parts of the system $|K + L_0|$; we find that $(\theta_1 \cdot \Gamma_C) = (\overline{D}_\nu \cdot \Gamma_C) = 0$, that $(\overline{D}_\nu^2) = 0$, and that all the curves of the system $|\overline{D}_\nu|$ lie in fibers. Hence $(\overline{D}_\nu \cdot L) = 0$, and thus also $(L \cdot \overline{L}) = 0$, where $\overline{L}$ denotes the irreducible pencil of curves of which the system $|\overline{D}_\nu|$ is composed. The equation $(L \cdot \overline{L}) = 0$ implies that the pencils L and $\overline{L}$ coincide. We now have $L_0 + K = L_1 + \cdots + L_s + \theta_1$, where $L_1, \cdots, L_s$ are the curves of the pencil L.

If L_0 is a component of the divisor θ_1, then

$$K = L_1 + \ldots + L_s + (\theta_1 - L_0).$$

Since the genus of the curve Γ parametrizing the pencil L does not exceed two, it follows from this that the divisor of $3K$ must change in the linear system and this contradicts the assumption $\kappa = 0$.

If L_0 is not a component of the divisor θ_1, then one of the divisors $L_1, \cdots, L_s$ coincides with L_0, and thus $K - \theta_1 \geq 0$. The case $K - \theta_1 > 0$ is analogous to the one already considered, and we can therefore assume that $K = \theta_1$. It follows from this that $|\overline{D}_\nu| = |L_0|$ and L is a linear pencil on the surface F, all of whose curves lie in fibers. But in this case the mapping α induces a nontrivial mapping of the line P^1 parametrizing the pencil L into the torus T^2, which is also impossible. Theorem 3 is thus completely proved.

Remark. We note that the condition $\kappa = 0$ is not used by us to the full extent of its meaning; namely, for the validity of this proof it is sufficient to assume that $P_4 = p_g = 1$.

Theorem 4. *A Kähler variety F with $\kappa = 0$, $q = 2$ and without exceptional curves of the first kind is biregularly equivalent to a complex torus.*

Proof. We denote by $t(F)$ the degree of transcendence of the field $k(F)$ of meromorphic functions on the surface F. If $t(F) = 2$, our theorem reduces to Theorem 3, since the surface F is algebraic in this case. If $t(F) = 0$, our theorem is proved by Kodaira ([25], Theorem 5.3).

We consider the case when $t(F) = 1$. It is known (cf. [25], §4) that in this case there exists a unique regular mapping Φ of the surface F onto a nonsingular curve Δ inducing an isomorphism of the fields of functions $k(F)$ and $k(\Delta)$. Only a finite number of the fibers of this mapping are reducible and a generic fiber is a nonsingular elliptic curve; every irreducible divisor on the surface F is a component of one of the fibers.

According to Lemma 1, §1, and its corollary, $p_g(F) \geq 0$, $(K^2) = 0$. Let K_0 be a connected component of the canonical class. Since $(K_0^2) = 0$, it follows from Theorem 2 of Chapter VII, §2, that $K_0 = r\Gamma_{C_0}$, where Γ_{C_0} is some fiber of the mapping Φ, and $r \geq 0$ is a rational number. If $r > 0$, there clearly exists an n such that the system $|nK|$ has a positive dimension (it is sufficient to choose n large enough so that the system $|nrC_0|$ on the curve Δ has a positive dimension). Since $\kappa = 0$, it follows from this that $K = 0$.

We now study the Albanese mapping α of the surface F. We consider the cases considered in the proof of Theorem 3.

The first case, if it occurs, leads to the desired result.

The second case is impossible, as was proved.

If the third case occurs, both holomorphic differentials of first degree existing on the surface F vanish on fibers of the mapping Φ (since these fibers must be components of the fibers of the mapping α). But this implies that $\dim H^1(F, \Omega(\Gamma_C)) = 2$ ([25], Theorem 2.3) and, by the Riemann-Roch theorem, that $\dim |\Gamma_C| = 1$. This, however, contradicts Theorem 2.5 of the same work of Kodaira, which states that $\dim H^1(F, \Omega(K + \Gamma_C)) = 0$ if the curve Γ_C is irreducible and $\dim |\Gamma_C| \geq 1$. This completes the proof of Theorem 4.

CHAPTER IX

THE SPACE OF MODULI OF A COMPLEX SURFACE WITH $q = 0$ AND $K = 0$

In this chapter we study the local structure of the space of moduli of a compact complex Kähler surface for which $q = 0$ and $K = 0$. Let us consider some examples of such surfaces.

1. Algebraic surfaces. In §3, Chapter VIII it was proved that an algebraic surface for which $q = 0$ and for which all the plurigenera are equal to one is birationally equivalent to a surface of the type we are considering. Examples of surfaces with $q = 0$ and $K = 0$ are a nonsingular surface of fourth degree in three-dimensional projective space $\mathbf{P}^3(\mathbf{C})$, the intersection of three quadrics in $\mathbf{P}^5(\mathbf{C})$, etc. In the book of Enriques [59] it is stated that for any $\pi \geq 2$ one can find a surface with $q = 0$ and $K = 0$ on which there exists a curve of genus π and on which there do not exist curves of a smaller genus. (This statement will be proved in §5 from other considerations.)

2. Surfaces without meromorphic functions. Kodaira [25] proved that if on a surface V with $q = 0$ there do not exist meromorphic functions other than the constant ones, and moreover, if the surface V does not contain exceptional curves of the first kind, then the canonical bundle of the surface V is trivial ($K = 0$). It will be proved in §4 that such surfaces actually exist, and moreover that any surface with $q = 0$, $K = 0$, can be deformed into a surface without meromorphic functions.

3. Kummer surfaces. The group of second order whose generator g takes a point $t \in T^2$ into the point $-t$ operates on the two-dimensional complex torus T^2. This group has 16 fixed points. Let us identify the points t and $g(t)$. We obtain a variety with 16 singular points, which can be resolved with the use of a σ-process; the nonsingular surface obtained is called a Kummer surface $K(T^2)$. The cohomology groups of the surface are known. In particular, $q(K(T^2)) = 0$. It is possible to show that for the surface $K(T^2)$, as for the torus, the canonical bundle is trivial.

It will be proved at the end of this chapter that all compact Kähler surfaces with $q = 0$ and $K = 0$ are diffeomorphic.

This chapter begins with a general theorem about the subset of algebraic structures in the family of complex structures of compact Kähler varieties. It is proved that the points of the base of the family of complex structures that correspond to algebraic varieties can be distinguished by means of complex analytic conditions.

It is proved in §2 that a complex structure on a variety of complex dimension n with a trivial canonical bundle is in some sense uniquely determined by the cohomology class of an n-dimensional holomorphic form in this structure. More precisely, the space of moduli of such a variety can be locally realized as a subvariety in a projective space $\mathbf{P}^N(\mathbf{C})$ whose homogeneous coordinates are integrals of the holomorphic forms k_t of the variety V_t over the fixed basis of the n-dimensional cohomologies.

In the case of a surface with $q = 0$ and $K = 0$, the space of moduli can be locally realized as a 20-dimensional quadric in $\mathbf{P}^{21}(\mathbf{C})$ whose matrix coincides with the matrix of the intersection of the surfaces. In the last two sections, several theorems about algebraic surfaces of the type under consideration are proved with the use of this realization. In conclusion it is proved that all compact Kähler surfaces with $q = 0$ and $K = 0$ are c-homotopic, and hence mutually diffeomorphic.

§1. Deformations of complex structures of algebraic varieties

A complex analytic family of complex structures is a triple $(\mathcal{V}, M, \pi)$, where π is a holomorphic mapping of the complex variety $\mathcal{V}$ onto the complex variety M which is a differentiably locally trivial fibering [26]. In particular, it follows from the definition that the preimage V_t under the mapping π of each point $t \in M$ is always diffeomorphic to the same variety X.

In this section we study the set $A \subset M$ of those points $t \in M$ such that $V_t = \pi^{-1}(t)$ is an algebraic variety.

1. Since the family $(\mathcal{V}, M, \pi)$ is a differentiable locally trivial fibering, the restriction $\pi^{-1}(U)$ of the fibering of $\mathcal{V}$ over a contractible neighborhood $U \subset M$ is diffeomorphic to the direct product $X \times U$, and for any $t \in U$ the homomorphisms of the cohomology groups

$$i_t^*: H^i(\pi^{-1}(U), \mathbf{Z}) \longrightarrow H^i(V_t, \mathbf{Z}),$$

induced by the imbedding $i_t: V_t \longrightarrow \pi^{-1}(U)$ are isomorphisms.

By a theorem of Kodaira [24] a compact complex variety V is algebraic if and only if there exists on it a Hodge metric, i.e. the Kähler metric $ds^2 = 2\Sigma g_{\alpha\bar\beta}(dz^\alpha d\bar z^\beta)$ such that the differential form $\omega = i\Sigma g_{\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta$ associated with it belongs to an integral cohomology class. Let $t \in A$, and let $c_0 \in H^2(V_{t0}, \mathbf{Z})$ be the cohomology class containing the differential form ω_0 of type (1, 1) associated with the Hodge metric on the variety $V_0\ \pi^{-1}(t_0)$. We shall explain for which $t \in M$ the harmonic form ω_t on the variety V_t belonging to the class

$$c_t = i_t^* i_0^{*-1} c_0,$$

is associated with some Hodge metric. Clearly, it is necessary for this that the form ω_t be a form of type (1, 1), i.e.,

$$\Pi_t^{0,2}\omega_t = 0$$

where $\Pi^{p,q}\phi$, as always, denotes the component of type (p, q) of the form ϕ.

If t belongs to a sufficiently small neighborhood of the point t_0, then the condition

$$\Pi_t^{0,2}\omega_t = 0$$

is also sufficient. In fact, since the form ω_t is real, we have

$$\Pi_t^{2,0}\omega_t = \overline{\Pi_t^{0,2}\omega_t} = 0;$$

and hence the form ω_t has type (1, 1). We shall show that for all $t \in M$ which are sufficiently close to the point t_0, the form ω_t is associated with some Hodge metric. By [27], Theorem 15, for all t of a sufficiently small neighborhood of the point t_0 it is possible to choose a Kähler metric on the variety V_t depending differentiably on t and coinciding at $t = t_0$ with the given Hodge metric ds_0^2 on the variety V_0. On V_0 with this metric the form ω_0 associated with the metric ds_0^2 is harmonic and belongs to the class c_0.

Let W be a differential form on the variety $\pi^{-1}(U)$ belonging to the class $i_0^{*-1}c_0$. Then, by the definition of the form ω_t, we have

$$\omega_t = H_t i_t^* W,$$

where i_t^* is a homomorphism of the differential forms that is induced by the imbedding i_t. Since on the family of Kähler varieties the operators H_t depend differentiably on the parameter t [27], the forms ω_t are continuous in t. Moreover,

$$\omega_{t_0} = \omega_0$$

for the family of Kähler metrics chosen by us. Since the form ω_0 is associated with the metric ds_0^2, in view of the continuity of the family of forms the form ω_t is also associated with a positive definite metric ds_t^2 for all t lying in some neighborhood of the point t_0. Since the form ω_t belongs to the integral cohomology class c_t, the metric ds_t^2 is a Hodge metric.

2. Let $k_1(t), \cdots, k_l(t)$ be a basis of the space of holomorphic two-dimensional forms on the variety V_t. Then

$$\Pi_t^{0,2}\omega_t = \sum_{i=1}^{l} F_i(t)\,\overline{k_i(t)},$$

where the $F_i(t)$ are functions on the variety M. We will prove the following proposition.

Proposition 1.1. *The functions $F_i(t)$ are holomorphic in the variable t for an appropriate choice of the set of bases $k_i(t)$.*

For ease in calculation, we will assume in the future that the base M of the family $(\mho, M, \pi)$ is contractible. Moreover, the variety V_t will be assumed to be a Kähler variety. The following lemma will be used in the proof of Proposition 1.1.

Lemma. *Let β_0 be a harmonic form on the variety V_0. Then there exists on the variety $\mho$ a closed form β such that the form*

$$\beta_t = i_t^*\beta$$

is harmonic on the variety V_t for any $t \in M$, and moreover $\beta_0 = i_0^*\beta$.

Proof of the lemma. Let the form β_0 belong to a cohomology class of $b_0 \in H^*(V_0, C)$. We consider the closed form $\tilde{\beta}$ on the space $\mho$ belonging to the class

$$b = i_0^{*-1} b_0.$$

It is clear that

$$\beta_t = H_t i_t^* \tilde{\beta},$$

where H_t is a harmonic operator on the variety V_t. On the other hand,

$$H_t i_t^* \tilde{\beta} = i_t^* \tilde{\beta} - d_t \eta(t),$$

where

$$\eta(t) = \delta_t G_t i_t^* \beta.$$

As is known [27], on the family of Kähler varieties the operators H_t and G_t depend differentiably on t. Consequently, $\eta(t)$ is a family of forms on the varieties V_t which are differentiable with respect to t. Since $\mho$ is diffeomorphic to the product $X \times M$, there exists a differential form N on the variety $\mho$ such that $\eta(t) = i_t^* N$. Thus

$$\beta_t = i_t^* \tilde{\beta} - d_t i_t^* N = i_t^* (\tilde{\beta} - dN)$$

and thus we can set

$$\beta = \tilde{\beta} - dN.$$

The lemma is proved.

3. Proof of Proposition 1.1. Let $k_1(0), \cdots, k_l(0)$ be some basis of the

space of two-dimensional holomorphic forms on the variety V_0. We construct for the forms $\omega_0, \overline{k_1(0)}, \ldots, \overline{k_l(0)}$ closed forms $\omega, K_1, \ldots, K_l$ on the variety $\mathcal{V}$ satisfying the conditions of the lemma. It is then clear that

$$\omega_t = i_t^* \omega.$$

Moreover, for t sufficiently close to t_0 the forms

$$\overline{k_i(t)} = \Pi_t^{0,2} i_t^* K_i$$

are a basis of the space of two-dimensional antiholomorphic forms on the variety V_t because of their continuity with respect to t. Hence

$$\Pi_t^{0,2} \omega_t = \sum_{i=1}^{l} F_i(t) \overline{k_i(t)}.$$

We will show that the functions $F_i(t)$ are holomorphic with respect to t.

We consider the form

$$\sum_q \frac{\partial F_q(t)}{\partial \bar{t}_k} \overline{k_q}(t)$$

on the variety V_t. It is clear that this is an antiholomorphic form which is equal to zero if and only if

$$\frac{\partial F_q(t)}{\partial \bar{t}_k} = 0$$

for any q and any k. We will show that this form is d_t''-homologous to zero on the variety V_t for any t and any k. It will follow from this that all the functions $F_q(t)$ are holomorphic.

We consider the covering of the space $\mathcal{V}$ by coordinate neighborhoods W_i with coordinates $(z_1^i, \ldots, z_n^i; t)$, where

$$\begin{aligned} z_1^i &= f_{1ij}(z_1^j, \ldots, z_n^j, t) \\ &\cdots\cdots\cdots\cdots \\ z_n^i &= f_{nij}(z_1^j, \ldots, z_n^j, t) \\ t_k^i &= t_k^j. \end{aligned}$$

This can be done if one takes as a base M a sufficiently small neighborhood of the point t_0. In a neighborhood W_i in these coordinates the forms $\Pi^{0,2}\omega$ and $\Pi^{0,2}K_q$ have the form

$$(\Pi^{0,2}\omega)_i = \sum_q F_q(t) \overline{k_q(t)} + \sum_k P_i^k \wedge d\bar{t}_k + \ldots,$$
$$(\Pi^{0,2}K_q)_i = \overline{k}_q + \sum_k G_{iq}^k \wedge d\bar{t}_k + \ldots,$$

where P_i^k and $G_{i,q}^k$ are differential forms of the form $\Sigma a_i(z, t) d\bar{z}_i$ and are defined over the neighborhood W_i, but not over the whole variety. In this and all the following examples dots replace terms of the form

$$A(z, t)\, d\bar{t}_i \wedge dt_j,$$

where $A(z, t)$ is a function or a form.

For the rest of this proof we will denote by d and d'' differentials on the variety $\mathcal{V}$, and by d_t and d''_t differentials on the variety V_t. Since

$$d\omega = 0, \; dK_i = 0$$

for all i, we have, in particular,

$$d'' \Pi^{0,2} \omega = 0, \quad d'' \Pi^{0,2} K_i = 0.$$

We calculate these differentials in an obvious manner. Since the forms $\bar{k}_q(t)$ are harmonic, and thus $d''_t k_q(t) = 0$, we have

$$(d''\Pi^{0,2} K_q)_i = \sum_k \left(\frac{\partial \bar{k}_q}{\partial \bar{t}_k} + d''_t G^k_{i,q} \right) \wedge d\bar{t}_k + \dots$$

and analogously,

$$(d'' \Pi^{0,2} \omega)_i = \sum_k \left[\frac{\partial}{\partial \bar{t}_k} \left(\sum_q F_q(t)\, \bar{k}_q(t) \right) + d''_t P^k_i \right] \wedge d\bar{t}_k + \dots .$$

From this we obtain the system of equations

$$\left(\frac{\partial \bar{k}_q}{\partial \bar{t}_k} \right)_i = - d''_t G^k_{iq},$$

$$\left(\frac{\partial}{\partial \bar{t}_k} \right) \left(\sum_q F_q(t)\, \overline{k_q(t)} \right)_i = - d''_t P^k_i.$$

After differentiating the left-hand side of the last equation, we obtain

$$\frac{\partial}{\partial \bar{t}_k} \left(\sum_q F_q(t)\, \bar{k}_q(t) \right) = \sum_q \frac{\partial F_q(t)}{\partial \bar{t}_k} \cdot \overline{k_q(t)} + \sum_q F_q(t) \frac{\partial \bar{k}_q(t)}{\partial \bar{t}_k},$$

and hence

$$\left(\sum_q \frac{\partial F_q(t)}{\partial \bar{t}_k} \bar{k}_q(t) \right)_i = - d''_k \left(P^k_i - \sum_q F_q(t)\, G^k_{i,q} \right).$$

It remains to prove that for any t the forms

$$Q^k_i = P^k_i - \sum_q F_q(t)\, G^k_{i,q}$$

are the restriction on the neighborhoods $W_i \cap V_t$ of some form on the whole variety V_t. We consider for this the form

$$\Pi^{0,2}(\omega - \sum F_q(t)\, K_q).$$

In the neighborhood W_i this form can be written as

$$\Pi^{0,2}(W - \sum F_q(t)\, K_q)_i = \sum Q_i^k \wedge d\bar{t}_k + \ldots .$$

Let

$$Q_i^k = \sum \alpha_p^k d\bar{z}_p^i.$$

Then in the intersection $W_i \cap W_j$ we have

$$\left(\Pi_{0,2}\left(\omega - \sum F_q(t)\, K_q\right)\right)_j = \sum \alpha_p^k \left(\sum \frac{\partial \bar{z}_p^i}{\partial \bar{z}_l^j}\, d\bar{z}_l^j + \sum \frac{\partial \bar{z}_p^i}{\partial \bar{t}_l}\, d\bar{t}_l\right) \wedge d\bar{t}_k + \ldots$$

and hence

$$Q_j^k = \sum \alpha_p^k \sum \frac{\partial \bar{z}_p^i}{\partial \bar{z}_l^j}\, d\bar{z}_l^j = Q_i^k$$

in the intersection $W_i \cap W_j \cap V_t$. Proposition 1.1 is thus proved.

4. Let $c \in H^2(\mho, Z)$. We consider the set B_c of those points $t \in M$ such that the harmonic form ω_t on the variety V_t corresponding to the class $i_t^* c$ has type (1,1). Since the variety V_t is assumed to be a Kähler variety, and on Kähler varieties the operators H_t take forms of type (1,1) into forms of the same type, the set B_c is defined independently of the choice of the family of Kähler metrics on $(\mho, M, \pi)$. Let

$$\Pi^{0,2}\omega_t = \sum F_q(t)\, \overline{k_q(t)},$$

where a basis of the antiholomorphic forms is chosen as in the previous subsection. By Proposition 1, the set B_c is the complex subvariety of the base M which is given by the system of holomorphic equations

$$F_i(t) = 0.$$

We consider the set

$$B = \bigcup B_c, \quad c \in H^2(X, Z).$$

This set is the union of a countable set of complex varieties. It is easy to see that $A \subset B$. In fact, if $t \in A$ there exists on the variety V_t a form ω_t, associated with a Hodge metric, which is harmonic with respect to this metric, has type (1, 1) and belongs to the integral cohomology class of c_t. Hence the point t belongs to the set B_c, where $c = i_t^{*-1} c_t$. Conversely, as was proved in subsection 1, if V_t is an algebraic variety, then for some neighborhood U of t the subvariety B_c belongs to the set A. The following theorem summarizes all we have learned in this chapter about the structure of the set A.

Theorem 1. *Let $(\mathcal{V}, M, \pi)$ be a complex analytic family of complex structures, let $V_{t_0} = \pi^{-1}(t_0)$ be an algebraic variety, and let $h^{2,0}$ be the dimension of the space of two-dimensional holomorphic forms on V. Then in some neighborhood U of the point t_0 there exists an analytic subvariety $B_c \subset U$ with co-dimension not greater than $h^{2,0}$, such that for all $t \in B_c$ the variety V_t is algebraic (i.e. $B_c \subset A$). Moreover, the set $A \subset M$ of all points $t \in M$ such that V_t is an algebraic variety is contained in the union of not more than a countable number of such subvarieties.*

Example 1. We consider the complete, effectively parametrized family $(\mathcal{V}, M, \pi)$ (to be defined in §2.2) of complex tori T^n, where the base M is a set of complex matrices of nth order with a nondegenerate imaginary part, and the preimage of the matrix $Z \in M$ under the mapping is a complex torus with a matrix of periods (E, Z). For an n-dimensional torus the dimension of the space of two-dimensional holomorphic forms is equal to $n(n-1)/2$. Hence by Theorem 1, in order for an n-dimensional torus to be an abelian variety, it is necessary to impose on its periods $n(n-1)/2$ complex conditions. This assertion follows from the Riemann-Frobenius theorem on the relationship between the periods of abelian varieties.

Example 2. Let V be a Kummer surface. As will be shown in §4, for this surface the number of moduli exists and is equal to 20. Let M be the base of a complete, effectively parametrized family of deformations of a complex structure on a surface V. In the survey of Grauert on the number of moduli given at the International Colloquim on the theory of functions in Bombay (1960), the following problem was formulated: which of the points $t \in M$ correspond to algebraic varieties? It was proposed that these points form an analytic surface of dimension 19. It is not difficult to show that for a Kummer surface $h^{2,0} = 1$. It follows from Theorem 1 that either the set A coincides with M or it is the union of not more than a countable number of complex analytic subvarieites of dimension 19. It will be proved in §4 that A is the union of a countable number of 19-dimensional subvarieties, and moreover that the set A is everywhere dense in the base M.

In the two examples given the conditions $\{F_i = 0\}$ turned out to be independent if $(\mathcal{V}, M, \pi)$ is a complete family. This is not always true, however. For example, if V is an algebraic surface for which $K^2 > 0$, then any deformation of the surface V will also be an algebraic surface, although for a surface of that type we have

$$h^{2,0} = p > 0.$$

§2. Deformations of complex structures and integrals of holomorphic forms

1. Let $(\mathcal{V}, M, \pi)$ be a family of complex structures whose base M is contractible. As was noted in §1, the variety $\mathcal{V}$ is in this case diffeomorphic to the direct product $V_0 \times M$, where $V_0 = \pi^{-1}(t_0)$ for a fixed point t_0 of M. Thus for any $t \in M$

there exists a canonical diffeomorphism

$$b_t \colon V_t \to V_0,$$

which can be written as

$$b_t = p \cdot D \cdot i_t,$$

where D is some diffeomorphism

$$D \colon \mathcal{O} \to V_0 \times M,$$

and the mapping p is the projection of the direct product $V_0 \times M$ onto the variety V_0.

Let V_0 be a compact Kähler variety of dimension n such that its canonical bundle is trivial. This means that there exists on V_0 an n-dimensional holomorphic form k_0, unique up to proportionality, that does not vanish anywhere. In the local system $(z_1, \cdots, z_n)$ on the variety V_0 the form k_0 can be written as

$$k_0 = f_z dz_1 \wedge \ldots \wedge dz_n.$$

We consider the smooth family of forms

$$k_t = H_t \Pi_t^{n,0} b_t^* k_0$$

on the variety V_t. It is clear that k_t is an n-dimensional holomorphic form on the variety V_t. The form

$$k_{t_0} = k_0$$

vanishes nowhere. Because of the continuity with respect to t of the family of forms k_t, for a sufficiently small neighborhood U of the point t_0 the form k_t for $t \in U$ also vanishes nowhere. Hence the canonical bundle of the variety V_t is trivial, and k_t is an n-dimensional holomorphic form on the variety V_t that is unique up to proportionality.

Let b be the nth Betti number of the variety V_0 and let $c_1, \cdots, c_b$ be a basis of the free part of the cohomology group $H^2(V_0, \mathbf{Z})$. We consider the integrals of the form

$$b_t^{*-1} k_t$$

over the cycles c_i. The numbers

$$\alpha_i(t) = \int_{c_i} b_t^{*-1} k_t \ (i = 1, \ldots, b)$$

may be considered as the projective coordinates of a point in the projective space $\mathbf{P}^{b-1}$. We thus obtain a smooth mapping $F \colon U \to \mathbf{P}^{b-1}$, where U is some neighborhood of the point t_0 in the base M. This mapping can be described in another way. Let $\omega_1, \cdots, \omega_b$ be the basis of the space of the n-dimensional harmonic forms on the variety V_0 that is dual to the basis $c_1, \cdots, c_b$. Since the form k_t is harmonic on the variety V_t, and thus closed, we have the decomposition

$$b_t^{*-1} k_t = \sum_{i=1}^{b} \alpha_i \omega_i + d\eta_t. \qquad (2.1)$$

The numbers α_i coincide with the homogeneous coordinates of the image of the point t under the mapping F. Setting the decomposition (2.1) in the equation

$$\int_{V_t} k_t \wedge k_t = 0,$$

we obtain

$$\sum_{i,j} \int_V (\omega_i \wedge \omega_j)\, \alpha_i \alpha_j = 0.$$

Hence the image $F(U)$ lies on a hypersurface K in the projective space P^{b-1} that is given by the equation

$$zHz' = 0,$$

where $z = (z_1, \cdots, z_b)$ are the homogeneous coordinates of the space P^{b-1} and H is the matrix of intersections of the variety V_0.

Theorem 2. *Let a complex analytic family of complex structures* $(\mathcal{V}, M, \pi)$ *be effectively parametrized, and let* $V_0 = \pi^{-1}(t_0)$ *be an n-dimensional Kähler variety whose canonical bundle is trivial. Then the mapping* F *of some neighborhood* $U \subset M$ *of the point* t_0 *into the projective space* P^{b-1} *is holomorphic and is locally an imbedding.*

This theorem means, in particular, that if the mapping F takes the base of some complex analytic family into a point, then all the varieties V_t of this family are mutually biregularly equivalent. The proof of this theorem is given in subsection 4.

2. Let $(z_1, \cdots, z_n)_i$ be a system of local complex analytic coordinates on the variety V. Let there be given on this smooth variety some almost complex structure. This means that in each space of differentials

$$T_x = \mathrm{Hom}(E_x, \mathrm{C}),$$

where E_x is the tangent space at the point x, there is chosen an n-dimensional subspace L such that

$$T_x = L \oplus \bar{L}.$$

Let $u_1, \cdots, u_n$ be a basis of the space L. Since $dz_1, \cdots, dz_n, d\bar{z}_1, \cdots, d\bar{z}_n$ is a basis of the space T_x, we have the decomposition

$$u_i = \sum_{j=1}^{n} a_{ij}\, dz_j + \sum_{j=1}^{n} b_{ij}\, d\bar{z}_j.$$

Let the new almost complex structure be such that the determinant of the matrix $A = (a_{ij})$ does not vanish at any point of the variety V (condition (*)). Then such a structure is uniquely determined by the matrix

$$(\omega_{ij}) = A^{-1} \cdot (b_{ij}),$$

given at each point of the variety V. It is easy to verify that this matrix determines a differential form

$$\omega = \sum \left(\omega_{ij} \frac{\partial}{\partial z_i}\right) d\bar{z}_j$$

with coefficients in a sheaf Θ of germs of complex analytic vector fields on the variety V. It is known [41] that the condition for the integrability of the given almost complex structure is that

$$d''\omega + \frac{1}{2} [\omega, \omega] = 0. \qquad (2.2)$$

If the given almost complex structure is integrable and corresponds to some complex structure $\widetilde{V}$ with local coordinates $(\zeta_1, \cdots, \zeta_n)$, then the differentials $d\zeta_1, \cdots, d\zeta_n$ are a basis of the space L, where

$$d\zeta_i = \sum \frac{\partial \zeta_i}{\partial z_j} dz_j + \sum \frac{\partial \zeta_i}{\partial \bar{z}_j} d\bar{z}_j.$$

Consequently, the form $\omega = \omega_{\widetilde{V}}$ is, in this case, given by the matrix

$$\omega_{ij} = \left(\frac{\partial \zeta_i}{\partial z_j}\right)^{-1} \left(\frac{\partial \zeta_i}{\partial \bar{z}_j}\right).$$

We note the obvious fact: when the structure $\widetilde{V}$ coincides with the original structure V, then $\omega_{\widetilde{V}} = 0$.

Let $(\mathcal{V}, M, \pi)$ be a family of complex structures on a smooth variety X, and let $V_0 = V$. Then if $t \in U$, where U is some neighborhood of the point $t_0 \in M$, then the mapping b_t^{*-1} takes the system of local complex analytic coordinates on the variety V_t into some system of local coordinates on the variety V_0, which satisfies the condition (*) if the neighborhood U is sufficiently small. Thus we have an infinitely differentiable family $\omega(t)$ of forms of the type (0, 1) on the variety V_0, each of which satisfies equation (2.2) and $\omega(t_0) = 0$. Let T be the tangent space of the base M at the point t_0. We denote by $\rho'(L)$ the partial derivative $\partial\omega(t)/\partial L\,|_{t=0}$ of the form $\omega(t)$ in the direction $L \in T$. From (2.2) and from the fact that $\omega(t_0) = 0$, it immediately follows that $d''\rho'(L) = 0$.

Let $\rho(L)$ be the cohomology class of the cycle $\rho'(L)$. The mapping $\rho: T \to H^1(V_0, \Theta)$ is said to be an infinitesimal deformation at the point t_0 of the family of complex structures. The family $(\mathcal{V}, M, \pi)$ is said to be effectively parametrized if ρ is a monomorphism [26].

3. Let Θ and Ω^1 be sheaves of germs, respectively of the holomorphic vector fields and the one-dimensional holomorphic differential forms on the n-dimensional variety V. These sheaves are dual to each other by the definition of the sheaf Ω^1. Let us now assume that there exists on the variety V an n-dimensional holomorphic form k that is unique up to proportionality. Using this form, one can define a homomorphism σ of the sheaf Ω^{n-1} into the sheaf $\mathrm{Hom}(\Omega^1, \Omega^0)$ by the formula

$$(\sigma_{\omega_1})(\omega_2) = \frac{\omega_1 \wedge \omega_2}{k},$$

where ω_1 and ω_2 are sections of the sheaves Ω^{n-1} and Ω^1 respectively over some neighborhood, and the right-hand side of the formula is a function holomorphic in this neighborhood. It is easy to see that this homomorphism is an isomorphism. Thus on a variety V whose canonical bundle is trivial, the sheaves Θ and Ω^{n-1} are dual to the same sheaf, and hence are isomorphic. It is easy to verify that the isomorphism ϕ between the sheaves Θ and Ω^1 is given by the formula

$$\varphi\left(\sum_{i=1}^{n} A_i dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n\right) = \frac{1}{f_z} \sum_{i=1}^{n} (-1)^{i+n} A_i \frac{\partial}{\partial z_i},$$

where $f_z\, dz_1 \wedge \cdots \wedge dz_n$ is how one writes the form k in the local coordinates $(z_1, \cdots, z_n)$. In fact, let

$$\gamma = \Sigma A_i dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n$$

be a holomorphic differential form given in some neighborhood with the system of coordinates $(z_1, \cdots, z_n)$. Then

$$(\sigma_\gamma)(\alpha_1 dz_1 + \ldots + \alpha_n dz_n) = \frac{1}{f_z} \sum_{i=1}^{n} (-1)^{i+n} A_i \alpha_i.$$

On the other hand,

$$(\varphi\gamma)(\alpha_1 dz_1 + \ldots + \alpha_n dz_n) = \frac{1}{f_z} \sum (-1)^{i+n} A_i \alpha_i,$$

since the bases $\partial/\partial z_1, \cdots, \partial/\partial z_n$ and $dz_1, \cdots, dz_n$ are dual.

We denote by $A^p(V, \Theta)$ and $A^p(V, \Omega^{n-1})$ the spaces of the differential forms on the variety V of type $(0, p)$ with coefficients in the sheaves Θ and Ω^1. The isomorphism ϕ between these sheaves induces an isomorphism

$$\Phi: \quad A^p(V, \Omega^{n-1}) \to A^p(V, \Theta),$$

given by the formula

$$\Phi\Big(\sum (A_1^{i_1\dots i_p} dz_2 \wedge \dots \wedge dz_n + \dots + A_n^{i_1\dots i_p} dz_1 \wedge \dots \wedge dz_{n-1}) \times d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_p}\Big) = \frac{1}{f_z} \sum\Big((-1)^{1+n} A_1^{i_1\dots i_p} \frac{\partial}{\partial z_1} + \dots + (-1)^{n+n} A_n^{i_1\dots i_p} \frac{\partial}{\partial z_n}\Big) d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_p}.$$

Since the expression $1/f_z$ is holomorphic with respect to $(z_1, \dots, z_n)$, the isomorphism Φ commutes with the differential d'' (with accuracy up to the sign) and, if the forms depend on the parameter t, it also commutes with a differentiation with respect to the parameter.

4. Proof of Theorem 2. Let there be given on the complex variety V_0 with local coordinates $(z_1, \dots, z_n)$ another complex structure $\widetilde{V}$ with local parameters $(\zeta_1, \dots, \zeta_n)$ that satisfies the condition (*) of subsection 2. We consider the following differential form on the variety V_0 that has type $(n, 0)$ in the complex structure $\widetilde{V}$:

$$\widetilde{k} = \begin{vmatrix} \frac{\partial \zeta_1}{\partial z_1} & \dots & \frac{\partial \zeta_1}{\partial z_n} \\ \vdots & & \\ \frac{\partial \zeta_n}{\partial z_1} & \dots & \frac{\partial \zeta_n}{\partial z_n} \end{vmatrix}^{-1} f_z d\zeta_1 \wedge \dots \wedge d\zeta_n,$$

where

$$k_0 = f_z dz_1 \wedge \dots \wedge dz_n$$

is a holomorphic n-dimensional form in the structure V_0. We denote by $D_{i_1, \dots, i_k; j_1, \dots, j_l}$ the Jacobian

$$\frac{D(\zeta_1, \dots, \zeta_n)}{D(z_{i_1}, \dots, z_{i_k} \bar{z}_{j_1}, \dots, \bar{z}_{j_l})}$$

and set

$$D = D_{i_1 \dots i_n} = \frac{D(\zeta_1, \dots, \zeta_n)}{D(z_1, \dots, z_n)}.$$

Then the form

$$\widetilde{k} = D^{-1} f_z d\zeta_1 \wedge \dots \wedge d\zeta_n$$

is written in the following way in the system of coordinates $(z_1, \dots, z_n)$:

$$\widetilde{k} = k_0 + f_z \sum_{(i_1 \dots i_k) \neq \varnothing} D_{i_1 \dots i_k; j_1 \dots j_l} dz_{i_1} \wedge \dots \wedge dz_{i_k} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_l}, \quad (2.3)$$

Since the structure $\widetilde{V}$ satisfies condition (*), it is possible to construct a form $\omega_{\widetilde{V}}$ corresponding to $\widetilde{V}$. It has the form

$$\omega_{\widetilde{V}}\left(\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}\right)\begin{pmatrix}\frac{\partial\zeta_1}{\partial z_1} & \cdots & \frac{\partial\zeta_1}{\partial z_n}\\ \vdots & & \vdots\\ \frac{\partial\zeta_n}{\partial z_1} & \cdots & \frac{\partial\zeta_n}{\partial z_n}\end{pmatrix}^{-1}\begin{pmatrix}\frac{\partial\zeta_1}{\partial \bar z_1} & \cdots & \frac{\partial\zeta_1}{\partial \bar z_n}\\ \vdots & & \vdots\\ \frac{\partial\zeta_n}{\partial \bar z_1} & \cdots & \frac{\partial\zeta_n}{\partial \bar z_n}\end{pmatrix}\begin{pmatrix}dz_1\\ \vdots\\ dz_n\end{pmatrix}$$

Proposition 2.1. *The equation*

$$\omega_{\widetilde{V}} = \Phi\Pi^{n-1,1}\widetilde{k}.$$

holds.

This is proved by directly calculating the left- and right-hand sides of the equation. By definition of the form $\widetilde{k}$ we have

$$\Pi^{n-1,1}\widetilde{k} = f_z \sum_{i_1<i_2<\ldots<i_{n-1};\, j} D^{-1}D_{i_1\ldots i_{n-1};\, j}\, dz_{i_1}\wedge\ldots\wedge dz_{i_{n-1}}\wedge d\bar z_j$$

and hence

$$\Phi\Pi^{n-1,1}\widetilde{k} = \sum_j\Big(\sum_l(-1)^{n+l}D^{-1}D_{1,\ldots,\hat l,\ldots,n;\, j}\frac{\partial}{\partial z_l}\Big)d\bar z_j.$$

We now calculate the form $\omega_{\widetilde{V}}$. We denote by M_{ij} the associated minor of the element $\partial\zeta_i/\partial z_j$ for the matrix $(\partial\zeta_i/\partial z_j)$. Then the form $\omega_{\widetilde{V}}$ has the form

$$\omega_{\widetilde{V}} = \sum_j\Big(\sum_l\Big(\sum_k(-1)^{k+l}M_{kl}\frac{\partial\zeta_k}{\partial\bar z_j}\cdot D^{-1}\Big)\frac{\partial}{\partial z_l}\Big)d\bar z_j.$$

Since

$$\sum_k(-1)^{k+l}M_{kl}\frac{\partial\zeta_k}{\partial\bar z_j} = (-1)^{n+l}D_{1,\ldots,\hat l,\ldots,n;j},$$

we obtain for the form $\omega_{\widetilde{V}}$ the expression

$$\omega_{\widetilde{V}} = \sum_j\Big(\sum_l(-1)^{n+l}D^{-1}D_{1,\ldots,\hat l,\ldots,n;\, j}\frac{\partial}{\partial z_l}\Big)d\bar z_j,$$

from which it is evident that the forms $\omega_{\widetilde{V}}$ and $\Phi\Pi^{n-1,1}\widetilde{k}$ coincide. Proposition 2.1 is proved.

In the complex structure $\widetilde{V}$ the form $\widetilde{k}$ is a d'_ζ-cycle, and hence

$$\widetilde{k} = H_\zeta\widetilde{k} + d'_\zeta\delta'_\zeta G_\zeta\widetilde{k}.$$

The harmonic part

$$H_\zeta k = k_{\widetilde{V}}$$

is a holomorphic form of type $(n, 0)$ in the structure $\widetilde{V}$. We denote the form

$\delta'_\zeta G_\zeta \widetilde{k}$ by $C_{\widetilde{V}}$. Then

$$k_{\widetilde{V}} = \widetilde{k} - d'_\zeta C_{\widetilde{V}}.$$

Now let $(\mathfrak{V}, M, \pi)$ be a family of complex structures where

$$V_0 = V_{t_0} = \pi^{-1}(t_0), \qquad t_0 \in M.$$

Then, as was noted in subsection 2, the mapping b_t^{*-1} takes the system of local complex analytic coordinates on the variety V_t into the system of local complex analytic coordinates $(\zeta_1(t), \cdots, \zeta_n(t))$ on the variety V_0. Since V_0 is assumed to be a Kähler variety it follows that the V_t, for all t sufficiently close to t_0, are also Kähler varieties [27], and in this case the operators H_t and G_t depend smoothly on t. Therefore all the forms considered in this subsection depend in an infinitely differentiable manner on the parameter t:

$$k_{V_t} = \widetilde{k}(t) - d'_t C(t),$$

where, since

$$k_{V_0} = \widetilde{k}_{V_0} = k_0,$$

we have

$$C(t_0) = 0.$$

By definition of the form k_{V_t} we have

$$k_{V_t} = b_t^{*-1} k_t,$$

where k_t is an n-dimensional holomorphic form on the variety V_t. We replace this form with its decomposition (2.1) over the basis $(\omega_1, \cdots, \omega_b)$ of integral harmonic forms on the variety V_0. Then

$$\sum_{i=1}^{b} \alpha_i(t)\,\omega_i = - d\eta(t) - d'_t C(t) + \widetilde{k}(t). \tag{2.4}$$

Let L be the tangent vector to the base M at the point t_0. We denote by ∂_L the operator of taking a partial derivative in the direction L at the point t_0. We apply the operator ∂_L to both sides of (2.4), obtaining

$$\sum_{i=1}^{b} (\partial_L \alpha_i(t))\,\omega_i = - d\partial_L \eta(t) - \partial_L d'_t C(t) + \partial_L \widetilde{k}(t). \tag{2.5}$$

Proposition 2.2. *The equation*

$$\partial_L \widetilde{k}(t) = \Phi^{-1} \rho'(L).$$

holds, where

$$\rho'(L) = \partial_L\,\omega(t).$$

Proof. Applying Proposition 2.1 to equation (2.3), we can write

$$\widetilde{k}(t) = k_0 + \Phi^{-1}\omega(t) + \sum_{l>1} f_z D^{-1} D_{i_1,\dots,i_k;\, j_1,\dots,j_l} dz_{i_1} \wedge \dots \wedge dz_{i_k} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_l}.$$

We will prove that

$$\partial_L (D^{-1} D_{i_1,\dots,i_k;\, j_1,\dots,j_l}) = 0,$$

if $l > 1$. Indeed,

$$\partial_L (D^{-1} D_{i_1,\dots,i_k;\, j_1,\dots,j_l}) = D^{-1} (\partial_L D_{i_1,\dots,i_k;\, j_1,\dots,j_l}),$$

since

$$D_{i_1\dots i_k;\, j_1\dots j_l}(t_0) = 0.$$

But in order to calculate $\partial_L D_{i_1\cdots i_k;\, j_1\cdots j_l}$ it is necessary to take the sum of the determinants $\Delta_m(t_0)$, where $\Delta_m(t)$ is obtained from $D_{i_1\cdots i_k;\, j_1\cdots j_l}$ by the application of the operator $\partial/\partial L$ to the column with number m. However, since $l > 1$, in each of the determinants Δ_m there will remain a column of the form $(\partial\zeta_i/\partial\bar{z}_j)$, which completely vanishes for $t = t_0$. Thus

$$\Delta_m(t_0) = 0$$

for any m, and thus

$$\partial_L (D^{-1} D_{i_1\dots i_k;\, j_1\dots j_l}) = D^{-1} \sum_m \Delta_m(t_0)) = 0$$

for $l > 1$. As a result we have

$$\partial_L \widetilde{k}(t) = \partial_L \Phi^{-1}\omega(t) = \Phi^{-1}\partial_L \omega(t),$$

since the mapping Φ commutes with differentiation with respect to the parameter. Proposition 2.2 is proved.

Proposition 2.3. *The equation*

$$\partial_L d_t' C(t) = d' \partial_L C(t)$$

holds.

Proof. Let the form $C(t)$ be written in the local coordinates $(\zeta_1(t), \cdots, \zeta_n(t))$ in the following manner:

$$C(t) = \sum_{i=1}^{n} C_i(t)\, d\zeta_1 \wedge \dots \wedge \widehat{d\zeta_i} \wedge \dots \wedge d\zeta_n.$$

We calculate the form $\partial_L C(t)$. We have

$$\partial_L C(t) = \sum_{i=1}^{n} [\partial_L C_i(t)\, dz_1 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge dz_n + C_i(t_0)\, \partial_L (d\zeta_1 \wedge \dots \wedge \widehat{d\zeta_i} \wedge \dots \wedge d\zeta_n)].$$

As was noted above

$$C(t_0) = 0$$

and hence

$$C_i(t_0) = 0.$$

For the form $\partial_L C(t)$ we obtain the expression:

$$\partial_L C(t) = \sum_{i=1}^{n} (\partial_L C_i(t))\, dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n.$$

From this we have

$$d' \partial_L C(t) = \sum_{i=1}^{n} \frac{\partial \partial_L C_i(t)}{\partial z_i} (-1)^{i+1} dz_1 \wedge \ldots \wedge dz_n$$

or

$$d' \partial_L C(t) = \partial_L \left[\sum_{i=1}^{n} \frac{\partial C_i(t)}{\partial z_i} (-1)^{i+1} dz_1 \wedge \ldots \wedge dz_n \right].$$

It remains to calculate the form $\partial_L d_t' C(t)$. Since

$$d_t' C(t) = \left(\sum_{i=1}^{n} (-1)^{i+1} \frac{\partial C_i(t)}{\partial \zeta_i} \right) d\zeta_1 \wedge \ldots \wedge d\zeta_n$$

and moreover

$$\frac{\partial C(t_0)}{\partial \zeta_i(t_0)} = \frac{\partial C(t_0)}{\partial z_i} = 0, \tag{2.6}$$

after the application of ∂_L to the form $d_t C(t)$ we obtain

$$\partial_L d_t' C(t) = \sum_{i=1}^{n} (-1)^{i+1} \partial_L \left(\frac{\partial C_i(t)}{\partial \zeta_i} \right) dz_1 \wedge \ldots \wedge dz_n.$$

We also have

$$\partial_L \frac{\partial C_i(t)}{\partial \zeta_i} = \partial_L \left(\sum \frac{\partial C_i(t)}{\partial z_j} \frac{\partial z_j}{\partial \zeta_i} + \sum \frac{\partial C_i(t)}{\partial \bar{z}_j} \frac{\partial \bar{z}_j}{\partial \zeta_i} \right) = \partial_L \frac{\partial C_i(t)}{\partial z_i},$$

again using equation (2.6). As a result we get

$$\partial_L d_t' C(t) = \left(\partial_L \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial C_i(t)}{\partial z_i} \right) dz_1 \wedge \ldots \wedge dz_n$$

and hence

$$\partial_L d_t' C(t) = d' \partial_L C(t).$$

Proposition 2.3 is proved.

We use these two propositions in formula (2.5), obtaining

$$\sum_{l=1}^{b} (\partial_L \alpha_i (t))\, \omega_l = - d\beta - d'\partial_L C(t) + \Phi^{-1} \rho'(L), \tag{2.7}$$

where we denote by β the form $\partial_C \eta_t$.

Proposition 2.4. *In order for the mapping $\dot{F}$ tangent to the mapping F of subsection* 1 *to take the vector L into zero, it is necessary and sufficient that*

$$\partial_L \alpha_i(t) = \lambda \alpha_i(t_0)$$

for all $i = 1, \cdots, b$, where λ does not depend on i.

Proof. Let $\alpha_l(t_0) \neq 0$. Then for some neighborhood of the point t_0 the numbers

$$\frac{\alpha_i(t)}{\alpha_l(t)} \quad i = (1, 2, \ldots, \hat{l}, \ldots, b)$$

are nonhomogeneous coordinates of the point $F(t)$, and the image of the vector L under the mapping $\dot{F}$ is a vector with the coordinates

$$\partial_L \frac{\alpha_i(t)}{\alpha_l(t)} = \alpha_l^{-2}(t_0)\, [(\partial_L \alpha_i(t))\, \alpha_l(t_0) - \alpha_l(t_0)\, \partial_L \alpha_l(t)].$$

It is clear from this that in order for $\dot{F}(L)$ to be equal to zero, it is necessary and sufficient that

$$\partial_L \alpha_i(t) = \lambda \alpha_i(t_0),$$

where

$$\lambda = \frac{\partial_L \alpha_l(t)}{\alpha_l(t_0)}.$$

Proposition 2.4 is proved.

Let $\dot{F}(L) = 0$ for some vector L that is tangent to the base M at the point t_0. Using Proposition 2.4 and the fact that

$$\sum \alpha_l(t_0)\, \omega_l = k_0,$$

we obtain from formula (2.7) that

$$\Phi^{-1}\rho'(L) = d\beta + d'\partial_L C(t) - \lambda k_0.$$

We apply the harmonic projection operator $H = H_{t_0}$ to both sides of the last equation. Since

$$H(d'\partial_L C) = H(d\beta) = 0,$$

we can write

$$H(\Phi^{-1}\rho'(L)) = -\lambda k_0.$$

Here, however, the left-hand side is of type $(n-1, 1)$ and the right-hand side is of type $(n, 0)$. It follows from this that $\lambda = 0$ and

$$H(\Phi^{-1}\rho'(L)) = 0. \tag{2.8}$$

As was noted in the end of subsection 2,

$$d''\rho'(L) = 0.$$

Since the mapping Φ commutes with the differential d'', we have

$$d''\Phi^{-1}\rho'(L) = 0$$

and hence by equation (2.8)

$$\Phi^{-1}\rho'(L) = d''\gamma.$$

From this we obtain

$$\rho'(L) = \pm d''\Phi\gamma,$$

i.e.

$$\rho(L) = 0.$$

Thus we have proved that the kernel of the mapping $\dot{F}$ is a subspace of the kernel of the mapping ρ of subsection 2. If ρ is a monomorphism (i.e. if the family $(\mathcal{V}, M, \pi)$ is effectively parametrized), then the mapping $\dot{F}$ is also a monomorphism at each point and the mapping F is locally an imbedding.

We will now show that the mapping F is holomorphic. Let $t_1, \cdots, t_m$ be complex coordinates in the neighborhood of the point t_0 of the base M. We denote by $\bar{\partial}_k$ the operator of taking a derivative with respect to $\bar{t}_k$ at the point t_0, i.e.

$$\bar{\partial}_k = \frac{\partial}{\partial \bar{t}_k}\bigg|_{t=t_0}.$$

It is known that for a complex analytic family of structures $(\mathcal{V}, M, \pi)$ the family of forms $\omega(t)$ is holomorphic with respect to the variable t. Thus

$$\bar{\partial}_k \omega(t) = 0.$$

Taking this into account, we obtain from equation (2.7)

$$\sum (\bar{\partial}_k \alpha_i(t))\, \omega_i = -d\beta - d'\bar{\partial}_k C(t).$$

Since the left-hand side of the last expression contains a harmonic form, we have

$$\sum (\bar{\partial}_k \alpha_i(t))\, \omega_i = 0$$

and thus

$$\bar{\partial}_k \alpha_i(t) = 0$$

for any i and any k, i.e. the homogeneous coordinates $\alpha_i(t)$ depend holomorphically on t. The proof of Theorem 2 is complete.

§3. Topological properties of a surface with $q = 0$ and $K = 0$

1. We consider a compact Kähler surface V such that $c_1(V) = 0$ and $q = H^1(V, \Omega^0) = 0$, where Ω^i is the sheaf of the germs of the i-dimensional holomorphic forms on the surface V, and $c_1(V)$ is the first Chern class of this surface. It is possible to calculate the homology groups and d''-homology groups of a surface of this type.

By the duality theorem of Serre [47]

$$h^{1,0} = h^{1,2} = h^{2,1} = h^{0,1} = 0$$

and

$$\dim H^2(V, \Omega^0) = \dim H^0(V, \Omega^0(K)),$$

where K is the canonical bundle on V. The characteristic class $c(K)$ of the canonical bundle K is equal to $-c_1(V)$, and, in the given case, it is equal to zero. Since $H^1(V, \Omega^0) = 0$, it follows from the equation $c_1(V) = 0$ that the canonical bundle is trivial, and hence

$$h^{2,0} = \dim H^0(V, \Omega^0) = 1.$$

The dimension of the group $H^1(V, \Omega^1)$ is calculated by Noether's formula, which is also valid for an arbitrary compact Kähler surface [25],

$$12\,(h^{2,0} - h^{1,0} + 1) = E + c_1(V)^2,$$

where E is the Euler characteristic of the surface V, and $c_1(V)^2$ is the value of the cohomology class $c_1(V) \cup c_1(V)$ on a fundamental cycle of the surface. Since $E = 2 - 4q + 2h^{2,0} + h^{1,1}$, we have $h^{1,1} = 20.$

Since the surface under consideration is a Kähler surface, we have

$$H^1(V, \mathbf{R}) = H^3(V, \mathbf{R}) = 0, \quad \dim H^2(V, \mathbf{R}) = 22, \quad E = 24.$$

Using the theorem of Hodge about the index, it is easy to calculate the index $\tau(V)$ of the surface V:

$$\tau(V) = 2\,(h^{2,0} + 1) - h^{1,1} = -16.$$

2. We will show that the group $H^i(V, \mathbf{Z})$ is torsion-free. For this it is sufficient to show, because of the Poincaré duality, that the group $H_1(V, \mathbf{Z})$ is torsion-free. We will show that the surface V does not have finite-sheeted coverings. Let $\pi: \widetilde{V} \to V$ be an n-sheeted covering of the surface V. Then $\widetilde{V}$ is a compact complex Kähler surface, where its Euler characteristic $\widetilde{E}$ is equal to $nE = 24n$, and the canonical bundle is trivial. In fact, the sections of the canonical bundle are holomorphic two-dimensional forms. Since the canonical bundle of the surface V is trivial, there exists on V a two-dimensional holomorphic form k unique up to proportionality that vanishes nowhere. Its image π^*k, a two-dimensional holomorphic form, does not vanish anywhere on the surface $\widetilde{V}$. Because of this the

canonical bundle of $\widetilde{V}$ has a section that vanishes nowhere and which is consequently trivial. Applying Noether's formula to the surface $\widetilde{V}$, we obtain

$$12\,(h^{2,0}(\widetilde{V}) - h^{1,0}(\widetilde{V}) + 1) = 12\,(2 - h^{1,0}(\widetilde{V})) = 24n.$$

From this we have

$$2 - 2n = h^{1,0}(\widetilde{V}) \geq 0$$

and, consequently,

$$n \leq 1,$$

i.e. nontrivial finite-sheeted coverings of the surface V do not exist.

3. We now calculate the matrix of intersections of the surface V. As follows from Theorems 1 and 2 of Milnor [37], the integral quadratic form $H(x, x)$ is determined uniquely up to equivalence by the rank r and the index τ, if

1) the form is not of fixed sign (i.e. $\tau \neq \pm r$), and

2) on integral vectors it takes only even values.

Since for the matrix of intersections of our surface $r = 22$ and $\tau = -16$, the first condition is satisfied. By Lemma 3 of the same work of Milnor, the matrix of intersections of a given four-dimensional simply-connected variety satisfies condition 2) if and only if the second Stiefel-Whitney class of the given variety is equal to zero. Since the proof only uses the formula of Wu for Stiefel-Whitney classes, the assertion of the lemma is also valid for varieties M^4, for which the group $H^1(M, Z_2)$ is equal to zero.

We will now prove this. Let $W_2(M) = 0$. By the formula of Wu [13],

$$W(M) = \mathrm{Sq}(V),$$

where the class V is uniquely determined by the equation

$$\langle \alpha \cup V, \bar{\mu} \rangle = \langle \mathrm{Sq}\ \alpha,\ \bar{\mu} \rangle$$

for any class $\alpha \in H^*(M, Z_2)$. Here $\bar{\mu}$ is the generator of the group $H_4(M, Z_2)$. Since $W_2(M) = 0$ we have

$$\mathrm{Sq}\,(V) = 1 + W_3\,(M) + W_4\,(M).$$

We find V from this. Since $H^1(M, Z_2) = 0$, the class V has the form

$$V = \alpha_0 + \alpha_2 + \alpha_3 + \alpha_4, \quad \text{where} \quad \alpha_i \in H^i\,(M, Z_2)$$

and hence

$$\mathrm{Sq}\,V = \alpha_0 + \alpha_2 + (\mathrm{Sq}^1\alpha_2 + \alpha_3) + (\mathrm{Sq}^2\alpha_2 + \mathrm{Sq}^1\alpha_3 + \alpha_4).$$

Equating homogeneous components in Wu's formula, we obtain $\alpha_2 = 0$. Consequently, for any $\alpha \in H^2(M, Z_2)$ we have

$$<\alpha \cup V, \bar{\mu}> = 0.$$

But by Wu's formula

$$\langle \alpha \cup \alpha, \bar{\mu}\rangle = \langle \mathrm{Sq}\,\alpha, \quad \bar{\mu} \rangle = \langle \alpha \cup V, \quad \bar{\mu} \rangle = 0.$$

The sufficiency is proved. The necessity is proved by going in the opposite direction.

For a surface V of the type under consideration

$$W_2(V) = [c_1(V)]_2 = 0, \dim H^1(V, Z_2) = \dim H^1(V, R) = 0.$$

Thus condition 2) is satisfied for the matrix of intersections, and the matrix is in some integral basis a block-diagonal matrix in which there are three blocks of second order with the matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and two blocks of eighth order with the matrix $-V$, where V is a positive definite unimodular matrix of the form

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

This form of the matrix of intersections will be essential in §5.

§4. 19 moduli

1. In this section we study the space of moduli of the complex analytic surface V described in §3.

Let Θ be the sheaf of germs of the complex analytic vector fields on the surface V. By the duality theorem

$$\dim H^q(V, \Theta) = \dim H^{n-q}(V, \Omega^1(K)).$$

Since for our surface $K = 0$, we have

$$\begin{aligned} \dim H^0(V, \Theta) &= h^{1,2} = 0, \\ \dim H^1(V, \Theta) &= h^{1,1} = 20, \\ \dim H^2(V, \Theta) &= h^{1,0} = 0. \end{aligned}$$

By the basic theorem of [28] it follows that there exists an effectively parametrized family $(\mathfrak{V}, M, \pi)$ of complex structures complete at each point such that its base M is a 20-dimensional complex variety and $V_0 = V$, where $t_0 \in M$ (i.e. the number of moduli of the surface V exists and is equal to 20). If U is a sufficiently small neighborhood of the point t_0 in the base M of this family, then, by Theorem 2, using the mapping F the neighborhood U can be identified with some neighborhood $F(U)$ of the point $F(t_0)$ on a 20-dimensional hypersurface K_{20} in the projective space $\mathbf{P}_{21}$. In fact, as was proved in §2, the mapping F is locally an imbedding of the base M into the quadric K_{20}, and the dimensions of the varieties M and K_{20} coincide.

Let C be a curve (a one-dimensional complex subvariety) on the surface V_t. Since

$$\int_C k_t = 0,$$

we have

$$\beta_1 \alpha_1 (t) + \ldots + \beta_{22} \alpha_{22} (t) = 0,$$

where the integers $\beta_1, \cdots, \beta_{22}$ are the coefficients of the decomposition of the cycle C with respect to the homology basis, $c_1, \cdots, c_{22}$, which was chosen in §2.1. Thus, in order for a curve to lie on the surface V_t, it is necessary that the point $F(t)$ lie on the intersection of the quadric K_{20} and the hyperplane $\Sigma \beta_i z_i = 0$, where the β_i are integers. This intersection has dimension 19. Since at least one curve (hyperplane section) always lies on an algebraic surface, it follows from this that there does not exist an effectively parametrized family of deformations of the structure of the surface V such that all the surfaces V_t are algebraic and the dimension of the base of the family is equal to 20. On the other hand, if V is an algebraic surface of the type under consideration, then for it $h^{2,0} = 1$, and hence by Theorem 1 there exists an effectively parametrized family $(\mathfrak{V}_c, M_c, \pi)$ of algebraic surfaces with base dimension of 19 and such that $V_{t_0} = V$. This family is uniquely determined (up to equivalence) by the homology class C of a hyperplane section of the varieties V_t. Moreover, this family is complete in the sense that any other family of algebraic varieties with the same class of hyperplane section and with $V_0 = V$ must be contained in it. Thus, following the definition of Kodaira and Spencer ([26], Definition 12.4), we can formulate the following proposition.

Theorem 3. *Let V be an algebraic surface for which $K = 0$ and $q = 0$, which is imbedded in a projective space $\mathbf{P}^n$, and let $C \in H_2(V, \mathbf{Z})$ be the homology class of a hyperplane section. Then the number of moduli $m_c(V)$ of the surface V with respect to the space $\mathbf{P}^n$ is equal to* 19.

Remark. From what has been said so far in this section it follows that if

M_{20} is a base of a complete effectively parametrized family of deformations of a surface V, then almost all (i.e. all except for the union of a denumerable set of 19-dimensional hypersurfaces) the points of the variety M_{20} correspond to surfaces on which there is no divisor, and hence no nonconstant meromorphic function.

2. We saw above that if there is a curve $C \sim \Sigma\beta_i c_i$ on the surface V_t, then the coordinates $\alpha_i(t)$ of the point $F(t)$ lie in the plane $\Sigma\beta_i z_i = 0$. Conversely, let V_t be an algebraic surface and let the coordinates $\alpha_i(t)$ satisfy the equation

$$\Sigma\beta_i\alpha_i(t) = 0, \tag{4.1}$$

where the β_i are integers. We consider the homology class $C = \Sigma\beta_i c_i$. The condition (4.1) is equivalent to

$$\int_C k_t = 0.$$

Let

$$\omega = \alpha k_t + \omega^{1,1} + \bar{\alpha}\bar{k}_t$$

be the harmonic form on the surface V_t that is dual to the class C. From condition (3.1) it follows that $\int_V k_t \wedge \omega = 0$. But, as is easily seen,

$$\int_V k_t \wedge \omega = \bar{\alpha}\int_V k_t \wedge \bar{k}_t$$

and hence $\alpha = 0$ and the form ω has type $(1, 1)$.

As is known, on an algebraic variety every integral cohomology class to which there belongs a harmonic form of type $(1, 1)$ is dual to some divisor. Since in our case the form ω is dual to the homology class C, it follows that some divisor belongs to the class C. Moreover, for the given surface the group $H^1(V_t, \Omega^0)$ is equal to zero, and hence if two divisors are homologous they are linearly equivalent. Thus the group of classes of divisors on the algebraic surface V_t is isomorphic to the group of integral 22-dimensional vectors $(\beta_1, \cdots, \beta_{22})$ such that

$$\Sigma\beta_i\alpha_i(t) = 0,$$

where the $\alpha_i(t)$ are projective coordinates of the point $F(t)$.

Thus the base number of the algebraic surface V_t (in the given case it coincides with the rank of the group of classes of divisors) will be equal to ρ, if and only if the point $F(t) \in K_{20}$ lies on the intersection of ρ linearly independent hyperplanes with integral coefficients. This intersection has dimension $20 - \rho$. It is evident from this that "almost all" (i.e. all with the exception of the union of a not more than denumerable set of subvarieties of smaller dimension) the points of the base M_c of the family $(\mathcal{V}_c, M_c, \pi)$ correspond to algebraic surfaces on which all the curves are multiples of a hyperplane section. Nevertheless, the following assertion is true.

Theorem 4. *The base number of a surface V with $K = 0$ and $q = 0$ can take all the values from 1 to 20.*

Proof. Let $(\mho_c, M_c, \pi)$ be a complete family of algebraic varieties with a homology class of a hyperplane section $C = \Sigma \beta_i c_i$. Let $\alpha = F(t_0)$, $t_0 \in M_c$. As we have already seen, the points of $F(M_c)$ lie on the intersection of the quadric K_{20} with the hyperplane S whose equation is $\Sigma \beta_i z_i = 0$. Moreover, by Theorem 1 there exists a neighborhood U of the point α such that the intersection $F(M_c) \cap U$ coincides with the intersection $U \cap S$. In any neighborhood of the point α in the projective space $\mathbf{P}^{21}$ we can choose a point $\widetilde{\alpha}$ such that the real and imaginary parts of its coordinates are rational, and it itself lies in the plane S. It is easy to see that these coordinates satisfy twenty linearly independent linear equations with integer coefficients. As one of these, we take the equation of the plane S. We consider the line $L(\widetilde{\alpha})$ in the space $\mathbf{P}^{21}$ given by these twenty equations. We use the following lemma.

Lemma. *For any neighborhood $U \subset K_{20}$ of the point $\alpha = F(t)$ there exists a neighborhood $U' \subset \mathbf{P}^{21}$ of the same point in the space $\mathbf{P}^{21}$ such that if $\widetilde{\alpha} \in U'$, then at least one point α' of the intersection of the line $L(\widetilde{\alpha})$ with the quadric K_{20} lies in the neighborhood U.*

The proof of this lemma will be given below. It is clear that the point $\alpha' \in L(\widetilde{\alpha}) \cap K_{20}$, existing by the lemma, lies on the intersection of the quadric K_{20} with the plane S, since the line $L(\widetilde{\alpha})$ lies in S. Since all the points α belonging to the intersection $U \cap S$ correspond to certain algebraic surfaces, then, in particular, the point α' corresponds to an algebraic surface with base number 20. In order to prove the existence of algebraic surfaces with a base number ρ, it is sufficient to choose from the 20 equations giving the line $L(\widetilde{\alpha})$ ρ linearly independent ones such that the ρ chosen equations contain the equation of the plane S, and then to consider the intersection with the neighborhood U of the linear variety given by these ρ equations. This intersection is not empty, for the point α' belongs to it. Hence it has dimension $20 - \rho$, and the algebraic surfaces corresponding to "points of general position" on this intersection have base number ρ. It remains to prove the lemma.

Proof of the lemma. Let $\alpha(n)$ be a sequence of points with complex rational coordinates $\alpha_i(n) = p_i(n) + iq_i(n)$ converging to the point $\alpha = F(t_0)$ with coordinates $\alpha_i = p_i + iq_i$. It is possible to assume that $\alpha_1 = 1$ and $\alpha_1(n) = 1$ for all n. Since the point α lies on the quadric K_{20}, its coordinates satisfy the condition $\Sigma h_{ij} \alpha_i \alpha_j = 0$, where $(h_{ij}) = H$ is the matrix of the intersections of the surface V_0, or

$$\Sigma h_{ij}p_ip_j - \Sigma h_{ij}q_iq_j = 0, \ \Sigma h_{ij}p_iq_j = 0.$$

Moreover, since

$$k_{t_0} \wedge \overline{k}_{t_0} \neq 0,$$

for the numbers α_i we have

$$\Sigma h_{ij}\alpha_i\overline{\alpha}_j \neq 0$$

or

$$\Sigma h_{ij}p_ip_j + \Sigma h_{ij}q_iq_j = 2\Sigma h_{ij}q_iq_j \neq 0.$$

From this, in particular, it follows that the vector $q = (q_2, \cdots, q_{22})$ is nonzero. It is easy to verify that it is possible to take as a direction vector of the line $\mathfrak{L}(\alpha(n))$ the vector $q(n) = (q_2(n), \cdots, q_{22}(n))$. This vector can be considered nonzero, since it tends to the nonzero vector q. In fact, let the point 1, $\alpha_2(n), \cdots, \alpha_{22}(n)$ satisfy the equation $\Sigma\beta_i\alpha_i(n) = 0$. Then the point

$$1, \alpha_2(n) + \lambda q_2(n), \ldots, \alpha_{22}(n) + \lambda q_{22}(n)$$

also satisfies this equation. We will now find the point of intersection of the line $\alpha(n) + \lambda q(n)$ with the quadric $H(z, z) = \Sigma h_{ij} z_i z_j = 0$. We obtain the number λ from the equation

$$H(\alpha(n) + \lambda q(n), \alpha(n) + \lambda q(n)) = 0.$$

Hence we obtain

$$H(\alpha(n), \alpha(n)) + 2\lambda H(\alpha(n), q(n)) + \lambda^2 H(q(n), q(n)) = 0.$$

When the point $\alpha(n)$ tends to the point α, the free member of this equation tends to zero at the same time as the highest coefficient tends to the expression $H(q, q) = \Sigma h_{ij} q_i q_j$, which, as we have already seen, is not equal to zero if the point $\alpha \in K_{20}$ corresponds to some complex surface. Hence one of the roots λ_1 of this equation tends to zero when $\alpha(n)$ tends to α, and the point $\alpha'(n) = \alpha(n) + \lambda_1 q(n)$ tends to the point α. The proof of the lemma is complete.

§5. Diffeomorphisms

The basic result of this section is the proof that all surfaces of the type under consideration (with $q = 0$ and $K = 0$) are diffeomorphic. First of all we will prove that an arbitrary compact Kähler surface V for which $q(V) = 0$ and $c_1(V) = 0$ becomes, under an arbitrary small deformation, an algebraic surface V_3 on which the group of classes of divisors is generated by a curve of genus 3. It will then be proved that all such algebraic surfaces are diffeomorphic.

1. Proposition 5.1. *In any neighborhood of the point* $(\alpha) = (\alpha_1, \cdots, \alpha_{22})$ *lying on the quadric* K_{20} *in 21-dimensional projective space, there exists a point* $(t) = (t_1, \cdots, t_{22})$ *on the same quadric satisfying the linear equation with integer coefficients*

$$\sum_{i=1}^{22} \beta_i z_i = 0, \beta_i = 0 \quad \textit{for} \quad i > 6,$$

such that the β_i *do not have a common factor and satisfy the relationship*

$$\beta_1\beta_2 + \beta_3\beta_4 + \beta_5\beta_6 = \pi - 1,$$

where π *is any integer larger than one.*

Proof. Obviously it is always possible to find three real numbers y_1, y_3 and y_5, not all equal to zero, such that

$$\alpha_1 y_1 + \alpha_3 y_3 + \alpha_5 y_5 = 0.$$

The last relationship means that the point (α) lies in the hyperplane M which is given by the equation

$$\sum_{i=1}^{22} y_i z_i = 0$$

where the coefficients y_i are equal to zero for $i \neq 1, 3, 5$, and y_1, y_3, and y_5 are the real numbers found by us. We use the following lemma.

Lemma. *For any real number* x_3 *and any integer* $\pi \geq 2$ *there exists a sequence of integer vectors*

$$(\beta_1(n), \beta_2(n), \beta_3(n), \beta_4(n)),$$

which satisfies the following four conditions:

1) $\beta_1\beta_2 + \beta_3\beta_4 = \pi - 1$,

2) $\lim_{n\to\infty}(\beta_3/\beta_1) = x_3$,

3) $\lim_{n\to\infty}(\beta_2/\beta_1) = \lim_{n\to\infty}(\beta_3/\beta_1) = 0$,

4) *for any* n *the numbers* $\beta_1(n)$, $\beta_2(n)$, $\beta_3(n)$, *and* $\beta_4(n)$ *do not have a common factor.*

Proof of the lemma. We will first prove the lemma in the case $\pi = 2$. We denote by N the set of sequences of integer vectors $(\beta_1, \beta_2, \beta_3, \beta_4)$ satisfying conditions 1) and 2) of the lemma (for $\pi = 2$) for which there exists a limit

$$\lim_{n\to\infty} \frac{\beta_4}{\beta_1}$$

and moreover $\beta_1(n) \longrightarrow \infty$.

The set N is not empty. In fact, let β_1 be an arbitrary integer, and let β_3

be the closest integer to $\beta_1 x_3$ that is relatively prime to β_1. In view of the relative primeness of the numbers β_1 and β_3 there exist integers u and v such that

$$u\beta_1 - v\beta_3 = 1$$

and

$$|u| < |\beta_3|, \quad |v| < |\beta_1|.$$

The sequence $P(\beta_1)$ of the vectors

$$(\beta_1, u, \beta_3, -v),$$

clearly satisfies conditions 1) and 2) and, moreover,

$$\left|\frac{-v}{\beta_1}\right| < 1.$$

Because of the last inequality, one can choose from the sequence $P(\beta_1)$ a subsequence belonging to the set N.

We consider the mapping

$$\phi: N \to \mathbf{R}$$

of the set N onto the real line $\mathbf{R}$, that puts into correspondence with each sequence $P_n \in N$ the limit of the ratio β_4/β_1. It is clear that the set ϕN is closed. Let us assume that the point O does not belong to ϕN. Then the greatest lower bound X_4 of the set $|\phi N|$ (the absolute values of the numbers of ϕN) is greater than zero.

We consider some sequence

$$(\beta_1(n), \beta_2(n), \beta_3(n), \beta_4(n))$$

of the set N. We set

$$\lim \frac{\beta_4}{\beta_1} = x_4.$$

Then, because of condition (1) and the fact that $\beta_1 \to \infty$, we have

$$\lim \frac{\beta_2}{\beta_1} = \lim \left(\frac{1}{\beta_1^2} - \frac{\beta_3}{\beta_1} \cdot \frac{\beta_4}{\beta_1}\right) = -x_3 x_4.$$

We choose an integer k such that

$$0 < \frac{1}{x_4 + k} < X_4,$$

and consider the sequence $Q(n)$ of vectors

$$(\beta_4 + k\beta_1, \beta_3, k\beta_3 - \beta_2, -\beta_1).$$

We will show that this sequence belongs to the set N. In fact,

$$(\beta_4 + k\beta_1)\beta_3 + (k\beta_3 - \beta_2)(-\beta_1) = \beta_1\beta_2 + \beta_3\beta_4 = 1.$$

Further, since

$$\lim \frac{\beta_4 + k\beta_1}{\beta_1} = x_4 + k \neq 0,$$

we have $\beta_4 + k\beta_1 \to \infty$, and also

$$\lim \frac{k\beta_3 - \beta_2}{\beta_4 + k\beta_1} = \lim \frac{k\beta_3 - \beta_2}{\beta_1} \cdot \lim \frac{\beta_1}{\beta_4 + k\beta_1} = x_3.$$

The image ϕQ is equal, by definition, to the limit

$$\lim \frac{-\beta_1}{\beta_4 + k\beta_1} = -\frac{1}{x_4 + k}$$

and hence

$$|\varphi Q| = \frac{1}{x_4 + k} < X_4,$$

which is impossible, since X_4 is the greatest lower bound of the absolute values of the numbers ϕN. Thus $O \in \phi N$, i.e. there exists a sequence $P_n \in N$ such that

$$\lim \frac{\beta_4}{\beta_1} = 0$$

and hence

$$\lim \frac{\beta_2}{\beta_1} = -\lim \frac{\beta_3}{\beta_1} \cdot \frac{\beta_4}{\beta_1} = 0.$$

The lemma is proved in the case $\pi = 2$. It is easy to go from this case to that of an arbitrary π. Thus, let

$$(\beta_1(n), \beta_2(n), \beta_3(n), \beta_4(n))$$

be a sequence satisfying the conditions of the lemma for $\pi = 2$, i.e.

1) $\beta_1\beta_2 + \beta_3\beta_4 = 1$,
2) $\lim(\beta_3/\beta_1) = x_3$,
3) $\lim(\beta_2/\beta_1) = \lim(\beta_4/\beta_1) = 0$.

It is evident from condition 1) that the numbers β_1 and β_3 cannot have a common factor. We consider the sequence

$$(\beta_1, (\pi - 1)\beta_2, \beta_3, (\pi - 1)\beta_4).$$

It is clear that this sequence satisfies the conditions of the lemma.

We now return to the proof of Proposition 5.1. At least one of the numbers y_1, y_3, y_5 is not equal to zero, say $y_1 \neq 0$. Then, according to the lemma there exists a sequence P_n of integer vectors $(\beta_i(n))$, $i = 1, \cdots, 4$, satisfying the following conditions:

1) $\beta_1\beta_2 + \beta_3\beta_4 = \pi - 1$ for any n,

2) $\lim(\beta_3/\beta_1) = \gamma_3/\gamma_1$,

3) $\lim(\beta_2/\beta_1) = \lim(\beta_4/\beta_1) = 0$.

We consider the sequence of hyperplanes M_n in the projective space $\mathbf{P}^{21}(\mathrm{C})$ whose coefficients $\beta_i(n)$, $i = 1, \cdots, 22$, are chosen in the following way.

1. $\beta_1(n)$, $\beta_2(n)$, $\beta_3(n)$, $\beta_4(n)$ coincide with the components $(\beta_i(n))$ in the nth member of the sequence P_n.

2. $\beta_5(n)$ is the closest integer to the number $\beta_1(n)\gamma_5/\gamma_1$. The remaining coefficients are set equal to zero.

We have thus constructed a sequence of hyperplanes M_n with integer coefficients $\beta_i(n)$. These coefficients do not have a common factor and satisfy the relationship

$$\beta_1\beta_2 + \beta_3\beta_4 + \beta_5\beta_6 = \pi - 1.$$

This sequence of hyperplanes converges to the hyperplane M. If M is not the tangent plane to the quadric K_{20} at the point (α), then it is clear that in any neighborhood of the point (α) on the quadric there is a point $t(n) = (t_1, \cdots, t_{22})$ lying on some plane M_n. If the hyperplane M turns out to be tangent to the quadric at the point (α), then we consider instead of it the plane M' whose coefficients y_i are equal to zero for $i \neq 2, 4, 6$, while the y_2, y_4, y_6 are not all equal to zero, and which are obtained from the equation

$$\alpha_2 y_2 + \alpha_4 y_4 + \alpha_6 y_6 = 0.$$

Proposition 5.1 is thus proved.

Theorem 5. *For any $\pi \geq 2$ there exists an algebraic surface V_π with an irregularity of zero and a zero canonical class on which there lies a curve of genus π and on which there does not lie a curve of lesser genus.* [*The points corresponding to the surfaces V_π are everywhere dense in the space of moduli of Kähler surfaces with $q = 0$ and $K = 0$.*]

Proof. Let V_0 be an arbitrary Kähler surface for which $q = 0$ and $K = 0$. Let, further, $c_1, \cdots, c_{22}$ be a basis of the two-dimensional homology group of the surface V_0 in which the matrix of intersections has the form indicated at the end of §3. We recall that, in particular, $c_1 \cdot c_2 = 1$, $c_3 \cdot c_4 = 1$, $c_5 \cdot c_6 = 1$, and $c_i \cdot c_j = 0$ for all the remaining pairs (i, j) for $i, j \leq 6$.

We denote by $(\alpha_1, \cdots, \alpha_{22})$ the integrals of the two-dimensional holomorphic forms k of the surface V_0 over the elements of the basis (c_i). These numbers can be considered as homogeneous coordinates of the point (α) in 21-dimensional projective space. As was mentioned in §4.1, it follows from Theorem 2 that in some neighborhood U of the point (α) there corresponds to each point $(t) = (t_1, \cdots, t_{22})$ lying on the 20-dimensional quadric K_{20} some complex structure V_t

obtained by a deformation of the structure V_0. Here the coordinates of the point (t) are equal to the integrals of the two-dimensional holomorphic form k_t of the surface V_t over the cycles $c_1, \cdots, c_{22}$. As we proved (Proposition 5.1), in the neighborhood U there exists a point $t(\pi)$ lying on the intersection of the quadric K_{20} and the hypersurface M given by the equation

$$\sum_{i=1}^{6} \beta_i z_i = 0$$

with integer coefficients without a common factor such that

$$\beta_1\beta_2 + \beta_3\beta_4 + \beta_5\beta_6 = \pi - 1.$$

We may assume that the point $t(\pi)$ does not belong to more than one hyperplane with integer coefficients.

We consider the homology class $C = \Sigma_{i=1}^{6} \beta_i c_i$ and the harmonic form ω dual to it on the surface V_π corresponding to the point $t(\pi) = (t_1, \cdots, t_{22})$. The condition

$$\sum_{i=1}^{6} \beta_i t_i = 0$$

is equivalent to $\int_C k = 0$ or to the form ω having type (1, 1) (cf. §4.2). Moreover,

$$\int_V \omega \wedge \omega = C \cdot C.$$

We calculate the index of selfintersection $C \cdot C$ of the class C.

$$C \cdot C = \sum \beta_i \beta_j c_i \cdot c_j = 2(\beta_1\beta_2 + \beta_3\beta_4 + \beta_5\beta_6) = 2\pi - 2.$$

On the given surface V_π there corresponds to each integer harmonic form of type (1, 1) some complex line bundle whose characteristic class is the cohomology class of this harmonic form. Let F be the line bundle with characteristic class $c(F)$ to which the form ω belongs. By the Riemann-Roch theorem for line bundles on Kähler varieties [7], we have for our surface V_π

$$l(F) + l(-F) \geqslant 2 + \frac{c(F)^2}{2},$$

where $l(F)$ is the dimension of the space of sections of the line bundle F. But $c(F)^2$ has already been calculated:

$$c(F)^2 = \int_V \omega \wedge \omega = 2\pi - 2 \geqslant 2.$$

Hence either the line bundle F or the line bundle $-F$ has a section. It is well known that the set of zeros of this section (which is a complex curve D on the surface V_π) belongs to the homology class that is dual to the characteristic class of this line bundle, i.e. the curve D belongs either to the class C or to $-C$.

Thus D has in any case an index of selfintersection $2\pi - 2$ and hence is a curve of genus π.

We have assumed that the point $t(\pi)$ does not belong to any hyperplane with integer coefficients other than the plane with coefficients β_i. Hence any divisor on the surface V_π is equivalent to an integer multiple of the curve D. Thus

$$\widetilde{D}^2 = (kD)^2 = k^2D^2 \geqslant 2\pi - 2,$$

where $\widetilde{D}$ is any divisor. We see that the index of selfintersection of any curve on the surface V_π is not less than $2\pi - 2$. Thus the genus of any curve lying on V_t is not less than π.

We will now show that all the curves of the linear system $|D|$ are irreducible. In fact, let the curve $\widetilde{D} \in |D|$ be reducible; then each of its components D_i is equivalent to a positive integer multiple $k_i D$ of the curve D, and we have

$$D = \Sigma D_i \sim (\Sigma k_i)\, D,\ k_i > 0$$

and hence $i = 1$, $k = 1$.

Thus there is an irreducible curve D on the surface V_π such that $D^2 = 2\pi - 2 > 1$. By Theorem 3.3 of [25] it follows that the surface V_π is algebraic.

2. It follows from Theorem 5, in particular, that any Kähler surface with $q=0$ and $K = 0$ is diffeomorphic to an algebraic surface V_3 on which the group of classes of divisors generates a single curve D of genus 3.

Proposition 5.2. *An algebraic surface with such properties can be realized as a nonsingular surface of fourth degree in three-dimensional projective space.*

Proof. As was shown in Chapter VIII, the curve D is nonsingular, and its linear system $|D|$ either regularly and birationally maps V_3 onto a surface of fourth degree in $\mathbf{P}^3$, or maps the surface V_3 onto a quadric in $\mathbf{P}^3$ in two sheets. We will show that the second case is impossible. In fact, a quadric in $\mathbf{P}^3$ is biregularly equivalent to the product of two rational curves $\mathbf{P}^1 \times \mathbf{P}'^1$, and its hyperplane section S is reducible and equal to $\mathbf{P}^1 \times p' + p \times \mathbf{P}^{1\prime}$, where p and p' are points of $\mathbf{P}^1$ and $\mathbf{P}^{1\prime}$. It is clear from this that the preimage of S under a two-sheeted mapping, which (preimage) is by definition equivalent to D, is also reducible. But there can be no reducible curves in the linear system $|D|$.

Thus there exists a regular and birational mapping f_D of the surface V_3 into $\mathbf{P}^3$. It is easy to see that this mapping must be biregular. In fact, otherwise, there would exist on the surface V_3 at least one curve θ which would contract into a point under the mapping f_D. But then $\theta^2 < 0$, and hence the curve θ is not equivalent to a divisor kD. Proposition 5.2 is proved.

Two complex varieties V_1 and V_2 are said to be c-homotopic if there exists a family of complex structures $(\mathcal{V}, M, \pi)$ with a connected base M such that

$$V_1 = \pi^{-1}(t_1), \quad V_2 = \pi^{-1}(t_2), \quad t_1, t_2 \in M$$

([26], Definition 1.5). Using this definition, it is possible to formulate the following theorem.

Theorem 6. *All compact complex Kähler surfaces with $q = 0$ and $K = 0$ are c-homotopic.*

Proof. As follows from Theorem 5 and Proposition 5.2, each such surface is c-homotopic to a nonsingular surface that is given by an equation of fourth degree in three-dimensional projective space. We will show that any two nonsingular surfaces of fourth degree in $\mathbf{P}^3(\mathbf{C})$ are c-homotopic. Let

$$\{t_{i_0, i_1, i_2, i_3}\}, \text{ where } \; i_k \geqslant 0, \; \Sigma i_k = 4$$

be the coordinates of the space $\mathbf{C}^N$, $N = 35$, and (x_0, x_1, x_2, x_3) be homogeneous coordinates in the projective space $\mathbf{P}^3(\mathbf{C})$. We denote by B the set of those points of the direct product $\mathbf{P}^3 \times \mathbf{C}^N$ which satisfy the equation

$$P_t(x) = \Sigma t_{i_0 i_1 i_2 i_3} x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3} = 0.$$

For a fixed $t \neq 0$ the equation $P_t(x) = 0$ gives a surface of fourth degree in $\mathbf{P}^3(\mathbf{C})$. In order for this surface to be nonsingular it is necessary and sufficient that all the partial derivatives of the polynomial $P_t(x)$ with respect to the coordinates of the projective space $\mathbf{P}^3(\mathbf{C})$ never vanish simultaneously at any point of the surface $P_t(x) = 0$. The subset C of the points of the set B in which all the partial derivatives of the polynomial $P_t(x)$ vanish is a closed complex subspace of the complex space B. Let π be the holomorphic mapping of the set $B \subset \mathbf{P}^3 \times \mathbf{C}^N$ onto the space $\mathbf{C}^N$ induced by the projection of the product $\mathbf{P}^3 \times \mathbf{C}^N$ onto $\mathbf{C}^N$. The image of the set C under the mapping will also be a closed complex subspace in $\mathbf{C}^N$, and hence its complement

$$M = \mathbf{C}^N - \pi C$$

is connected.

It is easy to see that

$$\mathcal{V} = (\pi^{-1}(M), M, \pi)$$

is a complex analytic family of complex structures, where for any nonsingular surface V of fourth degree in $\mathbf{P}^3$ there exists a $t \in M$ such that $V = \pi^{-1}(t)$.

Since the base M of this family is connected, any two nonsingular surfaces of fourth degree in $\mathbf{P}^3(\mathbf{C})$ are c-homotopic. Theorem 6 is proved.

It is clear from the definition that c-homotopic varieties are always diffeomorphic. The following assertion is an obvious corollary of Theorem 6.

Theorem 7. *Any two compact complex Kähler surfaces with $q = 0$ and $c_1 = 0$ are diffeomorphic.*

CHAPTER X

ENRIQUES SURFACES

In this chapter we will study surfaces with $\kappa = p_g = q = 0$. It will be proved that a generic surface of this type is birationally equivalent to a surface of 6th degree in $\mathbf{P}^3$ that passes twice through the edges of a tetrahedron.

In §1 we prove a series of propositions about linear systems on generic surfaces of the type under consideration.

In §2 we present the basic construction of the chapter (belonging to Enriques), and we prove that on a generic surface with $\kappa = p_g = q = 0$ there exists a pair of isolated (i.e. not changing in the linear system) elliptic curves (i.e. divisors of arithmetic genus 1) with a positive index of intersection.

In §3 the basic construction is again applied to obtain a pair of isolated elliptic curves (irreducible) with an index of intersection of 1. Using these curves we prove that every surface with $\kappa = p_g = q = 0$ is birationally equivalent to a double plane whose branch curve is constructed in a certain way.

In §4 we construct a birational mapping possessing all the desired properties.

Finally, in §5 we calculate the "number of moduli" of algebraic surfaces with $\kappa = p_g = q = 0$. It is equal to ten.

§1. Linear systems on a generic Enriques surface

By a rational (elliptic) curve we will always understand a divisor (possibly reducible) of arithmetic genus 0(1). If $|C|$ is some linear system, we will denote by $|C'|$ the linear system $|C + K|$ and call this system an adjoint system. We do not exclude the case when both the systems $|C|$, $|C'| = |C + K|$ each contain a single curve; such curves are said to be isolated.

Thus, let F be a nonsingular algebraic surface with $\kappa = p_g = q = 0$. As was proved in Lemma 4, §1, Chapter VIII, $P_2(F) = 1$, $2K(F) = 0$, and thus for any effective divisor C on the surface F we have

$$(C^2) = 2p_a(C) - 2,$$

$$\dim |C| \geqslant \frac{(C^2)}{2} = p_a(C) - 1,$$

$$2C \sim 2(C + K).$$

Proposition 1. *Let* θ, $\theta' \sim \theta + K$ *be effective divisors without common components. Then* $p_a(\theta) \geq 1$.

Proof. We have $p_a(\theta) = (\theta \cdot (\theta + K))/2 + 1 = (\theta \cdot \theta')/2 + 1 \geq 1$.

Corollary. *If* θ *is an irreducible rational curve, then the divisor* $\theta + K$ *is not equivalent to an effective divisor.*

Proof. Let $\theta' \sim \theta + K$ be an effective divisor. Then the divisors θ, θ' must have a common component, and since θ is irreducible we have $0 \leq \theta' - \theta \sim K$. This is a contradiction.

Proposition 2. *Let* C *be an irreducible elliptic curve and let* $\dim |C| = 1$. *Then:*

1) *the system* $|C'| = |C + K|$ *is not empty and consists of a single curve;*

2) *if* $C' \in |C'|$ *is an effective divisor, then* $C' = C/2 + (C/2)'$, *where* $C/2$, $(C/2)'$ *denote effective divisors whose carriers are connected and do not have common points. Moreover,* $(C/2)' \sim C/2 + K$, *and the divisors* $2 \cdot (C/2)$, $2 \cdot (C/2)'$ *belong to the system* $|C|$ *and are uniquely determined by this property, from which it follows that they are isolated.*

Proof. By a theorem of Kodaira ([25], Theorem 2.5) $H'(F, \Omega(C + K)) = 0$, and 1) follows directly from the Riemann-Roch theorem.

We will now prove 2). We fix an arbitrary point on the carrier of the divisor C'; then there exists a divisor $C_1 \in |C|$ passing through this point. Since $(C_1 \cdot C') = (C^2) = 0$, the divisors C_1 and C' have a common component, which we denote by $C/2$. Thus, $C_1 = C/2 + D$, $C' = C/2 + D'$, where the divisors D and D' are effective, nonzero, and do not have common components. It is also clear that $D' \sim D + K$. Since the system $|C|$ is irreducible, we have $(C \cdot (C/2))$ and $(C \cdot D) \geq 0$, and since $(C^2) = (C \cdot (C/2)) + (C \cdot D) = 0$ we have $(C \cdot (C/2)) = (C \cdot D) = 0$. Hence $((C/2)^2) + ((C/2) \cdot D) = (C \cdot (C/2)) = 0$ and $((C/2) \cdot D) + (D^2) = (C \cdot D) = 0$. Since $(D^2) = (D \cdot D') \geq 0$ (the divisors D and D' do not have common components), we have $((C/2) \cdot D) \leq 0$, $((C/2)^2) \geq 0$. We set $C/2 = \Sigma s_i A_i + \Sigma s_j' A_j'$, where all the curves A_i, A_j' are irreducible, A_i is a component of D, and $\Sigma s_j' A_j'$ does not have any common components with D. Since the carrier of the divisor C_1 is connected, $(D \cdot \Sigma s_j' A_j') > 0$ if $\Sigma s_j' A_j' \neq 0$. On the other hand, $(\Sigma s_i A_i \cdot D) = (\Sigma s_i A_i D') \geq 0$. Hence $\Sigma s_j' A_j' \neq 0$ implies that $((C/2) \cdot D) > 0$, i.e. leads to a contradiction. Thus the carrier of the divisor $C/2$ is a subset of the carrier of the divisor D, and hence $((C/2) \cdot D) = ((C/2) \cdot D') \geq 0$. We thus obtain that $((C/2) \cdot D) = ((C/2)^2) = (D^2) = 0$, and the divisors D and D' do not have common points.

We will now show that $D = C/2$. By the Riemann-Roch theorem we have

$$\dim H^0\left(F, \Omega\left(\frac{C}{2}-D\right)\right)+\dim H^0\left(F, \Omega\left(D'-\frac{C}{2}\right)\right)\geqslant 1, \qquad (*)$$

$$\dim H^0\left(F, \Omega\left(D-\frac{C}{2}\right)\right)+\dim H^0\left(F, \Omega\left(\frac{C}{2}-D'\right)\right)\geqslant 1. \qquad (**)$$

The second terms in the left-hand sides of the inequalities $(*)$ and $(**)$ are equal to zero, for otherwise we would have $\dim H^0(F, \Omega(C')) \geq 2$, which contradicts the already-proved assertion 1). Hence both first terms are nonzero and $D \sim C/2$. If $D \neq C/2$, then $\dim|C/2| \geq 1$, which contradicts the irreducibility of the curve C.

The connectedness of the carriers of the divisors $C/2$ and $(C/2)' = D'$ also follows directly from the irreducibility of the curve C.

If, finally, H is an effective divisor and $2H \in |C|$, then by the Riemann-Roch theorem

$$\dim H^0\left(F, \Omega\left(H-\frac{C}{2}\right)\right)+\dim H^0\left(F, \Omega\left(\left(\frac{C}{2}\right)'-H\right)\right)\geqslant 1,$$

and if the first term is not equal to zero, $H = C/2$, while if the second term is not equal to zero $H = (C/2)'$. Proposition 2 is thus proved.

Corollary. *If C is a curve possessing the properties given in Proposition 2, and θ is an arbitrary divisor, then the index of intersection $(C \cdot \theta)$ is even.*

Proof. We have $(C \cdot \theta) = 2((C/2) \cdot \theta)$.

Definition. An Enriques surface F is said to be a surface of special type if there exists on it an irreducible linear pencil of elliptic curves C and a nonsingular rational curve θ, such that $(C \cdot \theta) = 2$.

In the future we will always assume unless something is said to the contrary that the surface F under consideration is not a surface of special type.

Proposition 3. *Let $|C|$ be a complete linear system and let $p_a(C) > 1$. Then the nonfixed part $|D|$ of the system $|C|$ is irreducible and $p_a(D) > 1$.*

Proof. We first assume that the system $|C|$ does not have fixed components and prove that it is irreducible. Let the system $|C|$ be reducible. Then (by Bertini's theorem and the regularity of the surface F) it is composed of the curves of (an irreducible) linear pencil H ($|C| = |nH|$, $(H^2) > 0$, $n > 1$). By the Riemann-Roch theorem we have $n = \dim|C| \geq n^2(H^2)/2 \geq n^2$. Now let the system $|C|$ be arbitrary. If $p_a(D) > 1$ then the system $|D|$ is irreducible by what has already been proved. We now show that $p_a(D) - 1 = (D^2) = 0$ leads to a contradiction. If it were not true, then $|D| = |nH|$, where H is an irreducible pencil of elliptic curves. Since, moreover, $(C^2) > 0$, the fixed part θ of the system $|C|$ is nonzero. Since $(\theta^2) \leq 2\dim|\theta| = 0$, we have $(D \cdot \theta) = ((C^2) - (\theta^2))/2 > 0$. There thus exists an irreducible component $\overline{\theta}$ of the divisor θ such that $(H \cdot \overline{\theta}) > 0$. Moreover, if the curve $\overline{\theta}$ is elliptic $((\overline{\theta}^2) = 0)$, then $(H \cdot \overline{\theta}) \geq 2$ (by Proposition 2), and if $\overline{\theta}$ is rational $((\overline{\theta}^2) = -2)$, then $(H\cdot\overline{\theta}) \geq 4$. In any case, by the Riemann-Roch theorem we have

$$n = \dim |D| = \dim |D + \bar{\theta}| \geqslant \frac{2n(H\theta) + (\bar{\theta}^2)}{2} \geqslant 2n.$$

This is the desired contradiction.

Proposition 4. *Let* $|C|$ *be an irreducible linear system and let* $p_a(C) > 1$. *Then* $\dim |C| = |C + K| = p_a(C) - 1$, *and the system* $|C'| = |C + K|$ *is also irreducible.*

Proof. By the theorem of Kodaira already mentioned, $H'(F, \Omega(C + K)) = 0$, and thus $\dim C' = \dim H^0(F, \Omega(C')) - 1 = p_a(C') - 1 = p_a(C) - 1$. We set $|C'| = |D| + \theta$, where $|D|$ and θ respectively are the nonfixed and fixed parts of the system $|C'|$. By Proposition 3, the linear system $|D|$ is irreducible and $p_a(D) > 1$. On the other hand, $p_a(D) - 1 \leq \dim |D| = \dim |C'| = p_a(C) - 1$. Thus

$$1 < p_a(D) \leq p_a(C).$$

We will now prove our proposition by induction on the number $p_a(C)$. Let $p_a(C) = 2$. Then $p_a(D) = 2$, and hence $(D^2) = (C^2) = ((C + \theta)^2)$, $2(D\cdot\theta) + (\theta^2) = 0$.

If $(\theta^2) < 0$, then $(D\cdot\theta) > 0$, and hence $0 > (D\cdot\theta) + (\theta^2) = ((D + \theta)\cdot\theta) = (C\cdot\theta)$, which is impossible since the system $|C|$ is irreducible.

If $(\theta^2) = 0$ but $\theta \neq 0$, then by the Riemann-Roch theorem there exists an effective divisor $\theta' \sim \theta + K$. Since $\theta' \neq \theta$ and $2\theta \sim 2\theta'$, the linear system $|2\theta|$ is at least one-dimensional, and at least one curve of this system passes through each point of the surface F. Since $(D\cdot\theta) = 0$, and the system $|D|$ is irreducible, there exists a divisor $\widetilde{\theta} \in |2\theta|$ that can be represented in the form $\widetilde{\theta} = D_0 + H$, where $D_0 \in |D|$ and $H > 0$. Since $(D^2) > 0$ and $(\theta^2) = 0$ we have $(D^2) + 2(D\cdot H) + (H^2) = 0$, $2(D\cdot H) + (H^2) < 0$, $(D\cdot H) + (H^2) = ((D + H)\cdot H) < 0$ (since $(D\cdot H) \geq 0$ by the irreducibility of $|D|$). This means that $((D + H)\cdot D) = (D + H)^2 - ((D + H)\cdot H) = (\widetilde{\theta}^2) - ((D + H)\cdot H) > 0$, i.e. $(D\cdot\theta) > 0$, which contradicts our assumptions.

We note further that $(\theta^2) \leq 2\dim|\theta| = 0$. Thus, it is impossible that $(\theta^2) < 0$ and $(\theta^2) > 0$ and $(\theta^2) = 0$, for $\theta \neq 0$. Thus $\theta = 0$, and hence the system $|C'| = |D|$ is irreducible, $H'(F, \Omega(C' + K)) = H'(F, \Omega(C)) = 0$, from which we obtain our assertion for $p_a(C) = 2$ by using the Riemann-Roch theorem.

We will now show, using the induction assumption, that it is always true that $p_a(D) = p_a(C)$. In fact, let $p_a(D) < p_a(C)$; then by the induction assumption $\dim |D| = p_a(D) - 1 < p_a(C) - 1 = \dim |C'|$. This is a contradiction.

Now, repeating the argument given in the case $p_a(C) = 2$, we obtain that $\theta = 0$, the system $|C'|$ is irreducible, and, applying the theorem of Kodaira, that

$\dim |C| = p_a(C) - 1$. Proposition 4 is proved.

§2. The basic construction

1. Let C be an irreducible curve, and let $p_a(C) = \pi > 1$. By Propositions 3 and 4 of §1, $\dim |C| = \pi - 1$, $\dim |2C| = 4\pi - 4$ and the system $L = |2C|$ is irreducible. In the space of parameters $\mathbf{P}^{4\pi-4}$ of the system L we consider the subsets S_1 and S_2 corresponding to curves of the system $|2C|$ that are representable in the form $C_1 + C_2$, $C'_1 + C'_2$ respectively, where $C_i \in |C|$, $C'_i \in |C'|$, $i = 1, 2$.

A) *Each of the subsets S_1, S_2 is a $(2\pi - 2)$-dimensional algebraic subvariety in $\mathbf{P}^{4\pi-4}$.*

For the proof of this assertion we fix the divisors $C_1, C_2 \in |C|$. Let the functions $f_0 = 1, f_1, \cdots, f_{\pi-1}$ form a basis of the vector space $L(C_1)$ consisting of the functions f such that $(f) + C_1 > 0$.

Moreover, let us assume that the function f_1 brings about the equivalence of the divisors $C_1 \sim C_2$ and that a common component of these divisors is either empty or is an isolated curve. We consider the functions $g_0 = 1/f_1$, $g_1 = 1, \cdots$, $g_{\pi-1} = f_{\pi-1}/f_1$ as a basis of the space $L(C_2)$. It is not difficult to see that under our assumption about the divisors C_1 and C_2 the functions $1 = f_0$, $f_1, \cdots, f_{\pi-1}, g_0, g_2, \cdots, g_{\pi-1}$ are linearly independent, and thus the space $L(C_1 + C_2)$ admits the basis $h_0 = 1$, $h_1 = f_1, \cdots, h_{\pi-1} = f_{\pi-1}$, $h_\pi = g_0$, $h_{\pi+1} = g_2, \cdots, h_{2\pi-2} = g_{\pi-1}, h_{2\pi-1}, \cdots, h_{4\pi-4}$. A point $(\nu_0 : \cdots : \nu_{4\pi-4}) \in \mathbf{P}^{4\pi-4}$ belongs to the set S_1 if and only if there exist points $(\alpha_0 : \cdots : \alpha_{\pi-1})$, $(\beta_0 : \cdots : \beta_{\pi-1}) \in \mathbf{P}^{\pi-1}$, such that $\nu_0 h_0 + \cdots + \nu_{4\pi-4} h_{4\pi-4} = \gamma(\alpha_0 f_0 + \cdots + \alpha_{\pi-1} f_{\pi-1}) \times (\beta_0 g_0 + \cdots + \beta_{\pi-1} g_{\pi-1})$. Setting $f_i g_j = \sum_{k=0}^{4\pi-4} C_k^{ij} h_k$, removing parentheses and equating the coefficients of h_k in the left and right-hand sides, we obtain a parametric equation of the set S_1:

$$\nu_k = \gamma \sum_{i,j} C_k^{ij} \alpha_i \beta_j, \quad k = 0, \ldots, 4\pi - 4. \tag{$*$}$$

We note here that with our choice of bases these equations are symmetric with respect to the indices i, j (since $f_i g_j = f_j g_i = f_i f_j / f_1$) and that the coordinates of any divisor of $|C|$ in the chosen bases of the spaces $L(C_1)$ and $L(C_2)$ are identical.

Making an analogous construction for the set S_2, we find that both sets are algebraic and of dimension $2\pi - 2$.

We now study the sets of the singular points of the subvarities S_1 and S_2.

B) *Let C_1 and C_2 be curves of the system $|C|$, a common part of which is either empty or is an isolated curve. Let, moreover, the divisor $C_1 + C_2$ be written uniquely (up to permutation of the terms) in such a form. Then the point of* *the variety* S_1 *corresponding to the curve* $C_1 + C_2 \in |2C|$ *is nonsingular.*

For the proof we continue the analysis of the equations (*). We note that $C_k^{i1} = C_k^{1i} = \delta_k^i$, $C_k^{j0} = C_k^{0j} = \delta_{k-\pi+1}^j$ $(j \neq 0, 1)$, $C_k^{00} = \delta_k^\pi$. In the basis chosen in the space $L(C_1)$ the divisor C_1 has the coordinates $(1: 0: \cdots : 0)$, and the divisor C_2 the coordinates $(0: 1: 0: \cdots : 0)$; in the basis constructed for the space $L(C_1 + C_2)$ the divisor $C_1 + C_2$ has the coordinates $(1: 0: \cdots : 0)$.

We consider the symmetric square T of the space $\mathrm{P}^{\pi-1}$ and the projective space P^{π^2-1} with the homogeneous coordinates $(\mu_{ij}, 0 \leq i, j \leq \pi - 1)$. The formulas

$$\mu_{ij} = \alpha_i\beta_j + \alpha_j\beta_i,\ i \neq j,$$
$$\mu_{ii} = 2\alpha_i\beta_i$$

determine an imbedding of the variety T into P^{π^2-1}, where singular points of T can only be points of the diagonal. We can rewrite the parametric equations (*) in the form

$$\nu_k = \gamma \sum_{i,j} C_k^{ij} (\alpha_i\beta_j + \alpha_j\beta_i),\ k = 0, \ldots, 4\pi - 4;$$

then the formulas

$$\nu_k = \gamma \sum_{i,j} C_k^{ij}\mu_{ij},\ k = 0, \ldots, 4\pi - 4$$

determine a projection $\mathrm{P}^{\pi^2-1} \to \mathrm{P}^{4\pi-4}$ mapping the symmetric product T onto the variety S_1.

The point $P \in T \subset \mathrm{P}^{\pi^2-1}$ corresponding to the pair of divisors (C_1, C_2) does not lie on the diagonal of T, is nonsingular in T, and has in P^{π^2-1} the coordinates $\mu_{00} = 0$, $\mu_{10} = \mu_{01} = \gamma$, $\mu_{ij} = 0$ $(i^2 + j^2 > 1)$.

Since the point $(1: 0: \cdots : 0) \in \mathrm{P}^{4\pi-4}$ is the image of a single point of T (namely of the point P), it follows that for the proof of its regularity in S_1 we must establish that the hyperplane in P^{π^2-1} projecting the point P is not tangent to T at this point. Substituting the values of the coordinates ν_i, we find the equation of this hyperplane:

$$0 = \sum C_k^{ij}\mu_{ij},\ 0 < k \leqslant 4\pi - 4.$$

In the nonhomogeneous coordinates of P^{π^2-1} in a neighborhood of the point P these equations can be written in the form

$$\sum_{(i,j)\neq(0,1),\,(1,0)} C_k^{ij} u_{ij} = 0,\ k = 1, \ldots, 4\pi - 4, \qquad (**)$$

where $u_{ij} = \mu_{ij}/\mu_{01}$, and in the right-hand sides there are zeros, since

$C_k^{10} = C_k^{01} = \delta_k^0 = 0$ for $k \geq 1$. On the variety T

$$u_{ij} = \frac{\mu_{ij}}{\mu_{01}} = \frac{\alpha_i\beta_j + \alpha_j\beta_i}{\alpha_0\beta_1 + \alpha_1\beta_0} = \frac{x_iy_j + x_jy_i}{1 + x_1y_0},$$

where we have set $x_0 \equiv y_1 \equiv 1$, and x_i, y_j $(i = 1, \cdots, \pi - 1;\ j = 0, 2, \cdots, \pi - 1)$ are the nonhomogeneous coordinates in neighborhoods of the points $(1: 0: \cdots: 0)$ and $(0: 1: 0: \cdots: 0) \in P^{\pi-1}$. Let $u_{10}(t)$, $u_{ij}(t)$ $(i^2 + j^2 > 1)$ be the curve on T tangent to the projecting hyperplane at the point P. We note that $u_{ij}(0) = x_i(0) = y_j(0) = 0$ $(i^2 + j^2 \neq 1,\ i \neq 0,\ j \neq 1)$. As it is not difficult to calculate,

$$\frac{du_{1i}(0)}{dt} = \frac{du_{i1}(0)}{dt} = \frac{dx_i(0)}{dt} \quad (i \neq 0, 1),$$

$$\frac{du_{0j}(0)}{dt} = \frac{du_{j0}(0)}{dt} = \frac{dy_j(0)}{dt} \ (j \neq 0, 1),$$

$$\frac{du_{11}(0)}{dt} = 2\frac{dx_1(0)}{dt}, \quad \frac{du_{00}(0)}{dt} = 2\frac{dy_0(0)}{dt}, \quad \frac{du_{ij}(0)}{dt} = 0$$

in the other cases.

The coordinates $du_{ij}(0)/dt$ of the vector tangent at the point P to the curve $u_{ij}(t)$ must satisfy the system (**). After a substitution, we obtain

$$0 = \sum_{(i,j)\neq(0,1),(1,0)} C_k^{ij}\frac{du_{ij}(0)}{dt} = \sum_{j\neq 1}(C_k^{0j} + C_k^{j0})\frac{dy_j(0)}{dt}$$

$$+ \sum_{i\neq 0}(C_k^{1i} + C_k^{i1})\frac{dx_i(0)}{dt} = 2\sum_{j>1}\delta_{k-\pi+1}^j\frac{dy_j(0)}{dt} + 2\sum_{i\neq 0}\delta_k^i\frac{dx_i(0)}{dt} + 2\delta_{k-\pi}^0\frac{dy_0(0)}{dt},$$
$$k = 1, \ldots, 4\pi - 4.$$

Giving k the values from 1 to $\pi - 1$, we obtain $dx_i(0)/dt = 0$, $i = 1, \cdots, \pi - 1$. Giving k the values $\pi, \cdots, 2\pi - 2$, we obtain $dy_j(0)/dt = 0$, $j = 0, 2, \cdots, \pi - 1$. Thus our tangent vector turns out to be zero. Assertion B) is proved.

C) *The degree of the subvariety* S_1 *is not less than* $\frac{1}{2}\binom{2\pi - 2}{\pi - 1}$.

For the proof we choose on the surface $F(2\pi - 2)$ points $Q_1, \cdots, Q_{2\pi-2}$ such that

1) the hyperplane $E^{2\pi-2}$ in $P^{4\pi-4}$ corresponding to the curves of $L = |2C|$ passing through all these points has dimension $(2\pi - 2)$;

2) a single curve of the system $|C|$ passes through any $(\pi - 1)$ points of the system $Q_1, \cdots, Q_{2\pi-2}$. Moreover, different curves of the system $|C|$ must correspond to different choices of the $(\pi - 1)$ points.

It is not difficult to construct such a system of points $Q_1, \cdots, Q_{2\pi-2}$ by induction: 1) as Q_1 we take any point of the surface that is a base point for neither the system $|C|$ nor the system L; 2) let the points $Q_1, \cdots, Q_s$ already be chosen such that a) the system of curves of $|2C|$ passing through all these points has dimension $4\pi - 4 - s$, and b) the system of curves of $|C|$ passing through the points $Q_{i_1}, \cdots, Q_{i_k}$, $1 \le i_1 < \cdots < i_k \le s$, $k = 1, \cdots, \min(s, \pi - 1)$, has dimension $\pi - 1 - k$.

Then we can take as the point Q_{s+1} any point other than a) base points of the system of curves of L passing through all the points $Q_1, \cdots, Q_s$, and b) base points of the system of curves of $|C|$ passing through the points $Q_{i_1}, \cdots, Q_{i_k}$, $1 \le i_1 < \cdots < i_k \le s$, $k = 1, \cdots, \min(s, \pi - 1)$.

It is not difficult to calculate the number of points of the intersection of the hyperplane $E^{2\pi-2} \subset P^{4\pi-4}$ with the subvariety S_1. Such points are in one-to-one correspondence with the pairs $C_1, C_2 \in |C|$ that partition the set $Q_1 \cup \cdots \cup Q_{2\pi-2}$ into equivalent subsets, i.e. by the construction, the number of points is equal to the number of such partitions, which is clearly equal to $\frac{1}{2}\binom{2\pi-2}{\pi-1}$. We shall denote this quantity by $\mathfrak{S}$.

Hence, the degree of the subvariety S_1 (and, obviously, of the subvariety S_2) cannot be less than $\mathfrak{S}$, i.e. assertion C) is proved.

We note that, as follows from assertions A) and C), the subvarieties S_1 and S_2 have at least $\mathfrak{S}^2$ points of intersection (counting multiples). The curves of the system $|2C|$ corresponding to such points will be called special.

2. Now we show that on the surface F

1) *there exists a pair of elliptic curves with a positive index of intersection, each of which, being double, changes at an irreducible point;*

2) *every divisor with a positive arithmetic genus is equivalent to a sum of such pairs and some divisor* $\Gamma \ge 0$.

Remark. From assertion 2) and Propositions 3 and 4 of §1 it follows easily that the group of the classes of divisors on the surface F is generated by irreducible rational and elliptic curves.

Proof. Let C be a divisor with a positive arithmetic genus. Without loss of generality we can assume it to be irreducible (by Proposition 3 of §1). By the results of subsection 1 there exists a special curve $L_0 \in |2C|$, i.e. a curve representable both in the form

$$L_0 = C_1 + C_2, \quad C_1, C_2 \in |C|,$$

and in the form

$$L_0 = C_1' + C_2', \quad C_1', C_2' \in |C'|.$$

We denote by Γ the common component of the curves C_1, C_2, C_1', C_2', and we consider the linear system $|2C - 2\Gamma|$.

Since the curves $2(C_1 - \Gamma)$, $2(C_2 - \Gamma)$, $2(C'_1 - \Gamma)$, $2(C'_2 - \Gamma)$ belong to this system and do not have common components, the system $|2C - 2\Gamma|$ does not have fixed components (and is nonzero, since otherwise we would have $K \geq 0$, which is not true).

A) We consider the system of equations

$$D_1 + D_2 = C_1 - \Gamma, \quad D_1 + D_3 = C'_1 - \Gamma,$$
$$D_3 + D_4 = C_2 - \Gamma, \quad D_2 + D_4 = C'_2 - \Gamma \tag{*}$$

and set $C_1 - \Gamma = \Sigma q_i H_i$, $C_2 - \Gamma = \Sigma r_i H_i$, $C'_1 - \Gamma = \Sigma s_i H_i$, $C'_2 - \Gamma = \Sigma t_i H_i$, where the H_i are irreducible curves, $q_i, r_i, s_i, t_i \geq 0$, $q_i + r_i + s_i + t_i > 0$, $q_i + r_i = s_i + t_i$. We also write $D_1 = \Sigma x_i H_i$, $D_2 = \Sigma y_i H_i$, $D_3 = \Sigma z_i H_i$, $D_4 = \Sigma u_i H_i$. In this notation the system (*) takes the form

$$x_i + y_i = q_i, \quad x_i + z_i = s_i,$$
$$z_i + u_i = r_i, \quad y_i + u_i = t_i. \tag{**}$$

for each i.

Since the common component of the right-hand sides of the equations (*) is equal to zero, $q_i r_i s_i t_i = 0$, and this condition, as it is not difficult to show, is necessary and sufficient for the existence of a nonnegative integer solution $\tilde{x}_i, \tilde{y}_i, \tilde{z}_i, \tilde{u}_i$ of the equations (**) satisfying the conditions $\tilde{x}_i \tilde{u}_i = 0$, $\tilde{y}_i \tilde{z}_i = 0$. Setting $D_1 = \Sigma \tilde{x}_i H_i$, $D_2 = \Sigma \tilde{y}_i H_i$, $D_3 = \Sigma \tilde{z}_i H_i$, $D_4 = \Sigma \tilde{u}_i H_i$, we obtain a solution of the system (*) possessing the following properties:

a) $D_i \geq 0$,

b) the divisors D_1 and D_4 (D_2 and D_3) do not have common components.

On the other hand, we have

$$D_1 + D_3 \sim D_1 + D_2 + K, \quad D_2 + D_4 \sim D_2 + D_1 + K,$$

i.e. $D_3 \sim D_2 + K$, $D_4 \sim D_1 + K$. It now follows from Proposition 1 that $D_i > 0$ and $p_a(D_i) \geq 1$. If $p_a(D_1) > 1$ (or $p_a(D_2) > 1$), then we denote by $|{}_1C|$ the nonfixed part of the system $|D_1|$ ($|D_2|$). By Proposition 3 the system $|{}_1C|$ is irreducible and $p_a({}_1C) > 1$; by Proposition 4, $p_a({}_1C) < p_a(C)$ (since $\dim |{}_1C| < \dim |C|$).

B) Now let $(C - \Gamma)^2 > 0$. Then we have found either an irreducible curve ${}_1C$ whose arithmetic genus satisfies the inequalities $1 < p_a({}_1C) < p_a(C)$, or a pair of elliptic curves D_1, D_2 that do not have common components with their adjoint curves $D_4 \sim D_1 + K$, $D_3 \sim D_2 + K$.

In the last case we consider the systems $|2D_1|$, $|2D_2|$. Clearly, each of these systems does not have fixed components (since $2D_1, 2D_4 \in |2D_1|$, $2D_2$, $2D_3 \in |2D_2|$ do not have common components), and is thus composed of an irreducible pencil of elliptic curves ($|2D_1| = |n_1 L_1|$, $|2D_2| = |n_2 L_2|$). The curves $L_1/2$, $L_2/2$ (cf. Proposition 2) are elliptic, isolated, and have a positive index of intersection $L_1/2$, $L_2/2$ (since $(D_1 \cdot D_2) = ((C - \Gamma)^2/2) > 0$)). The curves $(L_1/2)'$, $(L_2/2)'$ possess the same properties. The carriers of the curves $L_1/2$, $(L_1/2)'$ $(L_2/2$, $(L_2/2)')$ do not have common points, and the linear systems L_1 and L_2 are one-dimensional and irreducible.

C) Let $(C - \Gamma)^2 = 0$ (since the system $|2C - 2\Gamma|$ does not have fixed components, $(2C - 2\Gamma)^2 \geq 0$). We have $(C^2) = ((D_1 + D_2 + \Gamma)^2) = ((C - \Gamma)^2) + 2(D_1 \cdot \Gamma) + 2(D_2 \cdot \Gamma) + (\Gamma^2) = 2(D_1 \cdot \Gamma) + 2(D_2 \cdot \Gamma) + (\Gamma^2) > 0$.

If $(\Gamma^2) > 0$, we denote by ${}_1C$ the nonfixed part of the system $|\Gamma|$. By Propositions 3 and 4 we have

$$1 < p_a({}_1C) < p_a(C).$$

If $2(D_1 \cdot \Gamma) > 0$, we denote by θ an irreducible component of the curve Γ for which $(D_1 \cdot \theta) > 0$, and by ${}_1C$ the nonfixed part of the system $|D_1 + \theta|$. Since the surface F is not a surface of special type, it follows easily from Propositions 3 and 4 that $1 < p_a({}_1C) < p_a(C)$.

D) Let us summarize the results given in A), B), and C): on a given irreducible curve C with $p_a(C) > 1$ we have found either a) an irreducible curve ${}_1C$ whose arithmetic genus satisfies the inequalities $1 < p_a({}_1C) < p_a(C)$ (from A) and part of B) and C)), or b) a pair of elliptic curves D_1, D_2 with a positive index of intersection possessing the following property: (I) *there exist irreducible pencils of elliptic curves* L_1, L_2 *such that* $D_1 = L_1/2$, $D_2 = L_2/2$ (cf. Proposition 2 of §1 and B)).

Repeating our process as long as we have case a) (because of the inequalities $1 < p_a({}_1C) < p_a(C)$ this can only be done a finite number of times), we finally arrive at case b).

§3. The representation of an arbitrary Enriques surface as a double plane

1. Here we will show that there exist on the surface F two elliptic curves with index of intersection 1 possessing the property (I).

Let D_1 and D_2 be the pair of elliptic curves obtained at the end of the preceding section, and let $(D_1 \cdot D_2) = s > 1$. We consider the linear system $|C| = |D_1 + D_2|$.

A) *The system* $|C|$ *is irreducible and* $p_a(C) > 2$. For, $p_a(C) = (C^2)/2 + 1 = (D_1 \cdot D_2) + 1 = s + 1 > 2$. Since the systems $L_1 = |2D_1|$, $L_2 = |2D_2|$ are irreducible and one-dimensional, the system $|2C| = |2D_1 + 2D_2|$ does not have fixed components and is irreducible by Proposition 3 of §1. By Proposition 4, $\dim|2C| = p_a(2C) - 1$.

We denote by $|\widetilde{C}|$ the nonfixed part of the system $|C|$. By Proposition 4, $p_a(C) - 1 \le \dim|C| = \dim|\widetilde{C}| = p_a(\widetilde{C}) - 1$. Hence $p_a(\widetilde{C}) \ge p_a(C)$. On the other hand, $\dim|2\widetilde{C}| \le \dim|2C|$, where equality is possible only when $|C| = |\widetilde{C}|$. We have

$$\dim|2\widetilde{C}| = 4\,(p_a(\widetilde{C}) - 1),\ \dim|2C| = 4\,(p_a(C) - 1).$$

Comparing the inequalities obtained, we conclude that $p_a(C) = p_a(\widetilde{C})$, $\dim|2C| = \dim|2\widetilde{C}|$, and, finally, $|C| = |\widetilde{C}|$.

We have thus proved that one can apply to the system $|C|$ the process described in §2.

One of the special curves of the system $|2C|$ is the curve $H = D_1 + D_2 + D'_1 + D'_2$. We will now show that the points corresponding to this curve on the varieties S_1, S_2 can be assumed to be nonsingular on these varieties.

B) *If at least one of the curves* D_1, D_2, D'_1, D'_2 *is reducible, there exists on the surface* F *an irreducible curve* ${}_1C < C$ *whose arithmetic genus satisfies the inequalities* $1 < p_a({}_1C) < p_a(C)$. *If all four curves* D_1, D_2, D'_1, D'_2 *are irreducible, the points on the varieties* S_1 *and* S_2 *corresponding to the special curve* $H \in |2C|$ *are nonsingular by assertion* B) *of* §2.1.

For example, let the curve D'_1 be irreducible and let θ be an irreducible component of it such that $(D'_2 \cdot \theta) > 0$. By the assumption of the nonspecialness of the surface F and from Proposition 2, $(D'_2 + \theta)^2 \ge 2$. Since $\dim|D'_2 + \theta| < \dim|C|$ the arithmetic genus of the irreducible nonfixed part $|{}_1C|$ of the system $|D'_2 + \theta|$ satisfies the inequalities $1 < p_a({}_1C) < p_a(C)$ (by Propositions 3 and 4).

C) *If the divisors* D_1, D_2, D'_1, D'_2 *are irreducible, then the curve* H *is not the only special curve of the system* $|2C|$.

Since by the formula of §2.1 C) the system $|2C|$ has at least $\frac{1}{4}\binom{2s}{s}^2 \geqslant 9$ (possibly multiple) special curves, it is sufficient for us to show that the point $Q \in S_1 \cap S_2$ corresponding to the curve H is not a multiple intersection of S_1 and S_2. We can assume here, as was shown above, that this point is nonsingular on the varieties S_1 and S_2.

Let us assume the contrary. Then there exist on the varieites S_1 and S_2 curves $Q_1(t), Q_2(t), 0 \le t \le 1$ tangent to each other at the point $Q = Q_1(0) = Q_2(0)$. Two curves $C_1(t)$ and $C_2(t)$ ($C'_1(t)$ and $C'_2(t)$) correspond to the curve $Q_1(t)$

$(Q_2(t))$ in the space of the parameters of the system $|C|$ $(|C'|)$. We can choose the indices in such a way that

$$C_1(0) = D_1 + D_2,\ C_2(0) = D_1' + D_2',\ C_1'(0) = D_1 + D_2',\ C_2'(0) = D_2 + D_1'.$$

We consider the ruled surface V_1 formed by the lines $L_1(t)$ passing through the points Q, $Q_1(t)$, and the analogously constructed surface V_2 formed by the lines $L_2(t)$. We denote by L the intersection of these surfaces (i.e. the line corresponding to the common tangent vector of the curves $Q_1(t), Q_2(t)$).

We will prove that all the divisors lying on the line L contain the curve H; this will be the desired contradiction.

Clearly, every divisor lying on the line $L_1(t)$ $(L_2(t))$ cuts out on the (irreducible) curve D_1 the same divisor as the curve $C_1(t) + C_2(t)$ $(C_1'(t) + C_2'(t))$. Thus, every divisor lying on the line L trivially cuts out on the curve D_1 the divisor

$$\text{least common multiple of } \{\lim_{t\to 0}(C_1(t) + C_2(t))D_1,\ \lim_{t\to 0}(C_1'(t) + C_2'(t))D_1\}.$$

Clearly, $H_1 = \lim_{t\to 0} C_2(t)\cdot D_1 = D_2'\cdot D_1$, $H_2 = \lim_{t\to 0} C_2'(t)\cdot D_1 = D_2\cdot D_1$. Since the curves D_2 and D_2' do not have common points $H_1 \cap H_2 = 0$.

If we can show that $\lim_{t\to 0} C_1(t)\cdot D_1 \neq D_2\cdot D_1$, it will then follow that the divisors of the line L do not have more than $2s = 2(C\cdot D_1)$ points in common with the curve D_1, i.e. these divisors contain D_1.

Let us assume, then, that $\lim_{t\to 0} C_1(t)\cdot D_1 = D_2\cdot D_1$. This means that the divisor $D_2\cdot D_1$ belongs to the linear system $|C\cdot D_1|$, i.e. $(C - D_2)\cdot D_1 = D_1\cdot D_1$ is a divisor equivalent to zero on D_1. From the exact sequence

$$0 \to H^0(F, \Omega(0)) \to H^0(F, \Omega(D_1)) \to H^0(D_1, \Omega_{D_1}(D_1|_{D_1})) \to H^1(F, \Omega(0)),$$

corresponding to the exact sequence of sheaves

$$0 \to \Omega(0) \to \Omega(D_1) \to \Omega_{D_1}(D_1|_{D_1}) \to 0,$$

we conclude, however, that since $\dim H^0(F, \Omega(0)) = \dim H^0(F, \Omega(D_1)) = 1$ (the curve D_1 is isolated), $\dim H^1(F, \Omega(0)) = 0$, it is also true that $\dim H^0(D_1, \Omega_{D_1}(D_1|_{D_1})) = 0$. (If the set $D_1 \cap D_2$ contains a singular point of the curve D_1, we can consider $\lim_{t\to 0} C_1'(t)\cdot D_1$ and the divisor $D_2'\cdot D_1$.)

It is proved analogously that the divisors lying on the line L contain the curves D_2, D_1', D_2'. Assertion C) is thus proved.

D) A special curve H_1 of the system $|2C|$ different from H permits one to find either an irreducible curve ${}_1C$ whose arithmetic genus satisfies the inequalities $1 < p_a({}_1C) < p_a(C)$, or a pair of elliptic curves D_3, D_4 with a positive index of intersection possessing property (I) of §2.2 D) and such that

$D_3 + D_4 + \Gamma \sim C$ (cf. A), B), C) of §2.2).

In the first case (here we eliminate also the curve ${}_1C$ that can have been obtained in B)), we apply to the curve ${}_1C$ our basic construction and obtain in the end a pair of elliptic curves possessing the property (I) with an index of intersection less than s.

We turn to the second case. We have

$$((D_1 + D_2)(D_3 + D_4)) = (D_1 \cdot D_3) + (D_2 \cdot D_4) + (D_2 \cdot D_3) + (D_1 \cdot D_4) = (C^2) - (C \cdot \Gamma) \le 2s. \quad (*)$$

If one of these indices of intersection, for example $(D_1 \cdot D_3)$, is equal to zero, then the corresponding irreducible pencils of elliptic curves ($L_1 = |2D_1|$, $L_3 = |2D_3|$ in our case) coincide.

It thus follows from Proposition 2 of §1 that $L_1/2 = D_1$, $L_3/2 = D_3$ either coincide or are adjoint to each other. Hence only one of the indices of intersection in equation (*) can be equal to zero (since the curves H and H_1 are distinct). Therefore one of these indices of intersection is positive, but does not exceed $s - 1$.

Thus for each pair of elliptic curves possessing property (I) with index of intersection greater than one, we have found another pair of elliptic curves possessing the same property and whose index of intersection is positive but less than the original one.

Repeating this process we finally find the curves whose existence is asserted in the beginning of this section.

2. Here we obtain a representation of the Enriques surface F in the form of a double plane with a branch curve of eighth degree that decomposes into a curve of sixth degree and two lines.*

A) We first assume that F is not a surface of special type.

We then consider the linear system $C = |D_1 + D_2 + D'_1|$, where D_1 and D_2 are elliptic curves with an index of intersection of 1, whose existence was proved above. It is not difficult to prove (and this will be done in the beginning of the next section), that each of the curves D_1, D_2, D'_1, D'_2 are irreducible, and $\dim |D_1 + D_2| = 1$. Since $(C^2) = 4$, we have $\dim |C| \ge 2$ and since the system C cannot have fixed components, $\dim |C| = 2$, by Proposition 4 of §1. Each of the three following curves belongs to the system C: $D_1 + D_2 + D'_1$, $2D_1 + D'_2$, $2D'_1 + D'_2$; on the other hand, these three curves do not belong to any

* *Added in proof*: More precise statements and detailed proofs of the theorems in this subsection, and also proofs of the converse existence theorems, will be found in my article: B. G. Averbuh, *On special types of Kummer and Enriques surfaces*, Izv. Akad. Nauk SSSR 29 (1965), 1095–1118. *Translator's note*: This article had been translated as the Appendix to the present volume.

one-dimensional subsystem of this system: the curves $2D_1 + D'_2$ and $2D'_1 + D'_2$ generate the pencil $|2D_1| + D'_2$, to which the curve $D_1 + D_2 + D'_1$ does not belong. Hence each of the points of $D_1 \cap D'_2$ and $D'_1 \cap D'_2$ is a base point of the system C. Since all such points must obviously lie on the curve D'_2, and $(C \cdot D'_2) = 2$, the system C has no other base points. Thus the number of variable points of the intersection of the curves of this system is equal to two.

Let the functions $1, x, y$ on the surface F form a basis of some space $L(C_0)$, where C_0 is an irreducible divisor of the system C. Since the system C maps the surface F onto a plane, the functions x, y generate in the field $k(F)$ of functions on our surface a rational subfield $k(x, y)$. By what has been said about the variable points of intersection, the field $k(F)$ is an extension of degree two of this subfield. Thus our surface is birationally equivalent to a double plane $z^2 = F^n(x, y)$. In order to determine n we note that every divisor $\{ax + by + c = 0\}$ belongs to the system C, and therefore has a genus of 3. Hence $n = 8$.

The images of the points of $D_1 \cap D'_2$ and $D'_1 \cap D'_2$ on our plane are lines belonging to the branch curve. Hence this branch curve decomposes into a curve of sixth degree and two lines.

B) Now let the surface F be a surface of special type, and let the irreducible pencil of elliptic curves L have an index of intersection of 2 with a rational curve θ. We will show that the system $|L + \theta + K + L/2|$ plays the same role in this case as the system C in the proof of A).

We first consider the linear system $D = |L + \theta + K|$. By the theorem of Kodaira already mentioned many times, $H^1(F, \Omega(D)) = 0$, and hence by the Riemann-Roch theorem $\dim |D| = (D^2)/2 = 1$. We will prove that the system D is irreducible. Let $|H|$ and G be its nonfixed and fixed parts respectively. Since the pencil L is irreducible, we have $(L \cdot H) > 0$ and $(L \cdot G) \geq 0$ (if $(L \cdot H) = 0$, the systems L and $|H|$ coincide, because of the equality of dimensions, but then $G \sim \theta + K$, which is impossible by the corollary to Proposition 1). Since $(D \cdot L) = 2$ and $(L \cdot H)$ is even (by the corollary to Proposition 2) we have $(L \cdot H) = 2$ and $(L \cdot G) = 0$. The curve θ cannot be contained in the fixed part of the system D, since $\dim L' = 0$ (by Proposition 2), and hence $(H \cdot \theta), (G \cdot \theta) \geq 0$. Since $(D \cdot \theta) \geq 0$, we have $(H \cdot \theta) = (G \cdot \theta) = 0$. Now

$$(H^2) + (H \cdot G) = (H \cdot D) = (H \cdot (L + \theta)) = 2, \ (H \cdot G) + (G^2) = (G \cdot D) = (G \cdot (L + \theta)) = 0.$$

The following cases are thus possible:

1) $(H^2) = 0, (H \cdot G) = 2, (G^2) = -2$ and 2) $(H^2) = 2, (H \cdot G) = 0, (G^2) = 0$.

In case 1) the system $|H|$ is an irreducible pencil of elliptic curves, which is impossible by the corollary of Proposition 2, since $(H \cdot (L/2)) = 1$.

Case 2) thus holds. We will show that $G = 0$. Since the divisor $L/2 + (L/2)' + \theta$ belongs to the system D and θ is not contained in G, we have $G \leq L/2 + (L/2)'$. Since $((L/2)^2) = 0$ and the carrier of the divisor $L/2$ is connected, this divisor does not contain a component (even a reducible one) with a zero index of self-intersection. Since the same may be said about the divisor $(L/2)'$ and the divisor G decomposes into two components (g.c.d. $(G, L/2)$, g.c.d. $(G, (L/2)')$) without common points and has a zero index of self-intersection, we have $G = 0$. The pencil D is thus irreducible.

We note that this pencil has two base points not lying on the curve θ $((D \cdot \theta) = 0)$: namely, if D_0 is an irreducible curve of this pencil, then the base points are points $D_0 \cap L/2$, $D_0 \cap (L/2)'$, since both these points also lie on the curve $L/2 + (L/2)' + \theta \in D$.

We consider, finally, the linear system $C = |D + L/2|$. Since $(C^2) = 4$, $\dim |C| \geq 2$. On the other hand, the dimension of this system must be smaller than the dimension of the irreducible system $|D + L| = |2L + \theta + K|$.

According to the theorem of Kodaira, $H'(F, \Omega(D + L)) = 0$, and since $((D + L)^2) = 6$, the dimension of this system is equal to three. Thus $\dim |C| = 2$.

The system C contains the one-dimensional subsystems $|D| + L/2$ and $|L| + \theta + (L/2)'$. Hence this system does not have fixed components, is not composed of a pencil, and thus is irreducible. The base sets of these subsystems intersect, but only at the points $D_0 \cap (L/2)'$, $L/2 \cap \theta$. These two points are the unique base points of the system C. Now repeating the corresponding parts of the arguments of part A), we can show that our Enriques surface of special type F is birationally equivalent to a double plane with a branch curve of eighth degree that decomposes into a curve of sixth degree and two lines.

§4. The representation of a general Enriques surface in the form of a surface of sixth degree in P^3 that twice passes through the edges of some tetrahedron

1. We now give some properties of the curves D_1 and D_2 that were constructed in §3.1.

A) *If two isolated elliptic curves* D_1 *and* D_2 *possessing property* (I) *of* §2.2 D) *have an index of intersection of* 1, *then* 1) *each of the curves* D_1, D_2, D'_1, D'_2 *is irreducible;* 2) *the linear systems* $|D_1 + D_2|$, $|D_1 + D'_2|$ *are irreducible, one-dimensional, and have two base points each:* $D_1 \cap D'_2$ *and* $D'_1 \cap D_2$ *in the first case, and* $D_1 \cap D_2$ *and* $D'_1 \cap D'_2$ *in the second.*

Proof. 1) Let us assume that the curve D_1 is reducible. Since $(D_1 \cdot D_2) = 1$, the curve D_1 cannot have the form $s\theta$, $s > 1$, where θ is an elliptic curve;

hence it has a rational component. Since the pencil $|2D_2|$ is irreducible, $(D_2 \cdot \theta) \geq 0$ for every component θ of the curve D_1. Moreover, among these components one can find one for which $(D_2 \cdot \theta) > 0$. But in this case we have $(D_2 \cdot D_1) \geq (D_2 \cdot \theta) \geq 2$ (we recall that the surface F under consideration is not a surface of special type). We obtain a contradiction. The irreducibility of the curves D_2, D'_1, D'_2 is proved analogously.

2) It follows from the first part of our assertion (just proved) that the system $|D_1 + D_2|$ does not have fixed components. Since $((D_1 + D_2)^2) = 2$, it follows from Propositions 3 and 4 of §1 that this system is irreducible and one-dimensional. The curves $D_1 + D_2$ and $D'_1 + D'_2$ belong to this system and have two common points: $D_1 \cap D'_2$ and $D'_1 \cap D_2$. Hence both these points are base points of the system $|D_1 + D_2|$. One proves analogously the assertion about the system $|D_1 + D'_2| = |D'_1 + D_2|$.

B) *The curves* D_1, D_2, D'_1, *and* D'_2 *form a quadrangle with the vertices* $P_1 = D_1 \cap D_2$, $P_2 = D_1 \cap D'_2$, $P_3 = D'_1 \cap D_2$, $P_4 = D'_1 \cap D'_2$. *If an isolated elliptic curve* D_3 *possessing property* (I) *of* §2.2 D) *has an index of intersection of one with each of the curves* D_1 *and* D_2 *and passes through one of the vertices of the quadrangle, then* 1) *there exists an effective divisor* Γ *such that*

$$D_3 + \Gamma \sim D_1 + D_2 + \varepsilon K, \quad \varepsilon = 0, 1, \quad D_1\Gamma = D_2\Gamma = 0;$$

2) *the curve* D_3 *also passes through the opposite vertex of the quadrangle, i.e. is a diagonal of it, and the curve* D'_3 *is its other diagonal.*

Proof. 1) Let us assume that the curve D_3 passes through the vertex $P_1 = D_1 \cap D_2$. Then the divisors $D_3 \cdot D_2$ and $D_1 \cdot D_2$ on the curve D_2 are equivalent, and hence $\dim H^0(D_2, \Omega_{D_2}(D_3 - D_1|_{D_2})) = 1$. From the exact sequence

$$0 \to H^0(D_2, \Omega_{D_2}(D_3 - D_1|_{D_2})) \to H'(F, \Omega(D_3 - D_1 - D_2)),$$

corresponding to the exact sequence of sheaves

$$0 \to \Omega(D_3 - D_1 - D_2) \to \Omega(D_3 - D_1) \to \Omega_{D_2}(D_3 - D_1|_{D_2}) \to 0$$

(clearly, $H^0(F, \Omega(D_3 - D_1)) = 0$), we conclude that

$$\dim H'(F, \Omega(D_3 - D_1 - D_2)) \geqslant 1.$$

By the Riemann-Roch theorem

$$\dim H^0(F, \Omega(D_3 - D_1 - D_2)) + \dim H^0(F, \Omega(D_1 + D_2 + K - D_3)) = \frac{((D_3 - D_1 - D_2)^2)}{2} + 1 + \dim H'(F, \Omega(D_3 - D_1 - D_2)).$$

Since $((D_3 - D_1 - D_2)^2) = -2$ and $H^0(F, \Omega(D_3 - D_1 - D_2)) = 0$, we have

$\dim H^0(F, \Omega(D_1 + D_2 + K - D_3)) \geq 1$, and hence there exists an effective divisor Γ such that $D_3 + \Gamma \sim D_1 + D_2 + K$. We consider the indices of the intersections of the left- and right-hand sides of this relationship with the divisor D_1 (D_2), and verify that $(D_1 \cdot \Gamma) = 0$ $((D_2 \cdot \Gamma) = 0)$.

2) The vertex $P_4 = D'_1 \cap D'_2$, opposite the vertex P_1, is a base point of the system $|D_1 + D_2 + K| = |D_1 + D'_2|$ and thus lies on the curve $D_3 + \Gamma$. Since $(\Gamma \cdot D_1) = (\Gamma \cdot D_2) = 0$, and D_1 and D_2 are irreducible and isolated, this point does not lie on the curve Γ, and hence lies on the curve D_3. It follows analogously from $D'_3 + \Gamma \sim D_1 + D_2$ that the curve D'_3 is the second diagonal of the quadrangle.

Assertion B) is thus proved.

2. In this subsection we construct a linear system mapping the surface F onto a surface of sixth degree in the space $\mathbf{P}^3$ that twice passes through the edges of a tetrahedron. For this it is necessary for us to find a triple of isolated (irreducible) elliptic curves D_1, D_2, D_3 such that $(D_i \cdot D_j) = 1 - \delta^i_j$, $i, j = 1, 2, 3$, and moreover such that no three of the curves D_i, D_i, $i = 1, 2, 3$, pass through one point.

A) *There exists on the surface* F *an isolated elliptic curve possessing property* (I) *of* §2.2 D) *and distinct from the curves* D_1, D_2, D'_1, D'_2.

Proof. Let us assume that such a curve does not exist. We will show that then every divisor on the surface F is equivalent to a divisor of the form $mD_1 + nD_2 + \epsilon K$, $\epsilon = 0, 1$.

The first step of the induction on the value of the arithmetic genus is the proof of the absence of irreducible rational curves on the surface F. Let θ be an irreducible rational curve on the surface F. If $(D_1 \cdot \theta) = 0$, then the curve θ is a component of some divisor of the pencil $|2D_1|$. In this case the other components of this divisor are also rational, and there thus exists on the surface F a rational curve θ_1 that has a positive index of intersection with one of the divisors D_1, D_2. We have $p_a(D_1 + \theta_1) > 1$, if $D_1 \cdot \theta_1 > 0$; moreover, the system $|D_1 + \theta_1|$ is irreducible in this case, and the basic construction can be applied to it. We obtain a pair of distinct elliptic curves possessing property (I), each of which is clearly different from the curves D_1, D'_1, which contradicts our assumption.

If $p_a(\theta) = 1$, and θ is an irreducible isolated divisor, then, by assumption, θ coincides with one of the divisors D_1, D'_1, D_2, D'_2.

If $p_a(\theta) = 1$, and θ changes in the pencil, then the same can be said about the divisor $\theta/2$.

If, finally, the arithmetic genus of the irreducible curve θ is greater than one, it is equivalent, by the basic construction, to some reducible divisor, all of whose components have, by Proposition 4 of §1, a smaller arithmetic genus, and we can apply the induction assumption.

We now obtain a contradiction from what has been shown. It is known that a form of type (1, 1) represents an algebraic cycle if and only if it represents an integral class of cohomologies. Since the geometric genus of our surface is equal to zero, every closed 2-form has type (1, 1). Knowing the values of $p_a(F)$ and (K^2), it is easy to calculate that the second Betti number of the surface F is equal to 10, and thus it contains 10 independent algebraic cycles.

B) *Let* D_1, D_2, D_3 *be isolated elliptic curves possessing property* (I). *Let* $(D_i \cdot D_j) = 1 - \delta_j^i$, *and let the curve* D_3 *pass through one of the vertices of the quadrangle formed by the curves* D_1, D_2, D_1', D_2'. *Then there exists on the surface* F *an isolated elliptic curve* D_4 *also possessing property* (I) *and distinct from the curves* D_1, D_2, D_3 *and their adjoint curves.*

Proof. We first note that by assertion B) 1) of subsection 1 the condition of our assertion is symmetric with respect to the curves D_1, D_2, D_3. We thus have $D_1 + \Gamma_1 \sim D_2 + D_3$, $D_2 + \Gamma_2 \sim D_1 + D_3$, $D_3 + \Gamma_3 \sim D_1 + D_2$, where $\Gamma_1, \Gamma_2, \Gamma_3 > 0$, $(\Gamma_i \cdot D_j) = 0$ for $i \neq j$ (if necessary we replace some of the curves D_1, D_2, D_3 by their adjoint curves). It is not difficult to see, moreover, that $(\Gamma_i \cdot D_i) = 2$ and $(\Gamma_i^2) = -2$. From the last equation it follows that each of the divisors contains rational components. Since each of the components of the divisor Γ_i is a component of some divisor of the pencil $|2D_j|$ $(j \neq i)$ that is distinct from the divisors $2D_j$, $2D_j'$, all of the components of the curve Γ_i are rational.

If the divisor Γ_i is reducible, we denote by θ_i its irreducible component for which $(D_i \cdot \theta_i) > 0$. It is possible to apply the basic construction to the irreducible linear system $|D_i + \theta_i|$, whose arithmetic genus is greater than one; we find isolated elliptic curves H_1, H_2 possessing the property (I) and a divisor $G \geq 0$ such that $D_i + \theta_i \sim H_1 + H_2 + G$. If the divisors H_1, H_2 were contained among the divisors D_1, D_2, D_3 and their adjoint divisors, we would obtain $\theta_i \sim \Gamma_i + G + \epsilon K$, $\epsilon = 0, 1$. For $\epsilon = 0$ this contradicts the reducibility of the curve Γ_i, and for $\epsilon = 1$ the corollary to Proposition 1 of §1.

We would argue analogously in the case when there exists on the surface F an irreducible rational curve θ different from Γ_i such that $(D_i \cdot \theta) > 0$.

We note, finally, that if the curves $\Gamma_1, \Gamma_2, \Gamma_3$ are irreducible, then arguing as in the proof of assertion A) we can show that there exists on the surface F either the desired elliptic curve, or one more irreducible rational curve. This rational curve θ must have, by what has been proved, a zero index of intersection with each of the curves D_1, D_2, D_3. Since $((D_1 + D_2 - \theta)^2) = 0$, there exists an effective divisor $\tilde{D}_4 \sim D_1 + D_2 - \theta$. By the basic construction we can assume that all the irreducible components of the divisor $\tilde{D}_4$ either are rational, or are elliptic and isolated. It is clear, moreover, that none of the curves $D_1, D_2, D'_1, D'_2, \Gamma_3$ can be a component of this divisor (since each of the divisors $D_i - \theta$, $D'_i - \theta$ is not equivalent to an effective divisor). Since $(D_3 \cdot \tilde{D}_4) = 2$, the divisor $\tilde{D}_4$ contains a rational component (different from Γ_3) having a positive index of intersection with the divisor D_3, or contains the desired elliptic curve D_4. This completes the proof of assertion B).

C) *Let D_1 and D_2 be isolated elliptic curves with an index of intersection of 1. If there exists an isolated elliptic curve M possessing the property* (I) *of* §2.2 D) *and such that one of the numbers $(M \cdot D_1)$, $(M \cdot D_2)$ is greater than one, then there exists a four-tuple of isolated elliptic curves H_1, H_2, H_3, H_4 possessing property* (I) *and such that $(H_i \cdot H_j) = 1 - \delta^i_j$ $((i, j) \neq (3, 4))$.*

Proof. Let $(M \cdot D_1) = a$, $(M \cdot D_2) = b$, $a \geq b$, $a > 1$. We consider the linear system $|M + D_1|$. By assertion A) of §3.1 this system is irreducible. By assertion B) of §3.1 either the divisors M, M' are irreducible, or there exists an irreducible linear system $|C| < |M + D_1|$ whose arithmetic genus satisfies the inequalities $1 < p_a(C) < p_a(M + D_1)$. In the first case the system $|2(M + D_1)|$ contains a special curve different from the curve $M + D_1 + M' + D'_1$ (cf. assertion C) of §3.1).

Now considering this special curve or a special curve of the system $|2C|$, we find by the basic construction a pair of isolated elliptic curves M_1, M_2 and a divisor $\Gamma \geq 0$ such that $M + D_1 \sim M_1 + M_2 + \Gamma$. We now have

$$(M_1 \cdot D_1) + (M_2 \cdot D_1) = a - (\Gamma \cdot D_1),\ (M_1 \cdot D_2) + (M_2 \cdot D_2) = b + 1 - (\Gamma \cdot D_2). \quad (*)$$

Each of the numbers $(M_1 \cdot D_1)$, $(M_2 \cdot D_1)$, $(M_1 \cdot M)$, $(M_2 \cdot M)$ is positive. For if, for example, $(M_1 \cdot D_1) = 0$, then, as was shown in the proof of assertion D) of §3.1, $M_1 = D'_1$ or D_1, and hence $M = M_2 + \Gamma$ or $M = M'_2 + \Gamma$. But the curve M cannot have proper elliptic components (Proposition 2 of §1), and hence $\Gamma = 0$, $M = M_2$ or M'_2, which contradicts our choice of a special curve. Since moreover $(\Gamma \cdot D_1)$, $(\Gamma \cdot D_2) \geq 0$, we have $(M_1 \cdot D_1)$, $(M_2 \cdot D_1) < a$. If both of the curves M_1, M_2 are different from D_2, D'_2, then the second of the equations of $(*)$ gives us that $(M_1 \cdot D_2), (M_2 \cdot D_2) \leq b$. Let the curve M_2 coincide with D_2. Then $(M_1 \cdot D_1) \leq a - 1$ and $(M_1 \cdot D_2) \leq b + 1$.

We consider the case when $(M_1 \cdot D_1) = (M_1 \cdot D_2) = 1$ (and $M_2 = D_2$). For suitable $r > 0$ and $s > 0$ we can assume that the difference between the divisor $\Gamma_1 \geq 0$ taken from the relationship $M + D_1 \sim rM_1 + sD_2 + \Gamma_1 + \epsilon K$ ($\epsilon = 0, 1$ and is chosen in a suitable manner) and each of the divisors M_1, D_2, M'_1, D'_2 is not equivalent to an effective divisor. Let us assume that one of the indices of intersection $(M_1 \cdot \Gamma_1)$, $(D_2 \cdot \Gamma_1)$ is positive, say the first of them. Then there exists an irreducible component θ of the divisor Γ_1 such that $(M_1 \cdot \theta) > 0$. Considering a special curve of the system $|M_1 + \theta|$, we arrive at the relationship $M_1 + \theta \sim \widetilde{M}_1 + \widetilde{M}_2 + \widetilde{\Gamma}$. By the construction, both of the divisors $\widetilde{M}_1, \widetilde{M}_2$ cannot be simultaneously included among the divisors D_1, D_2, M_1 and their adjoints. Thus in the case under consideration we can either set $H_1 = D_1$, $H_2 = D_2$, $H_3 = M_1$, $H_4 = \widetilde{M}_1$ (if $\widetilde{M}_1 \neq D_1, D'_1, M_1, M'_1, D_2, D'_2$) or without loss of generality assume that one of the indices of intersection $(M_1 \cdot D_1)$, $(M_1 \cdot D_2)$ is greater than one.

If $(M_1 \cdot \Gamma_1) = (D_2 \cdot \Gamma_1) = 0$, then, on the one hand, $(\Gamma_1^2) = (M \cdot \Gamma_1) + (D_1 \cdot \Gamma_1) \geq 0$, and, on the other hand, $(\Gamma_1^2) \leq 0$, for otherwise each of the indices of intersection $(M_1 \cdot \Gamma_1)$, $(D_2 \cdot \Gamma_1)$ would be positive. Hence $(\Gamma_1^2) = (M \cdot \Gamma_1) = (D_1 \cdot \Gamma_1) = 0$, and thus on the one hand, $(M \cdot D_1) = s + r$ and, on the other hand, $(M \cdot D_1) = sr$, i.e. $s = r = 2$. From this we have $(M \cdot M_1) = (M \cdot D_2) = 1$, and we can set $H_1 = D_2$, $H_2 = M_1$, $H_3 = D_1$, $H_4 = M$.

We can now assume that $(M_1 \cdot D_1)$ or $(M_1 \cdot D_2)$ is greater than one if $M_2 = D_2$ (or, analogously, D'_2). We assume that $(M_1 \cdot D_1) + (M_1 \cdot D_2) \geq (M_2 \cdot D_1) + (M_2 \cdot D_2)$ and we set $a_1 = (M_1 \cdot D_1)$, $b_1 = (M_1 \cdot D_2)$. If $a > b + 1$, then $a > \max(a_1, b_1)$, and moreover, $a_1 + b_1 \leq a + b$. We will repeat our construction, replacing in the first repetition M by M_1 and D_1 by the divisor D_1 or D_2, taking the one whose index of intersection with M_1 is greatest. Since a_i remains greater than $b_i + 1$, the maximum of the indices of intersection is reduced, and their sum is not increased, while it remains constant only in the case $M_2 = D_2$ or D'_2, $(\Gamma \cdot D_1) = (\Gamma \cdot D_2) = 0$.

We consider the critical moment when $a = b + 1$ or b, $M_2 = D_2$ or D'_2, $(\Gamma \cdot D_1) = (\Gamma \cdot D_2) = 0$. We will show that then $\Gamma = 0$, $a = b + 1$, $(M \cdot M_1) = 1$. We have

$$(\Gamma^2) = ((M + D_1 - M_1 - D_2)^2) = 2 - 2\,(M \cdot M_1),$$
$$a = (M \cdot D_1) = (M \cdot M_1) + b + (\Gamma \cdot M).$$

Since $(\Gamma \cdot M) \geq 0$, $(M \cdot M_1) > 0$, we have $a = b + 1$, $(M \cdot M_1) = 1$, $(\Gamma^2) = 0$. On the other hand, the divisor Γ, if it is effective, must consist of components of some divisors of the pencil $|2D_1|$. Since $(D_1 \cdot D_2) > 0$ and $(\Gamma \cdot D_2) = 0$, all these components are proper, and hence $(\Gamma^2) < 0$, which contradicts the equation already proved: $(\Gamma^2) = 0$.

Thus, for the critical moment we have $M + D_1 \sim M_1 + D_2$ (or $M_1 + D'_2$). Since $(M \cdot M_1) = 1$, the divisors M and M_1 are irreducible (by assertion A) 1)), and since

$a = b + 1 > 1$, we find ourselves in the conditions under which assertion C) 1) of §3 was proved. According to the proof of this assertion, the system $|2(M + D_1)| = |2(M_1 + D_2)|$ contains a special curve that is different from the curves $M + D_1 + M' + D'_1$, $M_1 + D_2 + M'_1 + D'_2$. This third special curve permits us to find isolated elliptic divisors M_3, M_4 possessing property (I) of §2.2 D) that are different from the divisors D_1, D_2, M, M_1, and their adjoints. Each of the divisors M_3, M_4 allows us to continue our process.

Our construction is now completely described. It is clear that the process described breaks off after some number n of steps, and we obtain in addition to the curves $H_1 = D_1$, $H_2 = D_2$ a pair of divisors $H_3 = M_{2n-1}$, $H_4 = M_{2n}$, whose existence was asserted.

D) *Let* H_1, H_2, H_3, H_4 *be isolated elliptic divisors possessing the property* (I) *of* §2.2 D), *let* $(H_i \cdot H_j) = 1 - \delta^i_j$ $((i, j) \neq (3, 4))$, *and let the divisors* H_3, H_4 *pass through vertices of the quadrangle formed by the curves* H_1, H_2, H'_1, H'_2. *Then there exist isolated elliptic divisors* D_1, D_2, D_3 *also possessing property* (I) *and such that* 1) $(D_i \cdot D_j) = 1 - \delta^i_j$, $i, j = 1, 2, 3$; 2) *the divisor* D_3 *does not pass through any vertex of the quadrangle formed by the curves* D_1, D_2, D'_1, D'_2.

Proof. Without loss of generality we can assume (if necessary, replacing the divisor H_3 by its adjoint), that the divisors H_3 and H_4 pass through one pair of vertices of the quadrangle $H_1 + H_2 + H'_1 + H'_2$ (cf. assertion A) 1)). Since $H_3 + \Gamma_3 \sim H_1 + H_2$, $H_4 + \Gamma_4 \sim H_1 + H_2$ (assertion B) 1)), it is easy to calculate that $(H_3 \cdot H_4) = 2$, $(H_3 \cdot \Gamma_4) = (H_4 \cdot \Gamma_3) = 0$. By assertions A) 1) of §4.1 and C) 1) of §3, the divisors H_3, H_4, H'_3 and H'_4 are irreducible, and the system $|2(H_3 + H_4)|$ contains a special curve different from the curve $H_3 + H_4 + H'_3 + H'_4$. Using this curve, we find divisors H_5, H_6 (isolated, elliptic, etc.) satisfying (for some $\Gamma \geq 0$) the relationship $H_3 + H_4 \sim H_5 + H_6 + \Gamma$. Since the divisors H_5, H_6 are different from the divisors H_3, H_4, H'_3, H'_4, and $(H_3 \cdot \Gamma)$, $(H_4 \cdot \Gamma) \geq 0$, we have $(H_3 \cdot H_5) = (H_3 \cdot H_6) = (H_4 \cdot H_5) = (H_4 \cdot H_6) = 1$, $(H_3 \cdot \Gamma) = (H_4 \cdot \Gamma) = 0$. Neither of the divisors H_5 and H_6 can be contained among the divisors H_1, H_2, H'_1, H'_2. For otherwise we would have $H_3 + H_4 \sim H_1 + H_2 + \epsilon K + \Gamma$, $H_3 + H_4 \sim H_4 + \Gamma_4 + \Gamma + \epsilon K$, $H_3 \sim \Gamma + \Gamma_4 + \epsilon K$, which contradicts the irreducibility and isolatedness of the divisors H_3, H'_3. Hence one of the divisors H_5 and H_6, say H_5, differs from the divisors H_1, H_2, H'_1, H'_2 and has an index of intersection of 1 with one of them, say with H_2. Now let us set $D_1 = H_2$, $D_2 = H_3$, $D_3 = H_5$. By assumption, the vertices of the quadrangle $D_1 + D_2 + D'_1 + D'_2$ coincide with the vertices of the quadrangle $H_1 + H_2 + H'_1 + H'_2$, and the curves H_4, H'_4 are diagonals of these quadrangles (cf. assertion B) 1)). If the curve D_3 were also a diagonal, we would have $(D_3 \cdot H_4) \geq 2$, which is impossible.

E) *There exists on the surface F a triple of irreducible isolated elliptic curves D_1, D_2, D_3 possessing property* (I) *of* §2.2 D) *and such that* 1) $(D_i \cdot D_j) = 1 - \delta^i_j$, $i, j = 1, 2, 3$; 2) *the curve D_3 does not pass through the vertices of the quadrangle formed by the curves D_1, D_2, D'_1, D'_2.*

Proof. By the results of §3.1 there exists on the surface F a pair of isolated elliptic curves D_1, D_2 with index of intersection 1. By assertion A) 2) there exists on F a third isolated elliptic curve, also possessing property (I). This curve is either the desired curve D_3 or satisfies the conditions of either assertion C) or assertion B). If it satisfies the conditions of assertion B), then there exists either the desired curve D_3, or a curve satisfying the conditions of assertion C), or a pair of curves satisfying (along with the curves D_1, D_2) the conditions of assertion D). If it satisfies the conditions of assertion C), in this case also there either exists the desired curve D_3 or a pair of curves satisfying the conditions of assertion D). Using the last assertion, we obtain a proof of assertion E).

3. We consider the linear system $|C| = |D_1 + D_2 + D_3|$, where D_1, D_2, D_3 are the curves constructed in assertion E) 2). We will show that this system regularly and birationally maps our surface F onto a surface of sixth degree in the space P^3 that passes twice through the edges of some tetrahedron.

A) *The linear system $|C|$ is irreducible, has a dimension of three and does not have base points, i.e. it regularly maps the surface F into the space P^3.*

Proof. Since $(C^2) = 6$, $\dim |D_1 + D_2| < 2 \leq \dim |D_1 + D_2 + D_3|$. Since the system $|D_1 + D_2|$ does not have fixed components, and the curve D_3 is irreducible (cf. assertion A) 1)) and is not a fixed component of the system $|C|$, it follows that $|C|$ does not have fixed components and is irreducible by Proposition 3 of §1. By Proposition 4 of §1, $\dim |C| = 3$. We will show that the system $|C|$ does not have base points. In fact, this system contains the curves $D_1 + D_2 + D_3$, $D_1 + D'_2 + D'_3$, $D'_1 + D'_2 + D_3$, $D'_1 + D_2 + D'_3$; if a base point P of the system $|C|$ lies on the curve D_2, then it does not lie on the curve D'_2, and hence lies on D_1 or D'_3. In the first case it does not lie on D'_1, and thus lies on D_3, i.e. $P = D_1 \cap D_2 \cap D_3$; in the second case it does not lie on D_3 and hence lies on D'_1, i.e. $P = D'_1 \cap D_2 \cap D'_3$.

Analogously, assuming that P lies on D_1 or D_3, we obtain two more possible values: $P = D_1 \cap D'_2 \cap D'_3$, $P = D'_1 \cap D'_2 \cap D_3$. There can be no other base points of the system $|C|$. But the four intersections obtained are empty, by the construction of the curves D_1, D_2, D_3 (cf. assertion E) 2) and assertion B) 1)). Our assertion is thus proved.

B) *The system $|C|$ maps the surface F birationally.*

Proof. We denote our mapping by ϕ. From the irreducibility of the system $|C|$ it follows that the image under the mapping ϕ is a surface. Since $(C^2) = 6$, and the system $|C|$ does not have base points, the degree of the mapping ϕ is a divisor of the number 6.

We consider the subsystem $|D_1 + D_2| + D_3$ of the system $|C|$ consisting of the curves containing D_3 as a component, and the subsystem $|D_2 + D_3| + D_1$. By assertion A) 1) the system $H_1 = |D_1 + D_2|$ is one-dimensional and has exactly two base points: the point $D_1 \cap D'_2$ and the point $D'_1 \cap D_2$. Analogously, the system $H_2 = |D_2 + D_3|$ is irreducible and has two base points: $D_2 \cap D'_3$, $D'_2 \cap D_3$. Thus by the construction of the curves D_1, D_2, D_3 the systems H_1, H_2 do not have common base points. Let Γ_1 (Γ_2) be a generic curve of the system H_1 (H_2). We have $((\Gamma_1 + D_3)\cdot(\Gamma_2 + D_1)) = 6$, where, of the six points of intersection three are points of intersection of Γ_1 and Γ_2, and the other three lie on the curves D_1 and D_3. On the curve Γ_1 there are only a finite number of points identified under the mapping ϕ with points of the curves D_1 and D_3, and none of these points is a base point of the system H_2. We can thus assume that none of the three points of the set $\Gamma_1 \cap \Gamma_2$ is joined with a point of the set $D_1 \cup D_3$. This means that these three points can be joined only with each other, and hence, by the generality of the construction, the degree of the mapping ϕ is equal to 1 or 3.

Since $(C \cdot H_1) = 4$, and the systems $|C|$ and H_1 do not have common base points, analogously, the degree of the mapping is a divisor of the number 4. Hence the degree of the mapping ϕ is equal to 1, i.e. the mapping is birational.

C) *The image under the mapping ϕ is a surface of sixth degree in* $\mathbf{P}^3$ *that twice passes through the edges of a tetrahedron.*

Proof. Since $\dim |C| = 3$, the system $|C|$ maps our surface into $\mathbf{P}^3$, and since $(C^2) = 6$, (the system $|C|$ does not have base points), the degree of the image is equal to six.

We consider the curves $D_1 + D_2 + D_3$, $D_1 + D'_2 + D'_3$, $D'_1 + D_2 + D'_3$, $D'_1 + D'_2 + D_3$ of the system $|C|$. The image of each of these is some plane section. Since $(C \cdot D_i) > 0$, the curves D_i, D'_i, $i = 1, 2, 3$ are not contracted under the mapping ϕ, and hence the intersections of the plane sections described are one-dimensional. This means that each of these intersections is a line, the image of one of the curves D_i, D'_i. Thus the surface $\phi(F)$ contains the six lines $\phi(D_i)$, $\phi(D'_i)$.

Clearly, the preimage of their union coincides with the set $(\bigcup D_i) \cup (\bigcup D'_i)$. We will show that no two of the six curves D_i, D'_i are joined under the mapping ϕ.

Indeed, let $P \in D_1$, $Q \in D_2$, $\phi(P) = \phi(Q)$. This means that every divisor of the system $|C|$ passing through Q also passes through P. One such divisor is the divisor $D'_1 + D_2 + D'_3$, and hence $P = D_1 \cap D_2$ or $D_1 \cap D'_3$. The proof is analogous for any other pair.

We will now show that the surface $\phi(F)$ passes twice through each of the lines $\phi(D_i)$, $\phi(D'_i)$. It is sufficient for us to show that the mapping is of two sheets on each of the curves D_i, D'_i. Since $(C \cdot D_i) = 2$, this mapping is not more than two-sheeted on D_i; on the other hand, it is ramified as a mapping of a nonsingular elliptic or singular rational curve onto a nonsingular rational curve $\phi(D_i)$.

It remains for us to prove that the six lines $\phi(D_i)$, $\phi(D'_i)$ form a tetrahedron, or, what is the same, that the four indicated plane sections do not pass through one point. If this were not true, the six lines $\phi(D_i)$, $\phi(D'_i)$ would all pass through one point, i.e. the mapping ϕ would identify the nineteen points $D_i \cap D_j$, $D_i \cap D'_j$, $D'_i \cap D'_j$ $(i \neq j;\ i, j = 1, 2, 3)$. We have already shown, however, that if the point $P \in D_1$ is identified with the point $Q \in D_2$, then $P = D_1 \cap D_2$ or $D_1 \cap D'_3$, i.e. the points $D_1 \cap D_3$ and $D_1 \cap D_2$ are not identified under the mapping ϕ.

This completes the proof of assertion C).

§5. The "number of moduli" of an Enriques surface

In this section we calculate the "number of moduli" of an Enriques surface. By a well-known theorem of Kodaira, Nirenberg, and Spencer [28], a complete space of moduli of a variety F that is effectively parametrized in a neighborhood of each of its points exists if the zero-dimensional and two-dimensional homology groups with coefficients in the sheaf Θ of germs of holomorphic vector fields are trivial. Moreover, the dimension of this space is equal in this case to the dimension of the one-dimensional homology group with coefficients in this sheaf.

Thus we will show that $H^0(F, \Theta) = H^2(F, \Theta) = 0$.

Since $2K(F) \sim 0$, there exists a two-sheeted algebraic unramified algebraic covering surface F_1, of F, with a zero canonical class $K(F_1)$. Clearly F_1 is regular.

If there existed on the surface F a nonzero holomorphic vector field, such a field would also exist on the surface F_1. Since $K(F_1) = 0$, it would follow by duality that $h^{2,1}(F_1) = q(F_1) \neq 0$. We obtain a contradiction. Let us assume that $H^2(F, \Theta) \approx H^0(F, \Omega(K \otimes T^*(F))) \neq 0$, where we denote by $T^*(F)$ the fiber bundle on F dual to the tangent space. Then $H^0(F_1, \Omega(p^*K \otimes p^*T^*(F))) =$ $H^0(F_1, \Omega(p^*T^*(F))) = H^0(F_1, \Omega(T^*(F_1))) \neq 0$ (where p denotes the projection $F_1 \to F$), which contradicts the regularity of F_1.

Thus the desired "number of moduli" exists. It is not difficult to calculate with the help of the Riemann-Roch theorem that it is equal to ten.

We note further that, according to a theorem of Kodaira ([25], Theorem 3.5), every Kähler surface with $p_g = 0$ is algebraic.

APPENDIX

ON SPECIAL TYPES OF KUMMER AND ENRIQUES SURFACES*

This work studies Kummer and Enriques surfaces of special types. It is proved that such Kummer surfaces exist and that they form a subset of codimension one in the space of moduli. A theorem (and its converse) about representation of the double plane is proved for Enriques surfaces of both special and general type.

Introduction

The classification of Kummer and Enriques surfaces found in Chapters VIII and X is not complete. In fact, these chapters do not consider at all particular cases of surfaces of the two kinds. Our goal is to fill this gap.

In speaking of classification we mean the following: starting from given values of the invariants, geometric models of the surfaces under consideration must be constructed and then the converse theorems must be proved. It is understood that an intermediate step in this program must be the determination of some geometric construction that is characteristic for surfaces with the given values of the invariants. The chapters mentioned above contain such constructions, apparently due to Enriques, for both Kummer and Enriques surfaces. These constructions, however, are not suitable for all surfaces of both types, and hence particular cases must be considered separately; here the fact that a given surface is a surface of a particular type is determined by various degeneracies in its internal structure.

The first section is concerned with Kummer surfaces, i.e., with regular algebraic surfaces with zero canonical class K. As was proved in Chapter VIII, almost every such surface is birationally equivalent to a surface of degree $2\pi - 2$ in π-dimensional projective space; here the number π is equal (for each specific surface) to the smallest of the dimensions of the complete linear systems on this surface increased by one. For a general Kummer surface such a system is unique (see Chapter IX) and determines the indicated birational mapping. The special case consists of surfaces for which this minimal linear system consists of

* Translator's note: This article, by B. G. Averbuh, appeared in Izv. Akad. Nauk SSSR 29 (1965), 1095–1118; it is the forthcoming article referred to by Averbuh in his "added in proof" notes to Chapters VIII and X. For this reason, and because of its own intrinsic merits, we have included it here as an appendix.

hyperelliptic curves. For $\pi = 3$ we shall construct all such surfaces; the question remains open for $\pi > 3$.

The second and third sections are concerned with Enriques surfaces, i.e. regular surfaces for which the double canonical class is equivalent to zero. Surfaces of special type here are those surfaces on which there exist a pencil of elliptic curves and a rational curve with an index of intersection equal to zero. We shall show that every such surface is birationally equivalent to a surface of sixth degree in P^3 whose ramification curve has completely determined singularities. For comparison an analogous theorem is also proved for Enriques surfaces of general type. Finally, the converse theorems (under certain additional restrictions) are studied in §3.

§1. Kummer surfaces of special type

1. Let V be a Kummer surface. As was proved in Chapter IX, the family of such surfaces is nineteen-dimensional. To a generic point of the base space of this family there corresponds a surface with a base number one, and to each increase of the base number by one, there corresponds a decrease of the dimension of the corresponding subfamily (see Chapter IX). We note that all Kummer surfaces are simply connected.

Let C be a divisor on the surface V. We denote by $|C|$ the complete linear system containing this divisor, by $p_a(C) = (C, C)/2 + 1$ the arithmetic genus of this divisor, and by $\dim|C|$ the dimension of the system $|C|$. By the Riemann-Roch theorem, $\dim|C| \geq p_a(C)$ if $(C, C) > 0$ and the system $|C|$ is irreducible.

Further, let $\pi > 1$ be an integer such that:

1) there exists on V an irreducible curve C whose arithmetic genus is equal to π;

2) the arithmetic genus of every irreducible curve on V is either less than two or greater than $\pi - 1$.

If the generic curve of such a system $|C|$ is not hyperelliptic, then this system birationally maps the surface V onto a surface of degree $2\pi - 2$ in a π-dimensional projective space.

If this generic curve is hyperelliptic, then for $\pi > 2$ such a map is given by the system $|2C|$, and the mapping associated with the system $|C|$ is two-sheeted (see Chapter VIII).

A Kummer surface V with $\pi > 2$ will be called a surface of special type if the generic curve of every complete irreducible linear system of curves of arithmetic genus π is hyperelliptic.

Now let the surface V be of special type, let $\pi = 3$, and let C be an irreducible curve of arithmetic genus 3. Let the functions $y_0, y_1 = 1, y_2, y_3$ form a basis for the space $\mathfrak{L}(C)$ of the factors of the divisor C. By what was said above, the field of functions on V is an extension of degree two of its subfield $K(y_0, y_1, y_2, y_3)$ generated by these functions; if, on the other hand, $y_0, y_1, y_2, y_3, \cdots, y_9$ is a basis of the space $\mathfrak{L}(2C)$, then the subfield generated by the functions $y_0, \cdots, y_9$ coincides with the whole field of functions on V. This means that the functions $y_i y_j$, $i, j = 0, 1, 2, 3$, of the space $\mathfrak{L}(2C)$ do not generate this space and are thus linearly dependent.

Let us assume that

$$y_4 = y_0^2, \quad y_5 = y_0 y_2, \quad y_6 = y_0 y_3, \quad y_7 = y_2^2, \quad y_8 = y_2 y_3;$$

then the function y_9 satisfies a quadratic equation over the field $K(y_0, y_1, y_2, y_3)$ and, as is easily seen, can be chosen so that this equation has the form

$$y_9^2 = P^4(y_0, y_1, y_2, y_3),$$

where $P^4(y_0, y_1, y_2, y_3)$ is a polynomial.

We now consider the image of the special Kummer surface V under the mapping associated with the system $|2C|$. If a basis of the space $\mathfrak{L}(2C)$ is chosen as indicated above, then this image, according to what has been said, lies within the set given by the equations:

$$\left\{\begin{array}{l} \varphi^2(x_0, x_1, x_2, x_3) = 0, \\ x_1 x_4 = x_0^2, \\ x_1 x_5 = x_0 x_2 \\ x_1 x_6 = x_0 x_3 \\ x_1 x_7 = x_2^2, \\ x_1 x_8 = x_2 x_3, \\ x_1^2 x_9^2 = \eta^4(x_0, x_1, x_2, x_3), \end{array}\right. (x_1 \neq 0), \qquad \left\{\begin{array}{l} \varphi^2(x_4, x_0, x_5, x_6) = 0, \\ x_4 x_1 = x_0^2, \\ x_4 x_2 = x_0 x_5 \\ x_4 x_3 = x_0 x_6 \\ x_4 x_7 = x_5^2, \\ x_4 x_8 = x_5 x_6, \\ x_4^2 x_9^2 = \eta^4(x_4, x_0, x_5, x_6), \end{array}\right. (x_4 \neq 0),$$

$$\left\{\begin{array}{l} \varphi^2(x_5, x_2, x_7, x_8) = 0, \\ x_7 x_4 = x_5^2, \\ x_7 x_0 = x_5 x_2 \\ x_7 x_6 = x_5 x_8 \\ x_7 x_1 = x_2^2, \\ x_7 x_3 = x_2 x_8, \\ x_7^2 x_9^2 = \eta^4(x_5, x_2, x_7, x_8). \end{array}\right. (x_7 \neq 0), \qquad (*)$$

Since, moreover, the intersection of the divisor C with the zero divisors of the functions $y_7 = y_2^2$ and $y_4 = y_0^2$ can be assumed to be empty (the system $|C|$ does not have any base point, as was proved in Chapter VIII §4), the image of the surface V lies in the union of the sets

$$U_1 = \{x_1 \neq 0\}, \quad U_2 = \{x_4 \neq 0\}, \quad U_3 = \{x_7 \neq 0\}.$$

2. We shall now investigate the equations (∗) and shall prove the assertion converse to the one proved above.

First of all, in the intersections of the neighborhoods U_1, U_2, U_3 these systems of equations determine the same sets (for example, for the intersection $U_1 \cap U_2$ this follows from the relations $x_0/x_4 = x_1/x_0 = x_2/x_5 = x_3/x_6$), and, taken together, some algebraic set. We shall show that its subset V, which is in the union $U_1 \cup U_2 \cup U_3$, is (for general forms ϕ^2 and η^4) a nonsingular compact surface. We shall begin with compactness. Let Q be some limit point of the given subset that does not belong to the subset. As it is easy to see, for every point $P \in V$ the equation $x_1(P) = 0$ implies

$$x_0(P) = x_5(P) = x_6(P) = 0,$$

$x_4(P) = 0$ implies

$$x_0(P) = x_5(P) = x_6(P) = 0,$$

and, finally, $x_7(P) = 0$ implies

$$x_2(P) = x_8(P) = 0.$$

Thus the point Q must have the form $(0:0:\cdots:0:1)$. Setting $y_i = x_i/x_9$, we can write the last equation of our first system in the form

$$y_1^2 = \eta^4(y_0, y_1, y_2, y_3).$$

We shall now approach the point Q along a curve lying on the set V, on the hyperplane $x_1 - x_3 = 0$, and outside the hyperplanes $x_1 = 0$, $x_4 = 0$, $x_7 = 0$. Then the coordinates y_0, y_2 will decrease faster than $\sqrt{y_1}$ (since $y_1 y_4 = y_0^2$, $y_1 y_7 = y_2^2$, $y_4(Q) = y_7(Q) = 0$), but the coordinates y_1 and y_3, with the same speed as $\sqrt{y_1}$. We have a contradiction.

For the proof of nonsingularity, we assume that the equation $\phi^2(x_0, x_1, x_2, x_3) = 0$ determines a nonsingular quadric in $\mathbf{P}^3$ and that the system

$$\begin{cases} \varphi^2(x_0, x_1, x_2, x_3) = 0 \\ \eta^4(x_0, x_1, x_2, x_3) = 0 \end{cases}$$

is a nonsingular curve. The proof is carried out in an analogous manner for each of the neighborhoods U_1, U_2, U_3, and we shall give it only for the neighborhood U_1. We set

$$y_0 = \frac{x_0}{x_1}, \quad y_i = \frac{x_i}{x_1}, \quad i \geqslant 2;$$

then the corresponding functional matrix will have the form

$$\begin{pmatrix} \frac{\partial\varphi^2}{\partial y_0} & \frac{\partial\varphi^2}{\partial y_2} & \frac{\partial\varphi^2}{\partial y_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -2y_0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -y_2 & -y_0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -y_3 & 0 & -y_0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2y_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -y_3 & -y_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{\partial\eta^4}{\partial y_0} & -\frac{\partial\eta^4}{\partial y_2} & -\frac{\partial\eta^4}{\partial y_3} & 0 & 0 & 0 & 0 & 0 & 2y_9 \end{pmatrix}.$$

If $y_9 \neq 0$, the rank of this matrix is equal to seven, since one of the derivatives $\partial\phi^2/\partial y_i$ is different from zero; if $y_9 = 0$ the rank is again seven because the curve

$$\begin{cases} \varphi^2 = 0, \\ \eta^4 = 0 \end{cases}$$

is nonsingular.

We consider on the surface V the linear system of curves

$$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0.$$

This system consists of a fixed part $x_0 = 0$, $x_4 \neq 0$, which we denote by C_0, and a nonfixed part $|C|$, which is given in the neighborhood U_1 by the equation

$$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0,$$

in the neighborhood U_2 by the equation

$$\lambda_0 x_4 + \lambda_1 x_0 + \lambda_2 x_5 + \lambda_3 x_6 = 0$$

and in the neighborhood U_3 by the equation

$$\lambda_0 x_5 + \lambda_1 x_2 + \lambda_2 x_7 + \lambda_3 x_8 = 0.$$

The divisor $2C_0$ is given by the equation $x_1 = 0$; hence the system $|2C|$ is a system of hyperplane sections of the surface V. Obviously, this surface is a two-sheeted covering of the quadric $\phi^2(u_0, u_1, u_2, u_3) = 0$ in $\mathbf{P}^3$ with ramification curve

$$\begin{cases} \varphi^2(u_0, u_1, u_2, u_3) = 0, \\ \eta^4(u_0, u_1, u_2, u_3) = 0. \end{cases}$$

The divisors of the system $|C|$ are the pre-images under this covering of hyperplane sections of the quadric, i.e., they are two-sheeted coverings of the rational curves with eight ramification points. Thus the curves of the system $|C|$ are hyperelliptic and have a genus of three.

It remains for us to prove that the surface V has the necessary values of the invariants and does not contain systems of genus two and three other than $|C|$.

We begin with a calculation of the canonical class of the surface V. Again we set

$$y_0 = \frac{x_0}{x_1}, \quad y_2 = \frac{x_2}{x_1}$$

and we find the zero divisors and the polar divisors of the differential $dy_0 \wedge dy_2$. As is clear from the functional matrix presented above, at points of the set $U_1 \cap V$ where $y_9 \neq 0$ and $\partial\phi^2/\partial y_3 \neq 0$, the functions y_0, y_2 are local coordinates and the differential $dy_0 \wedge dy_2$ has neither zeros nor poles. Let

$$\frac{\partial \varphi^2}{\partial y_3} = 0, \quad \frac{\partial \varphi^2}{\partial y_0} \neq 0, \quad y_9 \neq 0.$$

We have

$$\frac{\partial \varphi^2}{\partial y_0} dy_0 \wedge dy_2 = -\frac{\partial \varphi^2}{\partial y_3} dy_3 \wedge dy_2.$$

Hence the differential $dy_0 \wedge dy_2$ has a zero on the curve $\partial\phi^2/\partial y_3 = 0$. This curve, as can be easily seen, belongs to the system $|C|$.

Now let $y_9 = 0$. We have

$$\begin{cases} y_9\, dy_9 = \dfrac{\partial \eta^4}{\partial y_0} dy_0 + \dfrac{\partial \eta^4}{\partial y_2} dy_2 + \dfrac{\partial \eta^4}{\partial y_3} dy_3, \\ 0 = \dfrac{\partial \varphi^2}{\partial y_0} dy_0 + \dfrac{\partial \varphi^2}{\partial y_2} dy_2 + \dfrac{\partial \varphi^2}{\partial y_3} dy_3. \end{cases}$$

Since the isolated points do not interest us, we can assume that the determinant Δ formed from the coefficients of the first and third columns of the right-hand side is nonzero. This means in particular that the functions y_9, y_2 are local coordinates. Solving the system of equations

$$\begin{cases} \dfrac{\partial \eta^4}{\partial y_0} dy_0 \wedge dy_2 + \dfrac{\partial \eta^4}{\partial y_3} dy_3 \wedge dy_2 = y_9\, dy_9 \wedge dy_2, \\ \dfrac{\partial \varphi^2}{\partial y_0} dy_0 \wedge dy_2 + \dfrac{\partial \varphi^2}{\partial y_3} dy_3 \wedge dy_2 = 0, \end{cases}$$

we find that

$$dy_0 \wedge dy_2 = \frac{y_9 \dfrac{\partial \varphi^2}{\partial y_3} dy_9 \wedge dy_2}{\Delta},$$

i.e., the differential $dy_0 \wedge dy_2$ has a zero also on the curve $y_9 = 0$ which belongs to the system $|2C|$ of hyperplane sections.

In order to calculate the differential $dy_0 \wedge dy_2$ on the complement of the set $U_1 \cap V$ (it consists of the curve C_0), we set $x_i/x_4 = z_i$ $(i \neq 4)$.

We have

$$d\left(\frac{x_0}{x_4}\cdot\frac{x_4}{x_1}\right)\wedge d\left(\frac{x_2}{x_4}\cdot\frac{x_4}{x_1}\right)=\frac{dz_0\wedge dz_2}{z_1^2}-\frac{z_2\,dz_0\wedge dz_1}{z_1^3}-\frac{z_0\,dz_1\wedge dz_2}{z_1^3}.$$

Since $z_1 = z_0^2$, this expression can be put in the form

$$-\frac{dz_0\wedge dz_2}{z_1^2},$$

and then, since $z_2 = z_0 z_5$, in the form

$$-\frac{dz_0\wedge dz_5}{z_0^3}.$$

The functions z_0 and z_5 are local coordinates at a generic point of the set $U_2 \cap V$ (just as the functions y_0, y_2 are in $U_1 \cap V$), and under a general choice of the original equations, also at a generic point of the complement of the set $U_1 \cup V$. Thus the differential $dy_0 \wedge dy_2$ has a pole of third order on the curve C_0. Since the zero divisors and polar divisors of the differential $dy_0 \wedge dy_2$ belong to the system $|3C|$, the canonical class of the surface V is equivalent to zero. It also follows from this that this surface is a minimal model.

We now consider the system $|C|$. Since the index of intersection of the hyperplane sections of the quadric is equal to two, the index of intersection of two curves of the system $|C|$ is equal to four. Further, since by the theorem of Kodaira ([25], Theorem 2.5)

$$\dim H^1(V, \Omega(C)) = \dim H^1(V, \Omega(K + C)) = 0,$$

we obtain from the Riemann-Roch theorem

$$p_a(V) = \dim|C| + 1 - \frac{(C, C)}{2} \geqslant 2.$$

By Chapter VIII, §1, $p_a(V) \leq 2$ for $\kappa = 0$; consequently, $p_a(V) = 2$, $\dim|C| = 3$, $q(V) = 0$, i.e., the surface V actually has the necessary invariants. (We denote by $\kappa(V)$ the degree of transcendency of the subfield of the field of functions on V which is generated by the factors of positive multiples of the canonical class.)

We shall now show that for a general choice of the coefficients of the forms ϕ^2 and η^4 the surface V does not contain linear systems of genus two and three other than $|C|$. For this we shall find the number of parameters determining the birational equivalence classes of the surfaces we constructed. This number is equal to seventeen. It follows by a result of G. N. Tjurina (Chapter IX), that the base number of a generic one of our surfaces is equal to two, and this leads to the desired result without difficulty.

Thus the family of surfaces we constructed depends on ten coefficients of the quadric ϕ^2 (which when considered from the point of view of factors of proportionality, gives nine parameters), and on 35 coefficients of the form η^4. However, the forms η_1^4 and η_2^4, whose difference is divisible by ϕ^2, yield the same surface V, and hence the coefficients of the form η^4 yield 25 parameters, i.e., the whole family of examples depends on 34 parameters.

We now find the dimension of the class of surfaces projectively equivalent to V. For this we consider on the surface V the functions $y_i = x_i/x_0$. The functions y_0, y_2, y_3 have as their polar divisor the divisor C_0, and the remaining functions have the divisor $2C_0$. Moreover, the functions $1, y_0, y_2, y_3$ form a base of the space $\mathfrak{L}(C_0)$ of the factors of the divisor C_0, and the functions $1, y_0, y_2, \cdots, y_9$ form a base of the space $\mathfrak{L}(2C_0)$ (the last assertion follows from the Riemann-Roch theorem). From the first system of equations of the surface V we obtain

$$y_4 = y_0^2, \quad y_5 = y_0 y_2, \quad y_6 = y_0 y_3,$$

$$y_7 = y_2^2, \quad y_8 = y_2 y_3, \quad y_9^2 = \eta^4 (y_0, 1, y_2, y_3).$$

If $f_\alpha = 0$ is the local equation of the divisor C_0, we obtain

$$s_i = \{y_i f_\alpha^2\}, \quad i = 0, \ldots, 9,$$

and obtain a base of the space of the sections of the line bundles $[2C]$. Clearly we have

$$(s_0(P) : s_1(P) : \ldots : s_9(P)) = \gamma (x_0(P) : x_1(P) : \ldots : x_9(P)).$$

Thus to every imbedding of the surface V in the space $\mathbf{P}^9$ under which it falls into our family there corresponds some divisor C_0 and a base of the space $\mathfrak{L}(2C_0)$ such that its functions y_i satisfy the above relations. Since the sections s_i yield the original imbedding $V \subset \mathbf{P}^9$, this correspondence is one-to-one. Moreover, the imbeddings g_1 and g_2 are projectively equivalent if and only if the systems of hyperplane sections $|2C_0^1|$ and $|2C_0^2|$ corresponding to them coincide, and this last condition is equivalent to the coincidence of the systems $|C_0^1|$ and $|C_0^2|$, since the surfaces of the type under consideration are simply connected (see Chapter IX).

Since the dimension of the system $|C|$ is equal to three, the choice of the divisor C_0 depends on three parameters. The choice of the system of functions $1, y_0, y_2, y_3$, i.e. the basis of the space $\mathfrak{L}(C_0)$, depends on 12 parameters, and the functions y_4, y_5, y_6, y_7, y_8 are uniquely determined by this choice. As for the function y_9, it is determined up to proportionality as is easily proved.

Thus, the class of surfaces of our family which are projectively equivalent depends on not more than 16 parameters: the three parameters of the choice of the divisor C_0, the twelve in the choice of the functions y_0, y_2, y_3, and the proportionality factor in the choice of y_9.

Finally, the class of surfaces birationally equivalent to V consists of the collection of classes of projectively equivalent surfaces, each of which corresponds uniquely to a choice of the system $|C|$ on V. Since V is regular it contains no more than a denumerable number of such systems, and hence the number of parameters of the class of surfaces birationally equivalent to V is also equal to 16. Thus the family of examples we have constructed corresponds to no more than an 18-dimensional subset of the space of moduli of the Kummer surfaces, and hence the base number of a general example is not larger than two. This means that there exist two divisors F_1 and F_2 in terms of which every divisor may be expressed with integral coefficients (we recall that since our surface is simply connected, there are no cycles of finite order on it).

We now consider two families of straight lines on the quadric $\phi^2(u_0, u_1, u_2, u_3) = 0$. To each of them there corresponds on the surface V a pencil of elliptic curves (two-sheeted coverings over a line with four ramification points); we denote these pencils by L_1 and L_2. We have:

$$L_1 = m_1F_1 + n_1F_2, \quad L_2 = m_2F_1 + n_2F_2.$$

Denoting the determinant $\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}$ by Δ, we obtain

$$F_1 = \frac{n_2L_1 - n_1L_2}{\Delta}, \quad F_2 = \frac{m_1L_2 - m_2L_1}{\Delta}.$$

Since

$$(L_i, L_i) = 2p_a(L_i) - 2 = 0, \quad (L_1, L_2) = 2$$

(a single point of intersection of the straight lines on the quadric corresponds to two points of intersection in their pre-image), we have

$$(F_1, F_1) = \frac{-4n_1n_2}{\Delta^2}, \quad (F_2, F_2) = \frac{-4m_1m_2}{\Delta^2}.$$

Hence the numbers $4n_1n_2/\Delta^2$ and $4m_1m_2/\Delta^2$ are integers. Let p^{2k}, where p is a prime, be a factor of the number

$$\Delta^2 = m_1^2n_2^2 + m_2^2n_1^2 - 2m_1m_2\,n_1n_2.$$

If $p^k \neq 2$, then n_1n_2 and m_1m_2 are divisible by p. Since the numbers m_1, n_1 (m_2, n_2) are relatively prime (the pencils L_1, L_2 are clearly irreducible), it follows from the divisibility by p of the number n_1 that m_2 is divisible by p and m_1 and n_2 are not. But the divisibility by p of the number Δ^2 implies in this case the divisibility of the product m_1n_2, and, on the basis of the contradiction obtained, we can assert that $|\Delta|$ is equal to one or two.

Let $|D|$ be an arbitrary linear system of curves of genus two on the surface V.

Then

$$D \sim \frac{r_1}{\Delta} L_1 + \frac{r_2}{\Delta} L_2,$$

where r_1 and r_2 are integers. Since

$$(D, D) = 2p_a(D) - 2 = 2,$$

we have

$$\frac{2r_1r_2(L_1, L_2)}{\Delta^2} = 2, \quad r_1r_2 = \frac{\Delta^2}{2}, \quad r_1r_2 = 2.$$

Since $|C| = L_1 + L_2$ (a plane section of a quadric is equivalent to the sum of two lines, one taken from each family), we have $|C - D| \sim L_1/2$ (or $L_2/2$) and $(C - D, C - D) = 0$. By the Riemann-Roch theorem,

$$\dim|C - D| \geqslant \frac{(C - D, C - D)}{2} + 1 \geqslant 1,$$

and this contradicts the irreducibility of the pencil L_1 (L_2).

Let, further, $|D|$ be an arbitrary linear system of curves of genus 3 different from $|C|$. Setting

$$D \sim \frac{r_1}{\Delta} L_1 + \frac{r_2}{\Delta} L_2,$$

we obtain, as above, that $r_1r_2 = \Delta^2$. Since the solution $r_1/\Delta = r_2/\Delta = 1$ gives $|D| = |C|$, we conclude that $r_1 = 4$, $r_2 = 1$ (or $r_2 = 4$, $r_1 = 1$). Hence

$$D - C \sim L_1 - \frac{L_2}{2}, \quad (D - C, D - C) = -(L_1, L_2) = -2.$$

By the Riemann-Roch theorem,

$$\dim H^0(V, \Omega(D - C)) + \dim H^0(V, \Omega(C - D)) \geqslant \frac{(D - C, D - C)}{2} + 2 \geqslant 1.$$

Thus either the system $|D - C|$ or $|C - D|$ is not empty. Since the systems $|C|$ and $|D|$ have the same dimension, this contradicts their irreducibility.

We have thus proved the following theorem.

Theorem 1. *Every Kummer surface of special type with* $\pi = 3$ *is birationally equivalent to a surface of sixteenth degree in* $\mathbf{P}^9$ *that is given by equations* (*) *and lies in the complement of the hyperplane*

$$\left.\begin{array}{l} x_1 = 0 \\ x_4 = 0 \\ x_7 = 0 \end{array}\right\};$$

conversely, a generic surface of this kind is a special Kummer surface.

Moreover, the family of special Kummer surfaces with $\pi = 3$ *depends on* 18 *parameters, and a base number of a generic surface of this family is equal to two.*

§2. Representation of Enriques surfaces as double planes

We recall first of all that an Enriques surface is a regular surface whose double canonical class K is equivalent to zero. The Riemann-Roch formula on such a surface has the form

$$\dim |C| = \frac{(C, C)}{2} + \dim H^1(V, \Omega(C)),$$

where $\Omega(C)$ denotes the sheaf of germs of factors of the divisor C. If this divisor is equivalent to a divisor $D + K$, where D is a curve, then the last member is equal to the number of connected components of D reduced by 1 (cf. [25], Theorem 2.3).

We now present certain assertions proved in Chapter X which we shall need for the future.

Proposition 1. *Let V be an Enriques surface which is not a surface of special type, and let $|C|$ be a linear system without fixed component and such that $p_a(C) > 1$. Then $|C|$ is irreducible and*

$$\dim |C| = \dim |C + K| = p_a(C) - 1;$$

moreover, the system $|C'| = |C + K|$ is also irreducible.

Proposition 2. *Let V be an arbitrary Enriques surface, let L be an irreducible elliptic curve, and let $\dim |L| = 1$. Then:*

1) *the system $|L'| = |L + K|$ consists of a single curve;*

2) *if L' is that curve, then $L' = (L/2) + (L/2)'$, where $(L/2)$ and $(L/2)'$ are connected curves without common points, such that $(L/2)' \sim (L/2) + K$.*

Moreover, the divisors $2(L/2)'$ and $2(L/2)'$ belong to the system $|L|$ and are determined uniquely by it.

Corollary (to Proposition 2). *If $|L|$ is an irreducible pencil of elliptic curves and θ is an arbitrary divisor, then the index of intersection (L, θ) is even.*

Proposition 3. *On an Enriques surface which is not a surface of special type there exist irreducible elliptic curves D_1 and D_2 with index of intersection 1, each of which, when doubled, belongs to an irreducible pencil. Moreover, there exist irreducible curves $D'_1 \sim D_1 + K$ and $D'_2 \sim D_2 + K$, and the systems $|D_1 + D_2|$, $|D_1 + D_2 + K|$ are one-dimensional.*

We now establish our results.

1. Let us first consider a surface V which is not a surface of special type. We consider a linear system $|C| = |D_1 + D_2 + D'_2|$, where D_1 and D_2 are the elliptic curves with index of intersection 1 which exist by Proposition 3. Since $(C, C) = 4$ we have $\dim |C| \geq 2$, and since the system $|C|$ does not have a fixed component, $\dim |C| = 2$ by Proposition 1.

Each of the three following curves belongs to the system $|C|$:

$$D_1 + D_2 + D_1', \quad 2D_1 + D_2', \quad 2D_1' + D_2';$$

on the other hand, these three curves do not belong to any one-dimensional subsystem of this system, since the last two of them generate the pencil $|2D_1 + D_2'|$, which does not contain the first. Hence the points $P_1 = D_1 \cap D_2'$ and $P_2 = D_1' \cap D_2'$ lying on each of these curves are base points of the system $|C|$. Since the pencil $|2D_1|$ has no base points, all the base points of the system $|C|$ lie on the curve D_2', and since $(C, D_2') = 2$, the system $|C|$ has no base points other than the ones indicated. Thus the number of variable points of intersection of the curves of this system is equal to two.

We now perform a σ-process at each of the points P_1 and P_2 and denote the curves introduced by the σ-process by Γ_1, Γ_2, the surface obtained by $\widetilde{V}$, and the minimal pre-image of a curve $H \subset V$ under the projection $\widetilde{V} \to V$ by $\widetilde{H}$. We have

$$(\widetilde{C}, \widetilde{C}) = 2(\widetilde{C}, \widetilde{D}_1) = (\widetilde{C}, \widetilde{D}_1') = (\widetilde{C}, \widetilde{D}_2') = 0, \quad (\widetilde{C}, \widetilde{D}_2) = 2.$$

The system $|\widetilde{C}|$ maps the surface $\widetilde{V}$ onto $\mathbf{P}^2$ in two sheets; hence this surface can be mapped regularly onto the double plane $z^2 = F^n(x, y)$. Since each curve $ax + by + c = 0$ on V belongs to the system $|C|$ and hence has genus three, we have $n = 8$.

We proceed to the study of the ramification curve $F^8(x, y) = 0$. The mapping associated with the system $|\widetilde{C}|$ contracts each of the curves $\widetilde{D}_1, \widetilde{D}_1', \widetilde{D}_2'$, and thus the images of the curves Γ_1, Γ_2 coincide with the images of the curves

$$2\widetilde{D}_1 + \widetilde{D}_2' + 2\Gamma_1 \in |\widetilde{C}|, \quad 2\widetilde{D}_1' + \widetilde{D}_2' + 2\Gamma_2 \in |\widetilde{C}|,$$

i.e., they are straight lines. We denote these lines by p and q. Each of them belongs to the ramification curve $F^8(x, y) = 0$. In fact, if this is not so, the image of the curve Γ_1 (Γ_2) on the double plane is a two-sheeted covering of p (q). This two-sheeted covering must intersect a generic plane section of the form $ax + by + c = 0$, which is the image of some curve on $|\widetilde{C}|$, in two points. However, the indices of intersection $(\widetilde{C}, \Gamma_1)$, $(\widetilde{C}, \Gamma_2)$ are equal to one. Hence,

$$F^8(x, y) = p \cdot q \cdot F^6(x, y).$$

We denote by C_6 the zero divisor of the function $F^6(x, y)$ on the surface V. The divisor

$$(2\widetilde{D}_1 + \widetilde{D}_2' + 2\Gamma_1) + (2\widetilde{D}_1' + \widetilde{D}_2' + 2\Gamma_2) + C_6$$

is given by the equations $F^8(x, y) = 0$ and $z^2 = 0$; hence it belongs to the system $|8\widetilde{C}|$ and is divisible by two. Similarly, the divisor C_6: $C_6 = 2C_3$ belongs to the system $|6\widetilde{C}|$ and is divisible by two.

We now study the singular points of the ramification curve. Clearly, on the surface $\widetilde{V}$ there corresponds to each of them a curve that contracts to a point under the mapping associated with the system $|\widetilde{C}|$. Hence such a curve corresponds to each of the points

$$p \cap q, \quad p \cap \{F^6(x, y) = 0\}, \quad q \cap \{F^6(x, y)\} = 0.$$

These curves intersect Γ_1 or Γ_2, and thus their images on the surface V pass through one of the base points P_1, P_2 of the system $|C|$. On the other hand, every curve which is contracted under the mapping $V \to \mathbf{P}^2$ associated with the system $|C|$ intersects curves of this system only in their base points. If such a curve passes through the point P_1 and does not coincide with D_1 and D'_2, then its index of intersection with $2D_1 + D'_2 \in |C|$ is not smaller than three. Hence, of the curves intersecting Γ_1 (Γ_2) on $\widetilde{V}$, only $\widetilde{D}_1$ and $\widetilde{D}_2$ ($\widetilde{D}'_1$ and $\widetilde{D}'_2$) can contract.

Thus each of the lines p and q intersects the curve $F^6(x, y) = 0$ in no more than two points. We denote the image of the curve $\widetilde{D}_1$ (lying in the intersection $p \cap \{F^6(x, y) = 0\}$) by Q_1, the image of the curve $\widetilde{D}'_1$, which lies in the intersection $q \cap \{F^6(x, y) = 0\}$, by Q_2, and the image of the curve $\widetilde{D}'_2$, which lies in the intersection $p \cap q$ and, possibly, in $\{F^6(x, y) = 0\}$, by O.

We consider on the surface $\widetilde{V}$ the pencil $|2\widetilde{D}_1|$. Since $|2\widetilde{D}_1 + \widetilde{D}'_2| \in |\widetilde{C}|$, the images of the curves of this pencil on the plane $\mathbf{P}^2$ are lines passing through the point O. Since, on the other hand, $p_a(|2\widetilde{D}_1|) = 1$, their images on the double plane are also elliptic. Hence a generic line passing through the point O in the plane $\mathbf{P}^2$ must have exactly four geometrically distinct points of intersection of odd multiplicity with the curve $F^8(x, y) = 0$. Thus the point O is a point of intersection of multiplicity four or five and lies on the curve $F^6(x, y) = 0$. If this point has multiplicity five, then a generic curve of the pencil $|2\widetilde{D}_1|$ intersects the divisor C_3 on $\widetilde{V}$ in three points, which lie on the curve $\widetilde{D}'_2$. But this is impossible, since $|2\widetilde{D}_1|$ has no base points, and the index of intersection $(2\widetilde{D}_1, \widetilde{D}'_2)$ is equal to two. Hence the point O is a double point of the curve $F^6(x, y) = 0$.

We now consider the pencil $|\widetilde{D'_1 + D_2}|$, whose nonsingular generic curve has genus two. Since $|\widetilde{D'_1 + D_2}| + \widetilde{D}_1 \in |\widetilde{C}|$, the images of the curves of this pencil under the mapping associated with the system $|\widetilde{C}|$ are lines passing through the point Q_1. These lines must have six points of intersection of odd multiplicity with the curve $F^8(x, y) = 0$, and hence the point Q_1 is either a double or triple point of the curve $F^6(x, y) = 0$. This choice is determined by the multiplicity of the point $\widetilde{D}_1 \cap \widetilde{D}_2$ (a base point of $|\widetilde{D'_1 + D_2}|$) on the divisor C_3. We obtain analogous information about the point Q_2 by considering $|\widetilde{D_1 + D_2}|$.

We consider the pencil $|2\widetilde{D}_2|$. As can be easily seen, $\mathfrak{L}(2\widetilde{D}_2) \subset k(x, y)$, and thus curves of this pencil are mapped by the system $|\widetilde{C}|$ in two sheets. Since $(2\widetilde{D}_2, \widetilde{C}_1) = 4$, their images are quadrics. Since $(2\widetilde{D}_2, \widetilde{D}_1) = (2\widetilde{D}_2, \widetilde{D}'_1) = 2$, these quadrics pass through the points Q_1 and Q_2, and since $(2\widetilde{D}_2, \Gamma_1) = (2\widetilde{D}_2, \Gamma_2) = 0$, they have no other common points with the lines p and q. Hence Q_1 and Q_2 are either singular points of these quadrics or are points of contact with the corresponding lines p, q. On the other hand, a generic curve of each of the pencils $|\widetilde{D_1 + D_2}|$, $|\widetilde{D'_1 + D_2}|$ also intersects a generic curve of $|\widetilde{2D_2}|$ outside the singular set of the mapping given by the system $|\widetilde{C}|$, and this means that a generic curve passing through $Q_1 (Q_2)$ intersects the quadric under consideration not only in this point. Thus, a generic one of these quadrics is nonsingular at the points Q_1 and Q_2 and touches the lines p and q at those points. Since curves of $|\widetilde{2D_2}|$ are elliptic, their images have four points of intersection of odd multiplicity with the curve $F^8(x, y) = 0$. The index of intersection of $|\widetilde{2D_2}|$ with the divisor C_3 at points of the curves $\widetilde{D}_1, \widetilde{D}'_1$ is equal to the index of intersection of the quadrics under consideration and the curve $F^6(x, y) = 0$ at the points Q_1 and Q_2. Since $|2\widetilde{D}_2|$ has no base points and $(2\widetilde{D}_2, \widetilde{D}'_1) = (2\widetilde{D}_2, \widetilde{D}_1) = 2$, each of the indices of intersection at the points Q_1 and Q_2 is also even. However, they cannot be larger than four, since the points $\widetilde{D}_1 \cap \widetilde{D}_2$, $\widetilde{D}'_1 \cap \widetilde{D}_2$ are of multiplicity two on the divisor C_3. Hence each of them is actually equal to four, the points $\widetilde{D}_1 \cap \widetilde{D}_2$, $\widetilde{D}'_1 \cap \widetilde{D}_2$ are of multiplicity two on C_3, the points Q_1 and Q_2 are double on the curve $F^6(x, y) = 0$, and the curve $F^6(x, y) = 0$ has at these points an order of contact of not less than four not only with the quadrics under consideration, but also with their generic tangents, the lines p and q.

With this we have proved the following theorem.

Theorem 2. *An Enriques surface which is not of special type is birationally equivalent to a double plane with a ramification curve of eighth degree which is composed of a curve of sixth degree* $F^6(x, y) = 0$ *and two lines* p *and* q. *Further, the curve* $F^6(x, y) = 0$ *necessarily has three double points:*

O – *the point of intersection of the lines* p *and* q, *where its tangents are different from* p *and* q;

Q_1 *and* Q_2 – *double points, where the lines* p *and* q *are tangents and the order of contact with these tangents is equal to four. Here the indices of intersection at these points with the curves of the pencil of the quadrics passing through* *the points* Q_1 *and* Q_2 *and tangent to the lines* p *and* q *at these points are also equal to four.*

2. Now let the surface V be a surface of special type and let an irreducible system of elliptic curves L have index of intersection two with a rational curve Θ. We shall prove that the system $|L + \Theta + K + (L/2)|$ plays in this case the same role that the system $|C|$ did in the proof of Theorem 2.

We first consider the linear system $|D| = |L + \Theta + K|$. By a theorem of Kodaira (cf. [25], Theorem 2.5), $H^1(V, \Omega(D)) = 0$, and hence, by the Riemann-Roch theorem,

$$\dim |D| = \frac{(D, D)}{2} = 1.$$

We shall prove that the system $|D|$ is irreducible. Let $|H|$, G be its nonfixed and fixed parts respectively. Since the system $|L|$ is irreducible, we have

$$(L, H) > 0, \quad (L, G) \geq 0$$

(if $(L, H) = 0$, the systems $|L|$ and $|H|$ coincide because of the equality of dimensions, but then $G \sim \Theta + K$, which contradicts the rationality of the curve Θ). Since $(D, L) = 2$, and (L, H) is even (by the corollary to Proposition 2), we have

$$(L, H) = 2, \quad (L, G) = 0.$$

The curve Θ cannot occur in the fixed part of the system $|D|$, since $\dim |L'| = 1$ (by Proposition 2), and hence $(H, \Theta), (G, \Theta) \geq 0$. Since $(D, \Theta) = 0$, we have $(H, \Theta) = (G, \Theta) = 0$. Now we obtain

$$(H, H) + (H, G) = 2, \quad (H, G) + (G, G) = 0.$$

Therefore the following cases are possible:

1) $(H, H) = 0$, $(H, G) = 2$, $(G, G) = -2$;

and

2) $(H, H) = 2$, $(H, G) = (G, G) = 0$.

In case 1) the system $|H|$ is an irreducible (since $\dim |H| = 1$) pencil of elliptic curves, which is impossible in view of the corollary to Proposition 2, for $(H (L/2)) = 1$.

Thus we must have case 2). We will prove that $G = 0$. Since the divisor $(L/2) + (L/2)' + \Theta$ belongs to the system $|D|$, and Θ does not belong to G, we have $G \leq (L/2) + (L/2)'$. Since $((L/2), (L/2)) = 0$, and the carrier of the divisor $(L/2)$ is connected, this divisor does not contain a component (not even a reducible one) with zero index of selfintersection. Since the same can be said about the divisor $(L/2)'$, and the divisor G decomposes into two disjoint components G.C.D. $(G, (L/2))$, G.C.D. $(G, L/2)')$ and has zero index of selfintersection, we have

$C = 0$. Thus the system $|D|$ is irreducible.

We note that this pencil has two base points not lying on the curve Θ $((D, \Theta) = 0)$; namely, if D_0 is an irreducible curve of this pencil, then $D_0 \cap (L/2)$ and $D_0 \cap (L/2)'$ are base points since they both lie also on the curve $(L/2) + (L/2)' + \Theta \in |D|$.

We consider, finally, the linear system $C = |D + (L/2)|$. Since $(C, C) = 4$, we see that $\dim |C| \geq 2$. On the other hand, the dimension of this system must be smaller than the dimension of the irreducible system $|D + L| = |2L + \Theta + K|$. According to the theorem of Kodaira,

$$H^1(V, \Omega(D + L)) = 0,$$

and since $(D + L, D + L) = 6$, the dimension of this system is equal to three. Thus $\dim |C| = 2$.

The system $|C|$ contains the one-dimensional subsystems $|D| + (L/2)$ and $|L| + \Theta + (L/2)'$. Hence this system has no fixed component, is not composed of a pencil, and is thus irreducible. The base sets of this subsystem intersect, and do so only in the points

$$D_0 \cap \left(\frac{L}{2}\right)' = P_1, \quad \left(\frac{L}{2}\right) \cap \Theta = P_2.$$

These two points are the only base points of the system $|C|$.

At each of the points P_1 and P_2 we perform a σ-process and introduce notation analogous to that of subsection 1. We again obtain a regular mapping of the surface $\widetilde{V}$ onto the double plane $z^2 = F^8(x, y)$ with ramification curve $F^8(x, y) = 0$ which decomposes into two lines p and q and a curve $F^6(x, y) = 0$. The pre-image of the last can again be written in the form $2C_3$. Finally, as in subsection 1, the lines p and q will intersect the curve $F^6(x, y) = 0$ only in the images of the contracted curves, namely the curve $\widetilde{\Theta} + \widetilde{(L/2)}'$ which is contracted to the point $O = p \cap q$, and the curve $\widetilde{(L/2)}$, which is contracted to the point $Q \in p$.

We now study the behavior of the curve $F^8(x, y)$ at the points O and Q. First, it is clear that the point O is the only point of intersection of the line q and the curve $F^6(x, y) = 0$. Hence their index of intersection at this point is equal to six.

We consider, further, the pencil of elliptic curves $|\widetilde{L}|$ on the surface $\widetilde{V}$. Since

$$|\widetilde{C}| = |\widetilde{L}| + \widetilde{\Theta} + \left(\frac{\widetilde{L}}{2}\right)',$$

the images of the curves of this pencil are lines passing through the point O. Each of them must have four points of intersection of odd multiplicity with the curve $F^8(x, y) = 0$. The pre-image of the point O on the generic curve of $|\widetilde{L}|$ consists of two distinct points of its intersection with the curve $\widetilde{\Theta}$, and hence the point O is not a ramification point under the mapping of the curves of $|\widetilde{L}|$ onto their images. This means that a generic line passing through O has at this point an intersection of even multiplicity (namely, of multiplicity four) with the curve $F^8(x, y) = 0$. Hence the point O is a double point of the curve $F^6(x, y) = 0$, and this curve itself is tangent to the line q.

Analogously, considering the pencil $|\widetilde{D}|$ and the images of its curves, i.e. lines passing through Q, we see that Q is also a double point of the curve $F^6(x, y) = 0$.

We now turn to the system $|\widetilde{C}'|$ on the surface $\widetilde{V}$. Since $(\widetilde{C}, \widetilde{C}') = 4$ and $\mathfrak{L}(\widetilde{C}') \subset k(x, y)$, the images of the curves of the system $|\widetilde{C}'|$ on the plane are quadrics. Just as for the quadrics of subsection 1, we can prove that they intersect the line p in the points O and Q and the line q only in the point O. Hence the index of intersection of such a quadric with the curve $F^6(x, y) = 0$ at the point O is not less than three, and at the point Q is not less than two. Correspondingly, the indices of intersection with the curve $F^8(x, y) = 0$ at these points are not less than six and three. Further, as is not difficult to see, each of the points O, Q is a ramification point under the mapping of a generic one of our quadrics onto the plane (since $(\widetilde{C}', \widetilde{(L/2)}) = (\widetilde{C}', \widetilde{(L/2)}') = (\widetilde{C}', \widetilde{\Theta}) = 1$, and the point $\widetilde{(L/2)} \cap \widetilde{\Theta}$ is a base point for the system $|\widetilde{C}'|$); hence the indices of intersection with the curve $F^8(x, y)$ which we have been discussing are odd. Since the genus of a nonsingular generic curve of the system $|\widetilde{C}'|$ is equal to three, there are in all eight points of intersection of odd multiplicity, which is possible only in the case when the index of intersection at the point O is equal to seven and at the point Q is equal to three. Correspondingly, the indices of intersection with the curve $F^6(x, y) = 0$ at these points are equal to four and two, and outside them, to six.

The indices of intersection of the generic curve of the system $|\widetilde{C}'|$ with the divisors $n\widetilde{(L/2)}$, $m\widetilde{(L/2)}' + s\widetilde{\Theta}$, where m, n, s are the coefficients with which these curves occur in the divisor C_3, are also equal to two and four, and the index of intersection of a curve of the pencil $|\widetilde{L}|$ with the divisor $m\widetilde{(L/2)}' + s\widetilde{\Theta}$ is equal to two. Since

$$(\widetilde{L}, \widetilde{\Theta}) = 2, \quad \left(\widetilde{C}', \left(\frac{\widetilde{L}}{2}\right)\right) = \left(\widetilde{C}', \left(\frac{\widetilde{L}}{2}\right)'\right)$$

$$= (\widetilde{C}', \widetilde{\Theta}) = 1, \ \left(\widetilde{L}, \left(\frac{\widetilde{L}}{2}\right)\right) = \left(\widetilde{L}, \left(\frac{\widetilde{L}}{2}\right)'\right) = 0,$$

we have $s = 1$, $m = 3$, $n = 2$.

We consider finally the system $|2\widetilde{D}|$. As it is not difficult to prove, this system has no base points. Moreover, if $2\widetilde{D}_0$ is a divisor of this system, then

$\mathfrak{L}(2\widetilde{D}_0) \in k(x, y)$, and thus the curves of this system are mapped onto a plane in two sheets. Since $(2\widetilde{D}, C) = 6$, their images will be curves of the third degree which pass through the points O and Q, analogously to the quadrics considered above, and do not intersect the lines p and q in other points. The systems $|\widetilde{L}|$ ($|\widetilde{D}|$) and $|2\widetilde{D}|$ on $\widetilde{V}$ have index of intersection four, where the generic curves of these systems do not intersect the pre-image of the ramification curve $F^8(x, y) = 0$. Hence the lines passing through $O(Q)$ intersect a generic one of the cubics under consideration in two points which are different from $O(Q)$. Thus our cubics are nonsingular at O and Q and have at O a point of contact of third order with the line q and of first order with the line p, and the point Q is a point of contact of these cubics and the line p. The indices of intersection of a generic cubic with the curve $F^6(x, y) = 0$ and a generic curve of the system $|2\widetilde{D}|$ with the divisor C_3 coincide both at the point O and at the point Q and on its pre-image. Hence the index of intersection at the point O is equal to six, and at the point Q to four, since

$$\left(2\widetilde{D}, 3\left(\frac{\widetilde{L}}{2}\right)' + \widetilde{\Theta}\right) = 6, \quad \left(2\widetilde{D}, 2\left(\frac{\widetilde{L}}{2}\right)\right) = 4.$$

We have thus proved the following theorem.

Theorem 3. *An Enriques surface of special type is birationally equivalent to a double plane $z^2 = F^8(x, y)$ whose ramification curve decomposes into two lines, p and q, and a curve of sixth degree $F^6(x, y) = 0$. The line p intersects the curve $F^6(x, y) = 0$ at two points O and Q and the line q at the point O. Each of these points is double on the curve $F^6(x, y) = 0$. The order of contact of the curve $F^6(x, y) = 0$ with the line p at the point Q is equal to four, and with the line q at the point O to six. The same orders of contact with the curve $F^6(x, y) = 0$ at these points are possessed by the cubics passing through the points O and Q, tangent to the lines q and p respectively, and having no other points in common with these lines.*

We calculate now the number of parameters upon which the family of such curves $F^8(x, y) = 0$ on the plane depends (up to projective equivalence). We can assume here that the lines p, q and the point Q are given such that the equation of p is $x = 0$ and the equation of q is $y = 0$, and $Q = (0, 1)$. The total collection of the coefficients of the curve $F^8(x, y) = 0$ considered up to proportionality is equal to 27.

The conditions on the decomposition of this curve with respect to the lines p and q reduces the number of parameters to 11. These conditions automatically take into account that the point O is double; moreover, if they are satisfied, the singularity of the point Q on the curve $F^6(x, y) = 0$ decreases the number of free parameters by only one more.

We now turn to the condition for the cubic. A local uniformizer of this cubic at the point O is a variable x, and the expansion of the variable y with respect to this uniformizer starts with terms of third degree. Our condition is that the expansion of the function $F^6(x, y)$ in powers of x at the point O, being obtained after a substitution of expansion for y, must begin with terms of the sixth degree. All the coefficients of the polynomial $F^6(x, y)$ for powers of x that are of degree smaller than six are equal to zero (the condition of tangency of sixth order with q), and the coefficient for y is also equal to zero (the point O is double). Hence the powers of the uniformizer that are of degree less than six can be obtained only from the products xy, x^2y. This means that the coefficients of these products in the polynomial $F^6(x, y)$ are equal to zero (one cannot get rid of the x^5 term of the expansion at the expense of the relations, since the whole family of corresponding cubics is being considered).

In an analogous way, the condition on the contact at the point Q of the curve $F^6(x, y) = 0$ and the cubic decreases the number of free parameters by one more (the coefficient of the product $x(y-1)$ in the expansion of the polynomial $F^6(x, y)$ with respect to powers of x and $y - 1$ must be equal to zero).

We are thus left with 12 free parameters. However, the family of projective transformations of the space $\mathbf{P}^2$ taking into themselves the lines p, q, and the point Q is a subgroup of dimension three. We shall show that a general transformation of this subgroup operates on our set of curves without fixed points. In fact, if $(x_0 : x_1 : x_2)$ are homogeneous coordinates in $\mathbf{P}^2$ such that $x = x_0/x_2$, $y = x_1/x_2$, then the transformations of our subgroup can be written in the form

$$x_0' = ax_0, \quad x_1' = (c+d)\,x_1, \quad x_2 = bx_0 + cx_1 + dx_2$$

(we recall that they leave in place the lines $x_0 = 0$, $x_1 = 0$ and the point $(0:1:1)$). From this we obtain the relation

$$\alpha x_0' - x_1' + x_2' = d\,(\alpha x_0 - x_1 + x_2),$$

where $\alpha = b/(d-a)$. Thus the line $\alpha x_0 - x_1 + x_2 = 0$ also remains invariant. After a transformation of coordinates we can assume that this line is at infinity, and our transformation can be written in the form: $x' = ax$, $y' = y$. Clearly, such a transformation does not leave fixed any of our curves.

Hence, with accuracy up to projective equivalence, the family of curves $F^8(x, y) = 0$ *depends on nine parameters. It follows easily that an Enriques surface of special type corresponds in the space of moduli of Enriques surfaces (which has dimension ten, as was proved in Chapter* X, §5) *to a subset of dimension not greater than nine.*

§3. The converse of the theorem of the preceding section

1. Theorem 4. *Let* $z^2 = F^8(x, y)$ *be a double plane whose ramification possesses the properties indicated in the formulation of Theorem 2; let, moreover, the curve* $F^6(x, y) = 0$ *be irreducible and have no more than double singular points. Then* $\widetilde{V}$, *the relatively minimal nonsingular model of this double plane, is an Enriques surface.*

Proof. Let ϕ be the natural regular mapping of $\widetilde{V}$ onto $\mathbf{P}^2$. We shall calculate the canonical class of $\widetilde{V}$. We consider on $\mathbf{P}^2$ the meromorphic differential Ω, which has no zeros, such that the polar divisor of Ω, which does not pass through singular points of the curve $F^8(x, y) = 0$, is given by an equation of third degree $G^3(x, y) = 0$. We shall find a zero divisor of the differential $\phi^*(\Omega)$; for this it is sufficient to find zeros of the differential $dx \wedge dy$. If the point $P \in \widetilde{V}$ belongs to the pre-image of a singular point of the curve $F^8(x, y) = 0$, then this point is a zero of $dx \wedge dy$, as follows from the equalities

$$dx \wedge dy = \frac{2zdz \wedge dy}{\frac{\partial F^8}{\partial x}(P)} = \frac{2zdx \wedge dz}{\frac{\partial F^8}{\partial y}(P)}.$$

We shall show that if a curve Γ belongs to the pre-image of a singular point of the curve $F^6(x, y) = 0$, then the order of the zero on this curve of the differential $dx \wedge dy$ is no more than half the order of the zero of the function $F^6(x, y)$. It will follow from this that the zero divisor of the differential $(dx \wedge dy)^2$ contains a zero divisor of the function $F^6(x, y)$ which is equivalent to the polar divisor $(G^3(x, y))^2 = 0$ of this differential, i.e., it will follow that the double canonical class $2K$ of the surface $\widetilde{V}$ is equivalent to an effective divisor containing as a component the minimal pre-images of the lines p and q – the curves Γ_1 and Γ_2 – and that $\kappa(\widetilde{V}) \geq 0$.

Thus, let P be a nonsingular point of the curve Γ. We shall choose local coordinates u, v in a neighborhood of this point such that the equation $u = 0$ gives the curve Γ. We have

$$F^6(x, y) = u^k f(u, v), \quad x = u^{s_1} g(u, v), \quad y = u^{s_2} h(u, v),$$

where k, s_1, s_2 are the orders of the zeros that the corresponding functions have on the curve Γ (without loss of generality, we can assume that $x(P) = y(P) = 0$). Since, by assumption, the point $\phi(P)$ is a double point of the curve $F^6(x, y) = 0$, we can assume that

$$\frac{\partial^2 F^6}{\partial y^2}(\varphi(P)) \neq 0.$$

As is easily seen,

$$dx \wedge dy = u^{s_1+s_2-1} \cdot \Delta(u, v)\, du \wedge dv,$$

$$\frac{\partial F^6}{\partial y}\, dx \wedge dy = u^{k+s_1-1} \varphi(u, v)\, du \wedge dv .$$

Denoting by m and ϵ the orders of the zeros of the functions $\Delta(u, v)$ and $\phi(u, v)$ on Γ, we have

$$\frac{\partial F^6}{\partial y} = u^{k-s_2-m+\varepsilon} \psi(u, v).$$

Differentiating this identity and multiplying the result by dx, we obtain

$$\frac{\partial^2 F^6}{\partial y^2}\, dx \wedge dy = u^{k+s_1-s_2-m+\varepsilon-1} \eta(u, v)\, du \wedge dv ,$$

and, since $(\partial^2 F^6/\partial y^2)(P) \neq 0$,

$$s_1 + s_2 - 1 + m \geqslant k + s_1 - s_2 - m + \varepsilon - 1.$$

Hence $2s_2 + 2m \geq k$ and $2(s_1 + s_2 - 1 + m) \geq k$.

We shall prove that $\kappa(\widetilde{V}) \leq 0$. We consider on the surface $\widetilde{V}$ the linear system $|L|$, where L is the minimal pre-image of the generic line passing through the point O. As is easy to see, the geometric genus of this pre-image is equal to one. Let L_1 and L_2 be two divisors of this system. Since the index of intersection of the lines $\phi(L_1)$ and $\phi(L_2)$ is equal to one, the index of intersection $(\phi^{-1}\phi(L_1), L_2)$ is equal to two. For an appropriate divisor Θ that contracts to a point under the mapping ϕ,

$$\varphi^{-1}\varphi(L_1) = L_1 + \Theta,$$

where the carriers of the divisors L_2 and Θ intersect. On the other hand, the pre-image of the point O on L_2 consists either of two nonsingular points, or of one singular point. Thus

$$(L_2, \Theta) \geqslant 2, \quad (L_1, L_2) \leqslant 0.$$

Since the divisors L_1 and L_2 cannot have common components, $(L_1, L_2) = 0$, and the system $|L|$ is a pencil without base points, a generic curve of which is nonsingular. Hence the surface $\widetilde{V}$ contains a pencil of elliptic curves, and it follows that $\kappa(\widetilde{V}) \leq 1$ (see Chapter VI).

We consider, further, the linear system $|D|$, where D is the minimal pre-image of the generic quadric passing through the points Q_1 and Q_2 and tangent to the lines p and q. If D_1 and D_2 are two generic curves of this system, then each of the indices of intersection

$$(\varphi(D_1), \varphi(D_2))_{Q_1}, \quad (\varphi(D_1), \varphi(D_2))_{Q_2}$$

is equal to two, and hence the indices of intersection

$$(\varphi^{-1}\varphi(D_1), D_2)_{[\varphi^{-1}(Q_1)]}, \quad (\varphi^{-1}\varphi(D_1), D_2)_{[\varphi^{-1}(Q_2)]}$$

are equal to four. The divisor $\phi^{-1}\phi(D_1)$ is equal to the sum of the divisor D_1 and appropriate divisors Θ_1, Θ_2 which contract into the points Q_1, Q_2 under the mapping ϕ; here the indices of intersection (Θ_1, D_2), (Θ_2, D_2) are positive. If the index of intersection $(D_1, D_2)_{[\varphi^{-1}(Q_1)]}$ (or $(D_1, D_2)_{[\varphi^{-1}(Q_2)]}$) is positive, then the system $|D|$ has a base point on the curve $\phi^{-1}(Q_1)$ (or $\phi^{-1}(Q_2)$). If the curves D_1, D_2 are singular at this point, then their index of intersection at this point is not smaller than four. Hence the assumption that one of the numbers

$$(D_1, D_2)_{[\varphi^{-1}(Q_1)]}, \quad (D_1, D_2)_{[\varphi^{-1}(Q_2)]}$$

is positive leads to a contradiction. Since the quadrics $\phi(D_1)$, $\phi(D_2)$ intersect only at the points Q_1, Q_2, the index of intersection (D_1, D_2) is in general equal to zero. Hence the system $|D|$ is a pencil without base points, a generic curve of which is nonsingular. Since the geometric genus of a generic curve of the system $|D|$ is equal to one, the surface $\widetilde{V}$ contains two pencils of elliptic curves. If $\kappa(\widetilde{V}) = 1$, some multiple of the canonical class varies in a pencil. Since $(K, L) = (K, D) = 0$, this system must coincide simultaneously with the systems $|L|$ and $|D|$. We have obtained a contradiction.

Hence $\kappa(\widetilde{V}) = 0$. Since a divisor of $|2K|$ is equivalent to an effective one, on the absolutely minimal model V it is equivalent to zero (for all surfaces with $\kappa = 0$ other than the surfaces with $p_g = q - 1 = 0$, this follows from the results of Chapters VIII and X, and for surfaces with $p_g = q - 1 = 0$ it follows from Chapter VII).

We shall now prove that our surface is an Enriques surface.

First of all, it is not Abelian, for there can exist no linear system of elliptic curves on an Abelian surface.

Let, further, $K(V) \sim 0$, $q(V) = 0$. We denote by $|\widetilde{C}|$ the linear system on $\widetilde{V}$ of the pre-images of lines on $\mathbf{P}^2$. It is clear that

$$p_g(\widetilde{C}) = p_a(\widetilde{C}) = 3.$$

Since

$$(\widetilde{C}, \widetilde{C} + K) = 2p_a(\widetilde{C}) - 2 = 4,$$

and $(\widetilde{C}, \widetilde{C}) = 2$, it follows that $(\widetilde{C}, K) = 2$ and the canonical class of the surface $\widetilde{V}$ consists of exactly two exceptional curves (we recall that $\widetilde{V}$ was assumed to be relatively minimal, i.e., not to contain exceptional curves of the first kind that contract under the mapping ϕ associated with the system $|\widetilde{C}|$, in other words curves whose index of intersection with $|\widetilde{C}|$ is equal to zero). These curves are the curves $\Gamma_1 = \phi^{-1}(p)$ and $\Gamma_2 = \phi^{-1}(q)$. If we contract them, we find on the minimal model V a system $|C|$ of curves of genus three with two base points; here $(C, C) = 4$. Further, let $|H|$ be the system of minimal pre-images on $\widetilde{V}$ of lines on $\mathbf{P}^2$ passing through the point Q_2. The geometric genus of the curves of this system is two. If H_1 and H_2 are two generic divisors from $|H|$, the index of intersection $(\phi(H_1), \phi(H_2))_{Q_2}$ is equal to one, and hence the index of intersection $(\phi^{-1}\phi(H_1), H_2)_{[\varphi^{-1}(Q_2)]}$ is equal to two. Hence $(H, H) \leq 1$. We shall show that $(H, H) = 1$. In fact, let the index of selfintersection (H, H) be equal to zero. Since the curves of $|H|$ intersect Γ_1 in a single point, do not pass through the pre-image of the point Q_2 on Γ_2 (since $(H, H) = 0$ the system $|H|$ has no base points), and cannot pass through other points of Γ_2 (for their images on the plane do not do that either), we have

$$p_a(H) = \frac{(H, H + \Gamma_1 + \Gamma_2)}{2} + 1 = 1.5.$$

Since $(H, H) = 1$, the unique pre-image of the point Q_2 on a generic curve of the system $|H|$ is a nonsingular point (the argument is analogous to the one given above for the systems $|L|$ and $|D|$), and $p_a(H) = 2$. Now we have

$$(HK) = 1, \quad (H\Gamma_2) = 1, \quad (H\Gamma_1) = 0.$$

Since the index of intersection of the systems $|H|$ and $|\widetilde{C}|$ on $\widetilde{V}$ is equal to two, their index of intersection on V will be equal to three; moreover, the index of selfintersection of the system $|H|$ on V is not less than two. Since each of the systems $|H|$ and $|C|$ is irreducible,

$$\dim H^1(V, \Omega(C)) = \dim H^1(V, \Omega(K + C)) = \dim H^1(V, \Omega(D)) = 0,$$

by the theorem of Kodaira already cited frequently. Therefore by the Riemann-Roch theorem we find that $\dim|C| = 3$ and $\dim|D| = 2$. Since $(C - D, C - D) \geq 4 - 6 + 2 = 0$, we have $\dim|C - D| \geq 1$. But this contradicts the irreducibility of the system $|C|$.

It remains for us to prove that the surface $\widetilde{V}$ cannot be a surface with $\kappa = p_g = q - 1 = 0$ (see Chapter X, §1). We shall assume that it nevertheless is such a surface, and shall then show that the pencil $|L|$ contains four divisors of multiplicity two, and then obtain a contradiction. We consider on the surface V the system $|nL|$. This system is n-dimensional, and a generic curve of it consists of n connected components. According to the theorem of Kodaira, there exist no holomorphic differentials vanishing on this curve. Hence

$$\dim H^1(V, \Omega(nL + K)) = n - 1, \quad \dim H^0(V, \Omega(nL + K)) = n - 1.$$

Denoting the nonfixed part of the $(n - 2)$-dimensional system $|nL + K|$ by D_ν, and the fixed part by Θ, we obtain

$$(D_\nu, L) = (\Theta, L) = (\Theta, \Theta) = 0,$$

and thus the system $|D_\nu|$ is composed of the pencil $|L|$, and each component of the divisor Θ is also a component of some divisor of $|L|$. If Θ_1 is a connected component of Θ, then, using the standard algebraic lemma about nonpositive definite quadratic forms (see for example [35]), we find that $\Theta_1 = rL_1$, where the number r is rational and less than one, and the divisor L_1 belongs to $|L|$. Since $(C, L) = 2$, the number r is equal to $\frac{1}{2}$, and this means that the divisor L_1 is of multiplicity two. Now let $\Theta_1, \cdots, \Theta_s$ be the connected components of the divisor Θ. Since $2K \sim 0$, the systems $|2(D_\nu + \Theta)|$ and $|2nL|$ are equivalent. We now have

$$2(n-2)L + 2\Theta_1 + \ldots + 2\Theta_s \sim 2nL, \quad 2\Theta_1 + \ldots + 2\Theta_s \sim 4L.$$

Hence the system $|L|$ actually contains four divisors of multiplicity two.

We now consider the line $\phi(L_1)$. If its equation is $ax + by + c = 0$, then we have established that the function $ax + by + c$ on V has a zero of second (or higher) order on the support of the divisor L_1. This, however, is possible only when the line $\phi(L_1)$ belongs to the ramification curve $F^8(x, y) = 0$. Since this curve contains only two curves, we have obtained a contradiction. Theorem 4 is thus proved.

2. In this subsection we will prove for Enriques surfaces of special type an assertion analogous to Theorem 4.

As it is not difficult to see, all the above arguments remain valid in the case now under consideration. The only exception is the proof of the inequality $\kappa(V) \neq 1$. Moreover, we must prove that the surface V is a surface of special type.

We begin with the proof of the inequality, assuming it to be false. We consider on $\widetilde{V}$ the linear system $|B|$, where B is the minimal pre-image of the generic cubic occurring in the proof and formulation of Theorem 3. As is easily seen, the geometric genus of such a pre-image is equal to five (eight ramification points under the mapping onto the elliptic curve), and the index of intersection of two curves of the system $|B|$ outside the pre-images of the points O and Q is eight. Moreover, a generic curve of $|B|$ does not have a singularity outside the pre-images of these points. We shall prove that it has no singularities on these points either, or that otherwise $|B|$ has a base point there. So let there be no base point, and let Γ be a curve of the pre-image of the point $O(Q)$ at the intersection with which each divisor of the system $|B|$ has a singular point. Let f be an arbitrary function effecting an equivalence among the divisors of $|B|$. Then the differential of this function is zero at each point of the curve Γ, and it is thus constant on Γ. This in turn means that either the index of intersection (B, Γ) is equal to zero, or the system $|B|$ has a base point on Γ.

Hence curves of the system $|B|$ can have singularities only in base points lying in the pre-image of the points O and Q. We now resolve these singularities with the aid of σ-processes carried out at these base points. We obtain a surface

$\tilde{\tilde{V}}$ and on it a linear system $|\tilde{B}|$ of geometric genus five and with index of selfintersection equal to eight.

We now recall that on the surface $\tilde{V}$, and thus also on the surface $\tilde{\tilde{V}}$, there exists a pencil of elliptic curves $|L|$. Since $(B, L) > 0$, and some multiple of the canonical class is composed of the pencil $|L|$, we have $(\tilde{B}, K(\tilde{\tilde{V}})) > 0$. We obtain a contradiction by setting the values found for the invariants in the formula

$$p_a(\tilde{B}) = p_g(\tilde{B}) = \frac{(\tilde{B}, \tilde{B} + K(\tilde{\tilde{V}}))}{2} + 1.$$

We now turn to the last part of the proof, i.e. we find an irreducible rational curve Θ such that $(L, \Theta) = 2$. Considering the total and minimal pre-image of the generic line passing through the point O, we first find a divisor M, each of whose components contracts to a point under the transformation ϕ, and which is such that $M \sim C - L$. Considering, further, the line p on $\mathbf{P}^2$, we find a divisor N, each component of which is a component of the divisor M (only here do we use the difference of the ramification curves in the general and special cases), and such that $2\Gamma_1 + N \in L$. We now contract the curves Γ_1 and Γ_2 and consider the images of our divisors on the minimal model V. Since $(C, L) = 2$, we have

$$(M, M) = (C, C) - 2\,(C, L) + (L, L) = 0.$$

Since, further, the pencil $|L|$ contains two divisors of multiplicity two (by Proposition 2, §2) and since such divisors can arise only from pre-images of the lines p and q, the divisor N is of multiplicity two. We denote the divisor $N/2$ by $(L/2)'$. According to the classification of singular fibers of the elliptic pencil (see [65]), each of the components of N has the same multiplicity. Since $(C, L) = (C, N) = 2$, this multiplicity is exactly two, i.e., the divisor $(L/2)'$ is a curve. Hence the divisor $M - (L/2)'$ is effective, and since

$$\left(M - \left(\frac{L}{2}\right)', \left(\frac{L}{2}\right)'\right) = \frac{1}{2}(C, L) = 1,$$

it is nonzero. Further, since the index of intersection of the pencil $|L|$ with any divisor is nonnegative, and with any component of one of our divisors is equal to zero, the divisor $M - (L/2)'$ contains a component Θ not occurring in $(L/2)'$ and such that

$$(L, \Theta) = 2\left(\left(\frac{L}{2}\right)', \Theta\right) = 2;$$

here $M \geq (L/2)' + \Theta$. If the index of selfintersection (Θ, Θ) were nonnegative, i.e., if the curve Θ were not rational, we would have

$$\left(\left(\frac{L}{2}\right)' + \Theta, \left(\frac{L}{2}\right)' + \Theta\right) > 0,$$

$$\dim|C - L| = \dim|M| \geqslant \dim\left|\left(\frac{L}{2}\right)' + \Theta\right| \geqslant 1 = \dim|C| - \dim|L|,$$

i.e., the system $|C|$ would not be irreducible.

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