Geometry of Toric Varieties

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These notes present a course given during the Algebraic Geometry Summer School in Bilkent University (August 1995). This elementary introduction does not pretend to originality but to give examples and motivation for the study of toric varieties. These varieties play a prominent part giving explicit relations between combinatorial geometry and algebraic geometry. They provide also an important field of examples and models. The preface of [5] explains very well the interest of these objects "*Toric varieties provide a ... way to see many examples and phenomena in algebraic geometry... For example, they are rational, and, although they may be singular, the singularities are rational. Nevertheless, toric varieties have provided a remarkably fertile testing ground for general theories.*"

Basic references for toric varieties are [4], [5] and [7]. These references give proofs of the results and complete descriptions. They were used (a lot !) for writing these notes and the reader can consult them for useful complementary references and details. In particular, many points are omitted here, for lack of time during the course. For example, the reader will look at references for the notions and points of view such that character groups, one parameter subgroups, line bundles, Betti numbers etc...

I want to thank particularly Professor Sinan Sertöz and University of Bilkent for their hospitality during the Summer School.

1 From combinatorial geometry to toric varieties.

The procedure of the construction of (affine) toric varieties associates to a cone σ in $\mathbb{R}^n$ successively : the dual cone $\check{\sigma}$, a monoid S_σ, a finitely generated $\mathbb{C}$-algebra R_σ and finally the algebraic variety X_σ. In the following, we describe the steps of this procedure :

$$\sigma \quad \mapsto \quad \check{\sigma} \quad \mapsto \quad S_\sigma \quad \mapsto \quad R_\sigma \quad \mapsto \quad X_\sigma$$

and recall some useful definitions and results of algebraic geometry.

Let $A = \{x_1, \dots, x_r\}$ be a finite set of vectors in $\mathbf{R}^n$, the set

$$\{x \in \mathbf{R}^n \ : x = \lambda_1 x_1 + \cdots + \lambda_r x_r, \quad \lambda_i \in \mathbf{R}, \quad \lambda_i \geq 0\}$$

is called a polyhedral cone.

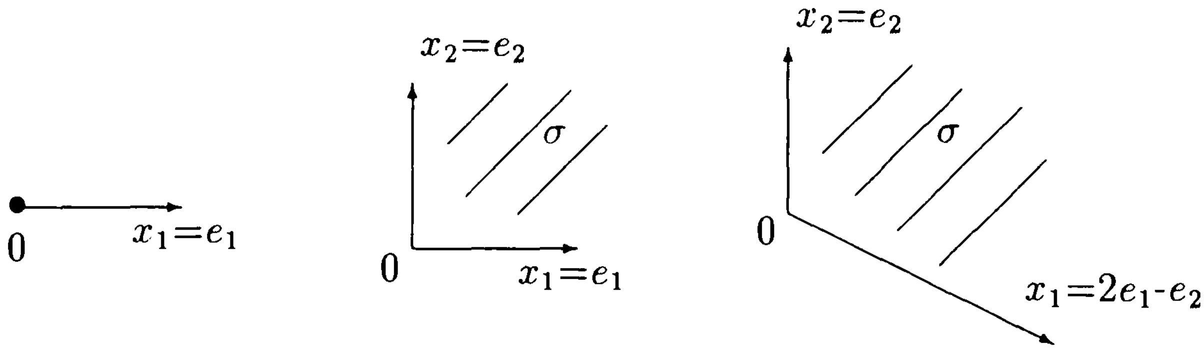
The *dimension* of σ, denoted $\dim \sigma$, is the dimension of the smallest linear space containing σ.

If $A = \emptyset$ then $\sigma = \{0\}$ is the zero cone.

In the following, N will denote a fixed lattice $N \cong \mathbf{Z}^n \subset \mathbf{R}^n$.

Definition. A cone σ is a *lattice* (or rational) cone if all the vectors $x_i \in A$ defining σ belong to N.

Examples 1. In $\mathbb{R}^2$ with canonical basis (e_1, e_2), we have the following cones :

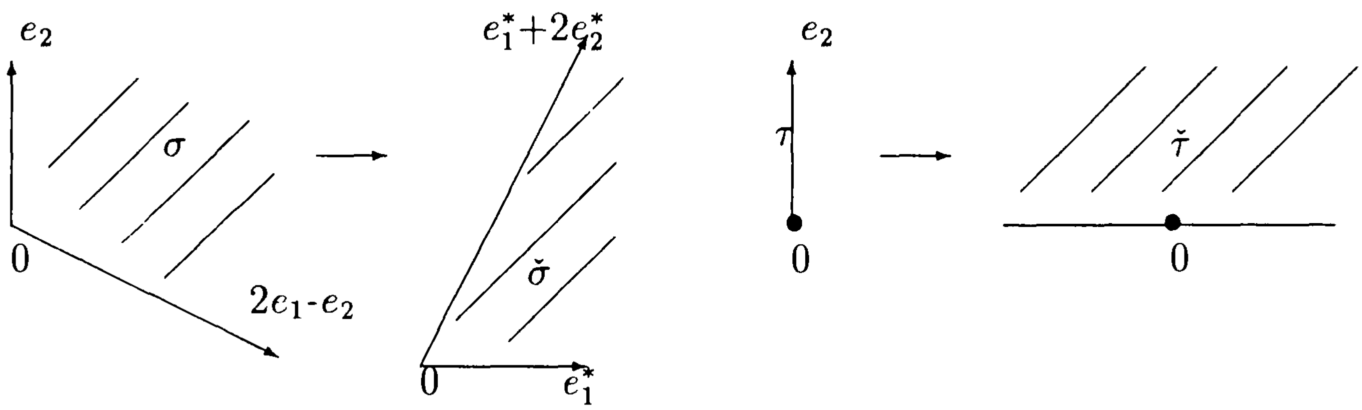


Definition. A cone is *strongly convex* if it does not contain any straight line going through the origin (i.e. $\sigma \cap (-\sigma) = \{0\}$).

One important step of the procedure of the construction of toric varieties is to define the dual cone associated to a cone. Let $(\mathbf{R}^n)^*$ be the dual space of $\mathbf{R}^n$ and $\langle\ ,\ \rangle$ the dual pairing. To each cone we associate the dual cone $\check{\sigma}$

$$\check{\sigma} = \{u \in (\mathbf{R}^n)^* \ : \ \langle u, v\rangle \geq 0 \quad \forall v \in \sigma\}$$

Examples 2. The canonical dual basis of $(\mathbf{R}^2)^*$ is denoted (e_1^*, e_2^*)



Giving a lattice N in $\mathbf{R}^n$, we define the dual lattice $M = \mathrm{Hom}_{\mathbb{Z}}(N; \mathbb{Z}) \cong \mathbb{Z}^n$ in $(\mathbf{R}^n)^*$ and we have the property :

Property. If σ is a lattice cone, then $\check{\sigma}$ is also a lattice cone (relatively to M).

Remark that if σ is a strongly convex cone, then $\check{\sigma}$ is not necessarily a strongly convex cone (see τ in examples 2).

Definition. Let σ be a cone and let $\lambda \in \check{\sigma} \cap M$, then $\tau = \sigma \cap \lambda^{\perp} =$

$\{v \in \sigma \ : \ \langle \lambda, v \rangle = 0\}$ is called a *face* of σ. We will write $\tau < \sigma$.

This definition coincides with the intuitive one. (Exercise).

A one dimensional face is called an *edge.*

Remark. If $\tau < \sigma$ then $\check{\sigma} \subset \check{\tau}$. (Easy exercise).

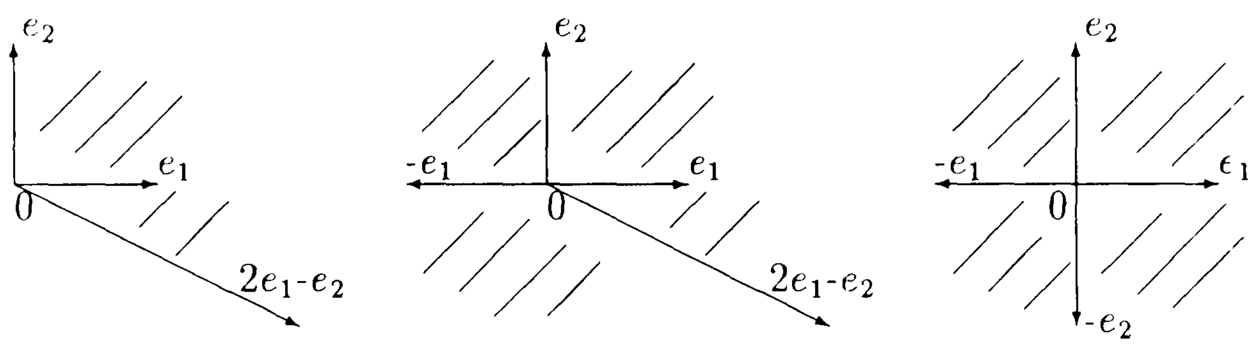
Definition. A *fan* Δ is a finite union of cones such that :

(i) every cone of Δ is a strongly convex, polyhedral, lattice cone.

(ii) every face of a cone of Δ is a cone of Δ,

(iii) if σ and σ' are cones of Δ, then $\sigma \cap \sigma'$ is a common face of σ and σ'.

Examples 3.



The aim of this course is to explain how we can associate an algebraic variety to each fan and to explain how properties of this algebraic variety can be read on the fan (in a combinatorial way). The first step is to associate to every cone σ a finitely generated monoid S_σ.

Definition. A *monoid* is a non empty set S with an associative and commutative law $+ : S \times S \to S$, with a zero element and satisfying the simplification law, i.e. :

$$s + t = s' + t \ \Rightarrow \ s = s' \text{ for } s, s' \text{ and } t \in S$$

Lemma. If σ is a cone, then $\sigma \cap N$ is a monoid.

Proof : If $x, y \in \sigma \cap N$, then $x + y \in \sigma \cap N$ and the rest is easily verified.

Definition A monoid S is *finitely generated* if there exists $a_1, \ldots, a_k \in S$ such that

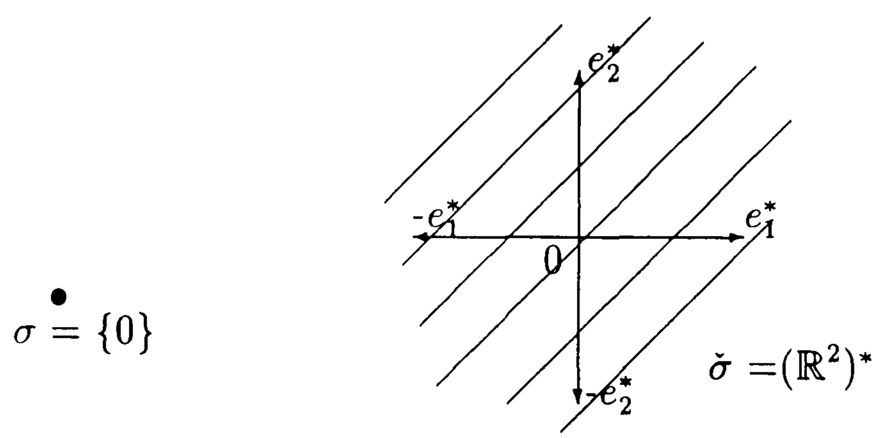
$$\forall s \in S, \; s = \lambda_1 a_1 + \cdots + \lambda_k a_k \text{ with } \lambda_i \in \mathbf{Z} \text{ and } \lambda_i \geq 0$$

Lemma. (Gordon's Lemma). If σ is a polyhedral lattice cone, then $\sigma \cap N$ is a finitely generated monoid.

Proof : Let $A = \{x_1, \cdots, x_r\}$ be the set of vectors defining the cone σ. Each x_i is an element of $\sigma \cap N$. The set $K = \{\sum t_i x_i, \quad 0 \leq t_i \leq 1\}$ is compact and N is discrete, therefore $K \cap N$ is a finite set. We show that it generates $\sigma \cap N$. In fact, every $x \in \sigma \cap N$ can be written $x = \sum (n_i + r_i) x_i$ where $n_i \in \mathbf{Z}_{\geq 0}$ and $0 \leq r_i \leq 1$. Each x_i and the sum $\sum r_i x_i$ belong to $K \cap N$, so we obtain the result.

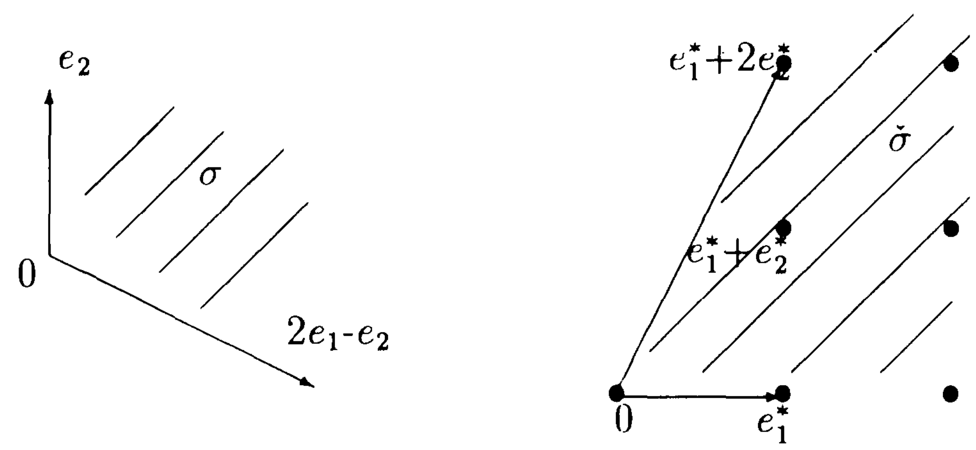
We will use this lemma for the polyhedral lattice cone $\check{\sigma}$ and will denote by S_σ the monoid $\check{\sigma} \cap M$.

Example 4. In $\mathbb{R}^2$, consider the cone reduced to the origin :



$S_\sigma = \check{\sigma} \cap M$ is generated by $(e_1^*, -e_1^*, e_2^*, -e_2^*)$.

Example 5. In $\mathbb{R}^2$, consider the following cone



$S_\sigma = \check{\sigma} \cap M$, marked •, is generated by $(e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^*)$.

Laurent polynomials

Denote by $\mathbf{C}[z, z^{-1}] = \mathbf{C}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$ the ring of Laurent polynomials. One of the important result in the definition of toric varieties, and key of the second step, is that the mapping

$$\begin{array}{rcl} \theta\ : \mathbf{Z}^n & \to & \mathbf{C}[z, z^{-1}] \\ a = (\alpha_1, \ldots, \alpha_n) & \mapsto & z^a = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \end{array}$$

is an isomorphism between the additive group $\mathbf{Z}^n$ and the multiplicative group of monic Laurent monomials. Monic means that the coefficient of the monomial is 1. This isomorphism is easy to prove and let as an exercise.

Definition. The support of a Laurent polynomial $f = \sum_{\text{finite}} \lambda_a z^a$ is defined by

$$\text{supp}(f) \ = \ \{a \in \mathbf{Z}^n \ : \ \lambda_a \neq 0\}$$

Proposition. For a lattice cone σ, the ring

$$R_\sigma = \{f \in \mathbf{C}[z, z^{-1}] \ : \text{supp}(f) \subset \check{\sigma} \cap M\}$$

is a finitely generated monomial algebra (i.e. is a $\mathbf{C}$-algebra generated by Laurent monomials).

This result is a direct consequence of the Gordon's Lemma.

The following section recalls how we can associate to each finitely generated $\mathbf{C}$-algebra (in particular to R_σ) a coordinate ring, then an affine variety.

2 Some basic results of algebraic geometry.

Let $C[\xi] = C[\xi_1, \ldots, \xi_k]$ be the ring of polynomials in k variables over C.

Definition. If $E = (f_1, \ldots f_r) \subset C[\xi]$, then

$$V(E) = \{x \in C^k \ : \ f_1(x) = \cdots = f_r(x) = 0\}$$

is the *affine algebraic set* defined by E.

Let $\mathcal{A}$ denote the ideal generated by E, then $V(\mathcal{A}) = V(E)$.

Definition. Let $X \subset C^k$, then

$$I(X) = \{f \in C[\xi] \ : \ f|_X = 0\}$$

is called the *vanishing ideal* of X.

Example 6. For a fixed point $x = (x_1, \ldots, x_k) \in C^k$, consider $E = \{\xi_1 - x_1, \cdots, \xi_k - x_k\}$. Then $V(E) = \{x\}$ and $I(\{x\}) = C[\xi](\xi_1 - x_1) + \cdots + C[\xi](\xi_k - x_k)$. It is a maximal ideal $\mathcal{M}_x$ (recall that an ideal $\mathcal{M}$ is maximal if for each ideal $\mathcal{M}'$ such that $\mathcal{M} \subset \mathcal{M}'$ then $\mathcal{M} = \mathcal{M}'$)

Theorem. (Weak version of the Nullstellensatz) : Every maximal ideal in $C[\xi]$ can be written $\mathcal{M}_x$ for a point x.

Corollary 1. There is a one-to-one correspondence between points in C^k and maximal ideals $\mathcal{M}$ of $C[\xi]$.

$$C^k \longleftrightarrow \{\mathcal{M} \subset C[\xi], \quad \mathcal{M} \text{ maximal ideal}\}$$

Lemma. Let $\mathcal{A}$ be an ideal of $C[\xi]$, then $V(\mathcal{A}) = \{x \in \mathbb{C}^k \ : \ \mathcal{A} \subset \mathcal{M}_x\}$.

Definition. Denote the vanishing ideal of $V(\mathcal{A})$ by $\mathcal{A}_V = I(V(\mathcal{A}))$, then $R_V = \mathbb{C}[\xi]/\mathcal{A}_V$ is the *coordinate ring* of the affine algebraic set $V(\mathcal{A})$. It is generated as a $\mathbb{C}$-algebra by the classes $\bar{\xi}_j$ of the coordinate functions ξ_j.

We remark that if $\mathcal{A} = \{0\}$, then $V(\mathcal{A}) = \mathbb{C}^k$ and $R_V = \mathbb{C}[\xi]$. The corollary 1, written for $\mathcal{A} = \{0\}$, is generalized for any ideal in the following way :

Corollary 2. There is a one-to-one correspondence

$$V \longleftrightarrow \{\mathcal{M} \subset R_V,\ \mathcal{M} \text{ maximal ideal}\} =: \mathrm{Specm}(R_V)$$

Defining the Zariski topology on each side (see, for example [4], VI.1), we obtain an homeomorphism

$$V \cong \mathrm{Specm}(R_V)$$

Remark. A finitely generated $\mathbb{C}$-algebra R can be written $\mathbb{C}[\xi_1, \ldots, \xi_k]/\mathcal{A}$, as a coordinate ring, for different k and ideals $\mathcal{A}$. It means that we associate by this way, different affine algebraic sets $V(\mathcal{A}) \in \mathbb{C}^k$. In fact, the corollary 2 shows that these representations $V(\mathcal{A})$ are all homeomorphic to an "abstract affine toric variety" X_σ. We will often write also X_σ for the representations of X_σ.

Remark. It can be useful to consider the spectrum $\mathrm{Spec}(R_V)$ whose elements are prime ideals of R_V. In our application, R_V is a domain, then $\mathrm{Spec}(R_V)$ is an irreducible variety and the prime ideals correspond to irreducible subvarieties of $V(\mathcal{A})$. For such a point of view and discussion, see [5], §1.3.

3 Affine toric varieties.

We are now able to define the affine toric variety associated to a cone σ :

Definition. The affine toric variety corresponding to σ is $X_\sigma := \mathrm{Specm}(R_\sigma)$.

The previous section shows that we can represent the finitely generated $\mathbb{C}$-algebra R_σ as a coordinate ring, according to a choice of generators of S_σ. Let us do this on an example, then give the general situation.

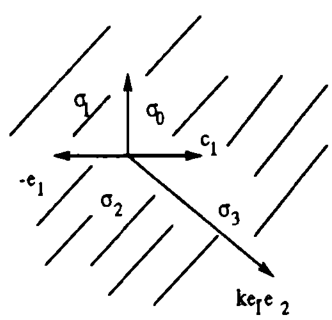
In the case of example 5, let $a_1 = e_1^*, a_2 = e_1^* + e_2^*$ and $a_3 = e_1^* + 2e_2^*$ be a system generators of S_σ. They correspond to monic Laurent monomials $u_1 = z_1, u_2 = z_1 z_2$ and $u_3 = z_1 z_2^2$ by the isomorphism θ. The $\mathbb{C}$-algebra R_σ can be represented as

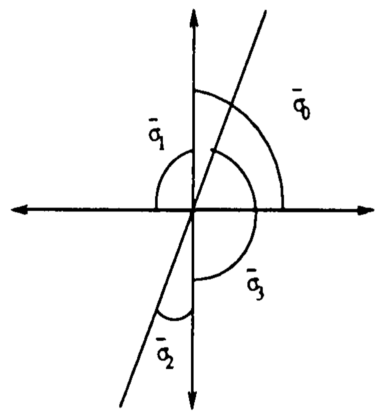
$$R_\sigma = \mathbb{C}[u_1, u_2, u_3] = \mathbb{C}[\xi_1, \xi_2, \xi_3]/\mathcal{A}_\sigma$$

where the relation $a_1 + a_3 = 2a_2$ provides the relation $u_1 u_3 = u_2^2$ between coordinates. The ideal $\mathcal{A}_\sigma$ is then generated by the binomial relation $\xi_1 \xi_3 = \xi_2^2$ and the affine toric variety corresponding to the cone σ is represented in $\mathbb{C}^3$ as the quadratic cone

$$X_\sigma = \{x = (x_1, x_2, x_3) \in \mathbb{C}^3 \ : \ x_1 x_3 = x_2^2\}$$

It has a singularity at the origin of $\mathbb{C}^3$. The following picture gives the real part of X_σ in $\mathbb{R}^3$.





In the general case, the situation is the same : Let $a_1, \ldots, a_k$ be a system of generators of S_σ, where each a_i is written $a_i = (\alpha_i^1, \ldots, \alpha_i^n) \in \check{\sigma} \cap M$. By the isomorphism θ, we obtain monic Laurent monomials $u_i = z^{a_i} \in \mathbb{C}[z, z^{-1}]$ for $i = 1, \ldots, k$. Writing R_σ as $\mathbb{C}[u_1, \ldots, u_k]$, this $\mathbb{C}$-algebra can be represented by

$$R_\sigma = \mathbb{C}[\xi_1, \ldots, \xi_k]/\mathcal{A}_\sigma$$

for some ideal $\mathcal{A}_\sigma$ that we must determinate.

Consider relations between generators of S_σ such that

$$(*) \qquad \sum_{j=1}^{k} \nu_j a_j = \sum_{j=1}^{k} \mu_j a_j \qquad \mu_j, \nu_j \in \mathbf{Z}_{\geq 0} \quad ,$$

we obtain the monomial relations

$$(z^{a_1})^{\nu_1}\cdots(z^{a_k})^{\nu_k}=(z^{a_1})^{\mu_1}\cdots(z^{a_k})^{\mu_k}$$

where $z^{a_i}=(z_1^{\alpha_i^1},\dots,z_n^{\alpha_i^n})$, i.e. relations

$$u_1^{\nu_1}\cdots u_k^{\nu_k}=u_1^{\mu_1}\cdots u_k^{\mu_k}$$

between the coordinates and finally the *binomial relations*

$$(**)\qquad \xi_1^{\nu_1}\cdots\xi_k^{\nu_k}=\xi_1^{\mu_1}\cdots\xi_k^{\mu_k}$$

that generate $\mathcal{A}_\sigma$.

Theorem. Let σ be a lattice cone in $\mathbb{R}^n$ and $A=(a_1,\dots,a_k)$ a system of generators of S_σ, the corresponding toric variety X_σ is represented by the affine toric variety $V(\mathcal{A}_\sigma)\subset\mathbb{C}^k$ where $\mathcal{A}_\sigma$ is an ideal of $\mathbb{C}[\xi_1,\dots,\xi_k]$ generated by finitely many binomials of the form (**) corresponding to relations (*) between elements of A.

Property. If σ is a lattice cone in $\mathbb{R}^n$, then $\dim_{\mathbb{C}} X_\sigma=n$.

The generators $u_1,\dots,u_k$ of R_σ are the coordinate functions on X_σ in $\mathbb{C}^k$. This means that a point $x=(x_1,\dots x_k)\in\mathbb{C}^k$ represents a point of X_σ if and only if the relation $x_1^{\nu_1}\cdots x_k^{\nu_k}=x_1^{\mu_1}\cdots x_k^{\mu_k}$ holds for all (ν,μ) appearing in the relation (*).

Example 7. Consider the cone $\sigma=\{0\}$, the dual cone is $\check{\sigma}=(\mathbb{R}^n)^*$. We can take as system of generators of S_σ for example :

$$A_1=(e_1^*,\dots,e_n^*,-e_1^*,\dots,-e_n^*)$$

or

$$A_2=(e_1^*,\dots,e_n^*,-(e_1^*+\cdots+e_n^*)).$$

Let us take the first system of generators. The corresponding monomial $\mathbb{C}$-algebra is

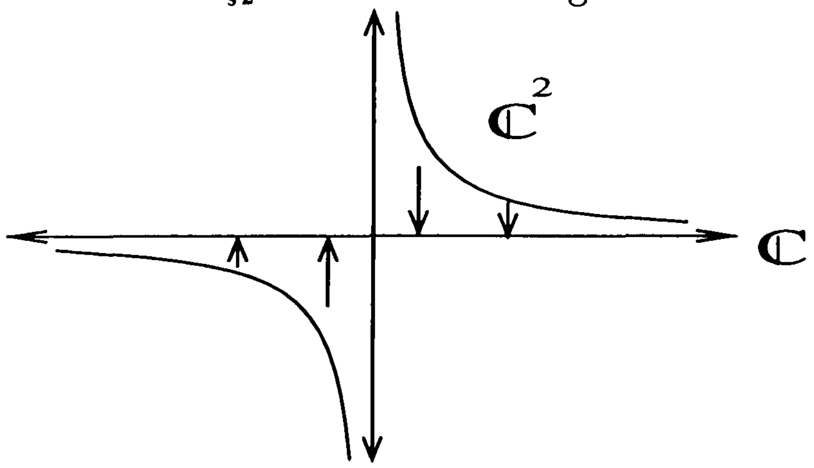
$$\mathbb{C}[z_1,\dots,z_n,z_1^{-1},\dots,z_n^{-1}]=\mathbb{C}[\xi_1,\dots,\xi_{2n}]/\mathcal{A}_\sigma$$

where

$$\mathcal{A}_\sigma=\mathbb{C}[\xi](\xi_1\xi_{n+1}-1)+\cdots+\mathbb{C}[\xi](\xi_n\xi_{2n}-1)$$

hence $X_\sigma = V((\xi_1\xi_{n+1} - 1), \cdots, (\xi_n\xi_{2n} - 1))$.

For $n = 1$, the obtained variety is a complex hyperbola whose asymptotes are the axis $\xi_1 = 0$ and $\xi_2 = 0$. It can be projected bijectively on the axis $\xi_2 = 0$ and the image is $\mathbb{C}^*$:



In the general case ($n \geq 1$), and by the same way, X_σ is homeomorphic to

$$\mathbf{T} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \ : \ z_i \neq 0, \quad i = 1, \ldots, n\} = (\mathbb{C}^*)^n$$

using the projection $\mathbb{C}^{2n} \mapsto \mathbb{C}^n$ on the first coordinates.

Definition. The set $\mathbf{T} = (\mathbb{C}^*)^n$ is called the complex algebraic n-torus.

Remarks 1. $\mathbf{T}$ includes the real torus as : $\mathbf{T} \cong (S^1)^n \times (\mathbb{R}_{\geq 0})^n$.

2. $\mathbf{T}$ is a closed set in $\mathbb{C}^{2n}$ but, as a subspace of $\mathbb{C}^n$, it is not closed.

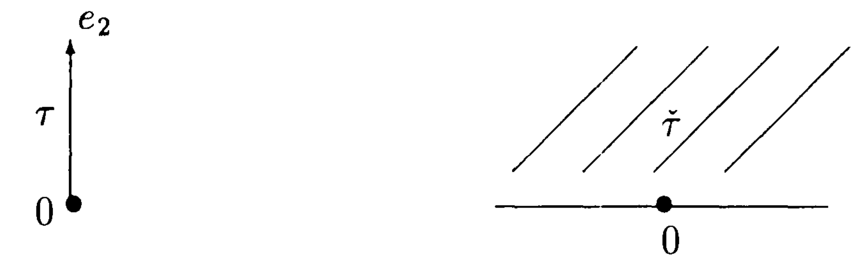
3. The second choice of generators A_2 for S_σ provides another realization of X_σ, now in $\mathbb{C}^{n+1}$. We let this as an exercise.

Example 8. Let $\sigma \in \mathbb{R}^2$ be the following cone

S_σ is generated by (e_1^*, e_2^*), $R_\sigma = \mathbb{C}[\xi_1, \xi_2]$, so $\mathcal{A}_\sigma = \{0\}$ and X_σ is

$\mathbb{C}^2$.

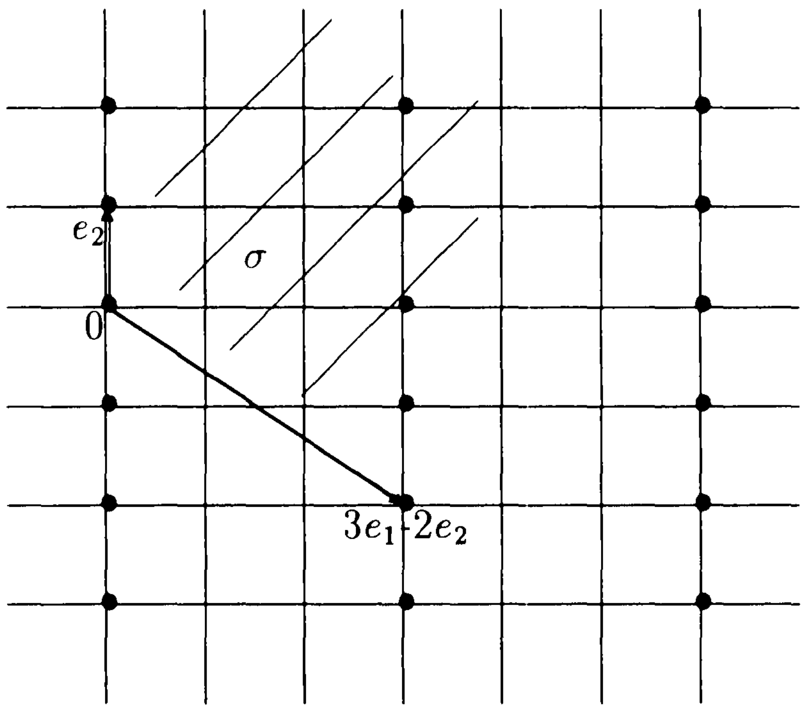
Example 9. Let $\tau \in \mathbb{R}^2$ be the following cone



S_τ is generated by $(e_1^*, -e_1^*, e_2^*)$, $R_\tau = \mathbb{C}[u_1, u_2, u_3]$ with $u_2 = u_1^{-1}$. We obtain $R_\tau = \mathbb{C}[\xi_1, \xi_2, \xi_3]/(\xi_1\xi_2 - 1)$ and X_τ is $\mathbb{C}^*_{\xi_1} \times \mathbb{C}_{\xi_2}$.

Example 10. This is the example of arbitrary 2-dimensional affine toric variety.

Let us consider in $\mathbb{R}^2$ the cone generated by e_2 and $pe_1 - qe_2$, for integers $p, q \in \mathbf{Z}_{>0}$ such that $0 < q < p$ and $(p, q) = 1$. In the following picture, $p = 3$, $q = 2$ and N' is pictured by the points $\bullet$.



Then $R_\sigma = \mathbb{C}[..., z_1^i z_2^j, ...]$ where the monoids $z_1^i z_2^j$ appear for all i and j such that $j \leq p/q\ i$. Let N' the sublattice of N generated by $pe_1 - qe_2$ and e_2, i.e. by pe_1 and e_2. Let us call σ' the cone σ considered in N' instead of N. Then σ' is generated by two generators of the lattice N', so $X_{\sigma'}$ is $\mathbb{C}^2$ (cf. example 3).

In such a situation, it is a general result that the inclusion $N' \subset N$ provides a map $X_{\sigma'} \to X_\sigma$. Here the group Γ_p of p-th roots of unity acts on $X_{\sigma'}$ by $\zeta \cdot (u, v) = (\zeta u, \zeta^q v)$ and then $X_\sigma = X_{\sigma'}/\Gamma_p = \mathbb{C}^2/\Gamma_p$. The map $X_{\sigma'} \to X_\sigma$ is the quotient map.

If $n = 2$ (and in general, if σ is simplicial), then singular affine toric varieties are quotient singularities.

4 Toric varieties.

The toric varieties associated to fans will be constructed by gluing affine ones associated to cones. Let us begin by recalling a very simple example, the one of the projective space $\mathbb{P}^2$.

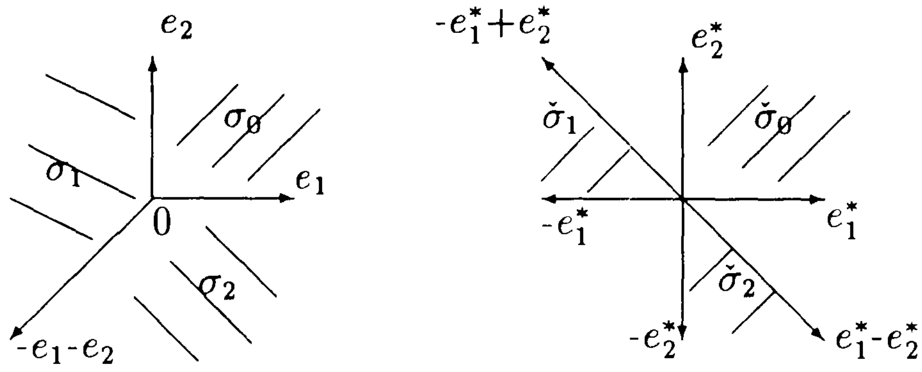
Let us denote by $(t_0 : t_1 : t_2)$ the homogeneous coordinates of the space $\mathbb{P}^2$. It is classically covered by three coordinate charts :

U_0 corresponding to $t_0 \neq 0$ and with affine coordinates $(t_1/t_0, t_2/t_0) = (z_1, z_2)$

U_1 corresponding to $t_1 \neq 0$ and with affine coordinates $(t_0/t_1, t_2/t_1) = (z_1^{-1}, z_1^{-1} z_2)$

U_0 corresponding to $t_0 \neq 0$ and with affine coordinates $(t_0/t_2, t_1/t_2) = (z_2^{-1}, z_1 z_2^{-1})$

Now consider in $\mathbb{R}^2$ the following fan :



then :

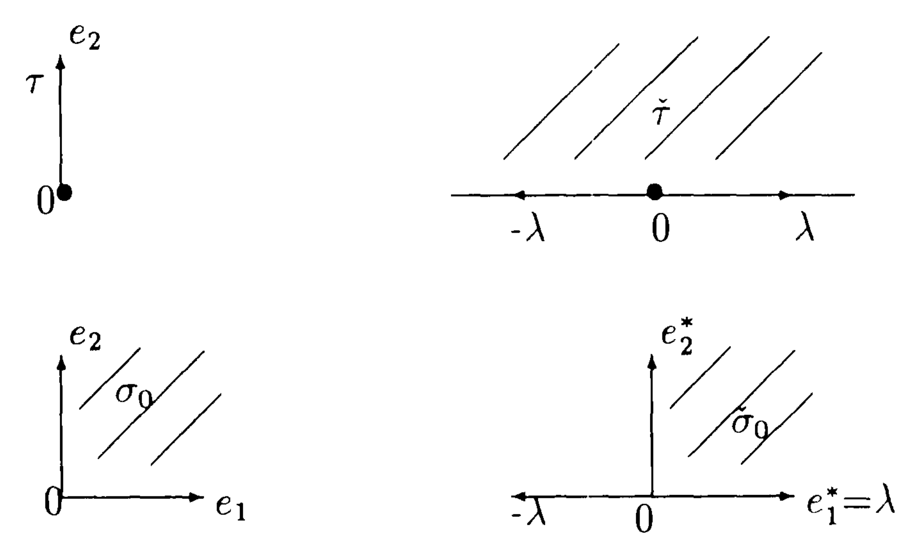
i) S_{σ_0} admits as generators (e_1^*, e_2^*), hence $R_{\sigma_0} = \mathbb{C}[z_1, z_2]$ and $X_{\sigma_0} = \mathbb{C}^2_{(z_1,z_2)}$;

ii) in the same way, S_{σ_1} admits as system of generators $(-e_1^*, -e_1^* + e_2^*)$, hence $R_{\sigma_1} = \mathbb{C}[z_1^{-1}, z_1^{-1}z_2]$ and $X_{\sigma_1} = \mathbb{C}^2_{(z_1^{-1}, z_1^{-1}z_2)}$;

iii) finally, S_{σ_2} admits as system of generators $(-e_2^*, e_1^* - e_2^*)$, hence $R_{\sigma_2} = \mathbb{C}[z_2^{-1}, z_1 z_2^{-1}]$ and $X_{\sigma_2} = \mathbb{C}^2_{(z_2^{-1}, z_1 z_2^{-1})}$.

We see that the three affine toric varieties correspond to the three coordinate charts of $\mathbb{P}^2$. In fact, the structure of the fan provides a gluing between these charts allowing to reconstruct the toric variety $\mathbb{P}^2$ from the U_{σ_i}. Let us explicit the gluing of X_{σ_0} and X_{σ_1} such that $\tau = \sigma_0 \cap \sigma_1$.

For seeing this let us first consider τ as a face of σ_0 :



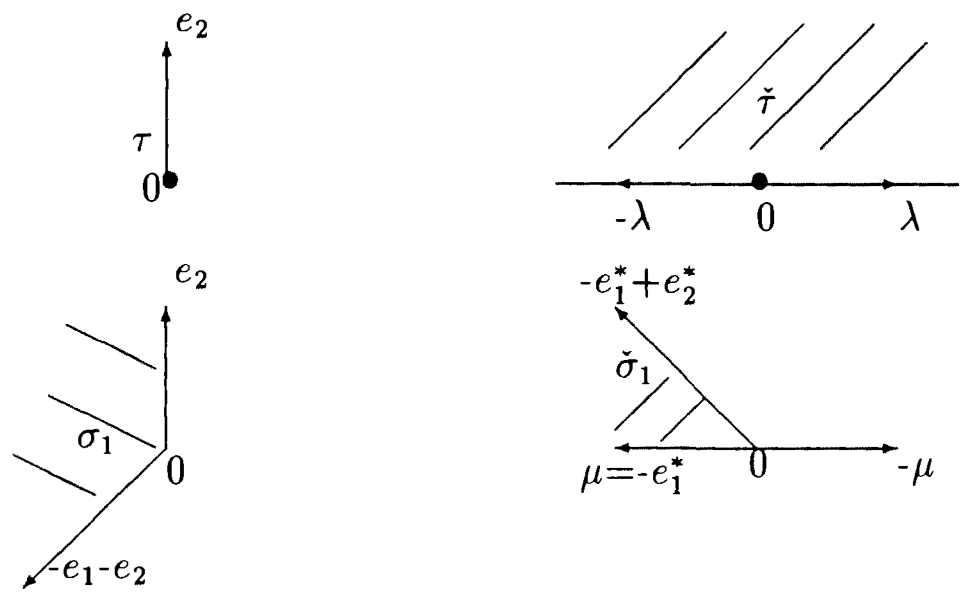
The vector $\lambda = e_1^*$ satisfies :

$$\lambda \in \check{\sigma}_0 \qquad \tau = \sigma_0 \cap \lambda^\perp$$

and we have

$$\check{\tau} = \check{\sigma}_0 + \mathbb{R}_{\geq 0}(-\lambda)$$

In the same way, let us now consider τ as a face of σ_1 :



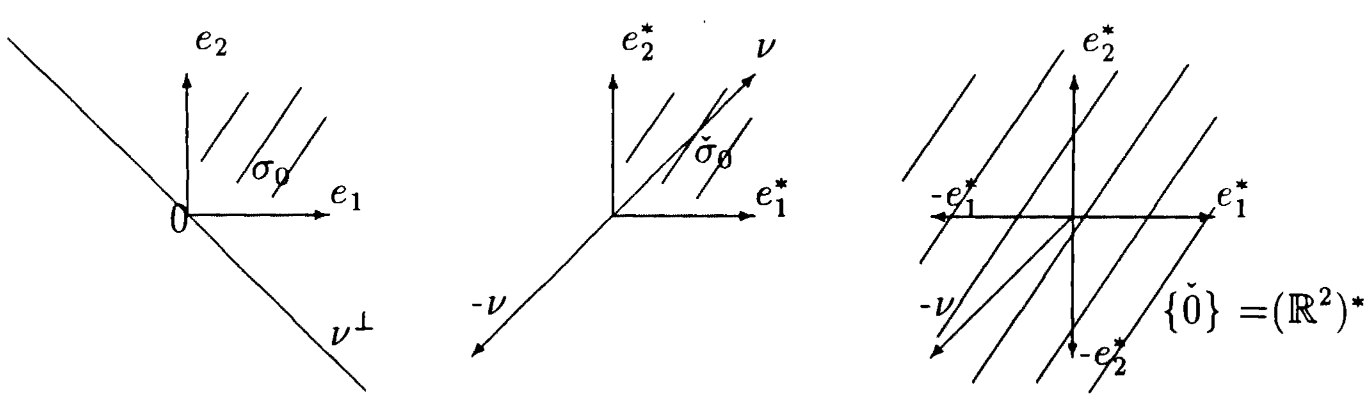
The vector $\mu = -e_1^*$ satisfies :

$$\mu \in \check{\sigma}_1 \qquad \tau = \sigma_1 \cap \mu^\perp$$

and we have

$$\check{\tau} = \check{\sigma}_1 + \mathbb{R}_{\geq 0}(-\mu)$$

Finally consider the origin $\{0\}$ as a face of σ_0 :



The vector $\nu = e_1^* + e_2^*$ satisfies :

$$\nu \in \check{\sigma}_0 \qquad \{0\} = \sigma_0 \cap \nu^\perp$$

and we have

$$\{\check{0}\} = (\mathbb{R}^2)^* = \check{\sigma}_0 + \mathbb{R}_{\geq 0}(-\nu).$$

The previous three examples are examples of a general result :

Proposition. Let τ be a face of a cone σ and let λ be a vector such that $\lambda \in \check{\sigma}$ and $\tau = \sigma \cap \lambda^{\perp}$, then $\check{\tau} = \check{\sigma} + \mathbb{R}_{\geq 0}(-\lambda)$.

The proof of this fact is a funny demonstration using properties of duality and orthogonality.

Corollary. Let τ be a face of a cone σ and let λ be a vector such that $\lambda \in \check{\sigma}$ and $\tau = \sigma \cap \lambda^{\perp}$, then $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-\lambda)$.

The monoid S_τ is thus obtained from S_σ by adding only one generator $-\lambda$. As λ is an element of a system of generators $(a_1, \ldots, a_k)$ for S_σ, we may suppose that $\lambda = a_k$. So for obtaining the relations between the generators of S_τ, we have to add only the supplementary relation $a_k + a_{k+1} = 0$ to the previous ones between the generators of S_σ.

This relation corresponds to the multiplicative relation $u_k u_{k+1} = 1$ in R_τ and it is the only supplementary relation we need defining R_τ from R_σ. As the generators u_i are precisely the coordinate functions on the toric varieties X_σ and X_τ, this means that the projection $\mathbb{C}^{k+1} \to \mathbb{C}^k$:

$$(x_1, \ldots, x_k, x_{k+1}) \mapsto (x_1, \ldots, x_k)$$

identifies X_τ with the open subset of X_σ defined by $x_k \neq 0$. This can be written :

Proposition. There is a natural identification

$$X_\tau \cong X_\sigma \setminus (u_k = 0)$$

This allows us to glue together X_σ and $X_{\sigma'}$ such that $\sigma \cap \sigma' = \tau$. Writing $(v_1, \ldots, v_l)$ the coordinates on $X_{\sigma'}$, there is an homeomorphism

$$X_\tau \cong X_{\sigma'} \setminus (v_l = 0)$$

and we obtain a gluing map

$$\psi_{\sigma,\sigma'} : X_\sigma \setminus (u_k = 0) \xrightarrow{\cong} X_\tau \xrightarrow{\cong} X_{\sigma'} \setminus (v_l = 0).$$

Theorem. (Definition of Toric Varieties). Let Δ be a fan in $\mathbb{R}^n$. Consider the disjoint union $\cup_{\sigma\in\Delta} X_\sigma$ where two points $x \in X_\sigma$ and $x' \in X_{\sigma'}$ are identified if $\psi_{\sigma,\sigma'}(x) = x'$. The resulting space X_Δ is called a *toric variety*. It is a topological space endowed with an open covering by the affine toric varieties X_σ for $\sigma \in \Delta$. It is an algebraic variety whose charts are defined by binomial relations.

Example 11. Let us return to the example of the projective space $\mathbb{P}^2$, we have, using the previous notations :

$S_\tau = S_{\sigma_0} + \mathbb{Z}_{\geq 0}(-e_1^*)$ and $X_\tau = X_{\sigma_0} \setminus (z_1 = 0) = \mathbb{C}_{z_2} \times \mathbb{C}^*_{z_1}$. Recall that $X_{\sigma_0} = \mathbb{C}^2_{(z_2,z_1)}$.

In the same way, $S_\tau = S_{\sigma_1} + \mathbb{Z}_{\geq 0}(e_1^*)$ and $X_\tau = X_{\sigma_1} \setminus (z_1^{-1} = 0) = \mathbb{C}_{z_1^{-1}z_2} \times \mathbb{C}^*_{z_1^{-1}}$ with $X_{\sigma_1} = \mathbb{C}^2_{(z_1^{-1}z_2, z_1^{-1})}$.

We can glue together X_{σ_0} and X_{σ_1} along X_τ using the change of coordinates $(z_2, z_1) \mapsto (z_1^{-1}z_2, z_1^{-1})$. We obtain $\mathbb{P}^2 \setminus \{(0:0:1)\}$.

Gluing this space by the same procedure with $X_{\sigma_2} = \mathbb{C}^2_{(z_2^{-1}, z_1 z_2^{-1})}$, we obtain the total space $\mathbb{P}^2$.

In fact, we have shown that, for cones $\tau < \sigma$, we have inclusions :

$$\begin{array}{ccc}
\tau & \overset{\cdot}{\hookrightarrow} & \tau \\
\check{\tau} & \hookleftarrow & \check{\sigma} \\
R_\tau & \hookleftarrow & R_\sigma \\
X_\tau & \hookrightarrow & X_\sigma
\end{array}$$

Before giving more examples, let us show a fundamental result :

Proposition. Every toric variety contains the torus $\mathbf{T}$ as a Zariski open dense subset.

Proof : The torus $\mathbf{T}$ corresponds to the zero cone, which is a face of every $\sigma \in \Delta$, i.e. $\mathbf{T} = X_{\{0\}}$. Let us explicit the embedding of the torus into every affine toric variety X_σ. Let $(a_1, \ldots, a_k)$ be a system of generators for the monoid S_σ and let $V(\mathcal{A}) \subset \mathbb{C}^n$ be a representation of X_σ. With the previous coordinates of $\mathbb{R}^n$, each a_i is written $a_i = (\alpha_i^1, \dots \alpha_i^n)$ with $\alpha_i^j \in \mathbb{Z}$ and $t \in \mathbf{T}$ is written $t = (t_1, \dots, t_n)$ with $t_j \in \mathbb{C}^*$. The embedding $\mathbf{T} \hookrightarrow X_\sigma$ is given by

$$\begin{aligned} t &= (t_1, \dots, t_n) \mapsto (t^{a_1}, \dots, t^{a_k}) \in V(\mathcal{A}) \cap (\mathbb{C}^*)^k \\ & \quad \text{where } t^{a_i} = t_1^{\alpha_i^1} \cdots t_n^{\alpha_i^n} \in \mathbb{C}^* \quad . \end{aligned}$$

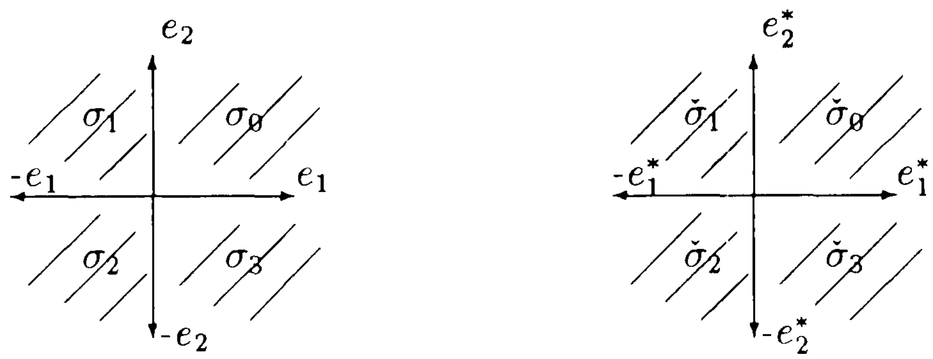
By the previous identifications, all the tori corresponding to affine toric varieties X_σ in X_Δ glue together in an open dense subset in X_Δ.

In the case of example 5, the embedding is given by

$$(t_1, t_2) \mapsto (t_1, t_1 t_2, t_1 t_2^2) \in V(\mathcal{A}) \cap (\mathbb{C}^*)^3$$

Here are some of the classical examples of toric varieties :

Example 12 Consider the following fan :



which gives the following monoids :

$$\begin{array}{ccc} S_{\sigma_1} \text{ gen. by } (-e_1^*, e_2^*) & \leftrightarrow & S_{\sigma_0} \text{ gen. by } (e_1^*, e_2^*) \\ \updownarrow & & \updownarrow \\ S_{\sigma_2} \text{ gen. by } (-e_1^*, -e_2^*) & \leftrightarrow & S_{\sigma_3} \text{ gen. by } (e_1^*, -e_2^*) \end{array}$$

and the following $\mathbb{C}$-algebra :

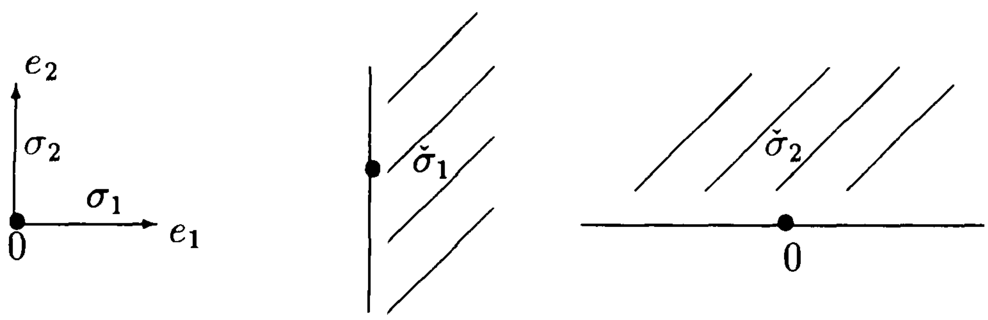
$$\begin{array}{ccc} R_{\sigma_1} = \mathbb{C}[z_1^{-1}, z_2] & \leftrightarrow & \mathbb{C}[z_1, z_2] = R_{\sigma_0} \\ \updownarrow & & \updownarrow \\ R_{\sigma_2} = \mathbb{C}[z_1^{-1}, z_2^{-1}] & \leftrightarrow & \mathbb{C}[z_1, z_2^{-1}] = R_{\sigma_3} \end{array}$$

The gluing of X_{σ_1} and X_{σ_0} gives $\mathbb{P}^1 \times \mathbb{C}$ with coordinates $((t_0 : t_1), z_2)$ where $(z_1 = t_0/t_1)$,

The gluing of X_{σ_2} and X_{σ_3} gives $\mathbb{P}^1 \times \mathbb{C}$ with coordinates $((t_0 : t_1), z_2^{-1})$,

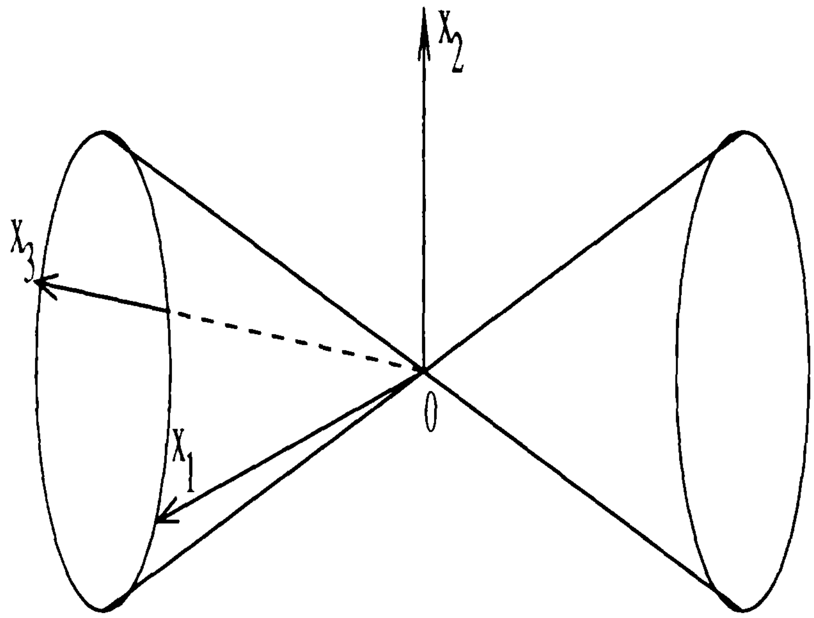
The gluing of these two gives $X_\Delta = \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $((t_0 : t_1), (s_0 : s_1))$ where $(z_2 = s_0/s_1)$.

Example 13. Consider the following fan :



then S_{σ_1} is generated by $(e_1^*, e_2^*, -e_2^*)$. The monoid S_{σ_2} is generated by $(e_1^*, -e_1^*, e_2^*)$ and $S_{\{0\}}$ is generated by $(e_1^*, -e_1^*, e_2^*, -e_2^*)$. The corresponding $\mathbb{C}$-algebras are respectively $R_{\sigma_1} = \mathbb{C}[z_1, z_2, z_2^{-1}]$, $R_{\sigma_2} = \mathbb{C}[z_1, z_1^{-1}, z_2]$ and $R_{\{0\}} = \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$. The corresponding affine toric varieties are $X_{\sigma_1} = \mathbb{C}_{z_1} \times \mathbb{C}^*_{z_2}$, $X_{\sigma_2} = \mathbb{C}^*_{z_1} \times \mathbb{C}_{z_2}$ and $X_{\{0\}} = \mathbb{C}^*_{z_1} \times \mathbb{C}^*_{z_2}$. The gluing of the affine toric X_{σ_1} and X_{σ_2} along $X_{\{0\}}$ gives $X_\Delta = \mathbb{C}^2 - \{0\}$.

Example 14. Consider the following fan :



Then the monoids S_{σ_i} are generated :

$$\begin{array}{ccc} S_{\sigma_1} \text{ by } (-e_1^*, e_2^*) & \leftrightarrow & S_{\sigma_0} \text{ by } (e_1^*, e_2^*) \\ \updownarrow & & \updownarrow \\ S_{\sigma_2} \text{ by } (-e_1^* - ke_2^*, -e_2^*) & \leftrightarrow & S_{\sigma_3} \text{ by } (e_1^* + ke_2^*, -e_2^*) \end{array}$$

and the corresponding affine varieties are

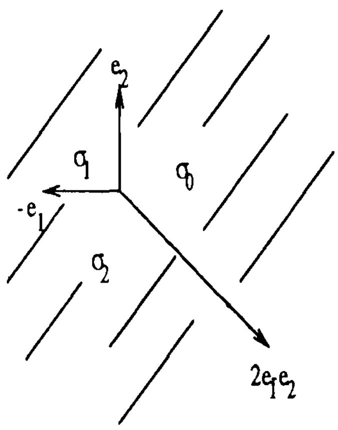
$$\begin{array}{ccc} X_{\sigma_1} = \mathbb{C}^2_{(z_1^{-1}, z_2)} & \leftrightarrow & X_{\sigma_0} = \mathbb{C}^2_{(z_1, z_2)} \\ \updownarrow & & \updownarrow \\ X_{\sigma_2} = \mathbb{C}^2_{(z_1^{-1} z_2^{-k}, z_2^{-1})} & \leftrightarrow & X_{\sigma_3} = \mathbb{C}^2_{(z_1 z_2^k, z_2^{-1})} \end{array}$$

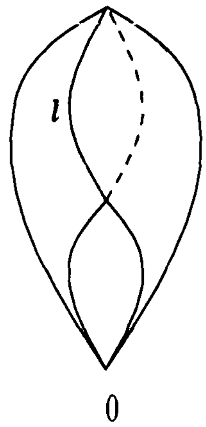
The gluing of X_{σ_1} and X_{σ_0} gives $\mathbb{P}^1 \times \mathbb{C}$ with coordinates $((t_0 : t_1), z_2)$ where $z_1 = t_0/t_1$, the gluing of X_{σ_2} and X_{σ_3} gives $\mathbb{P}^1 \times \mathbb{C}$ with coordinates $((s_0 : s_1), z_2^{-1})$ where $z_1 z_2^k = s_0/s_1$.

These two gluing, glued together, provide a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ (gluing the second coordinates), which is a rational ruled surface denoted $\mathbf{F}_k$ and called Hirzebruch surface. It is the hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$ defined by

$$\{(\lambda_0 : \lambda_1), (\mu_0 : \mu_1 : \mu_2) \ : \ \lambda_0^k \mu_0 = \lambda_1^k \mu_1\}$$

Example 15. Consider the following fan :





Then X_{σ_0} is the affine quadratic cone (cf example 5), X_{σ_1} and X_{σ_2} are affine planes (example 8). The affine quadratic cone is completed by a "circle at infinity" that represents a complex projective line. The real picture of X_Δ is a pinched torus.

Example 16. Generalizing the example of $\mathbb{P}^2$, we can consider the fan Δ whose cones are generated by all proper subsets of $(v_0, \ldots, v_n) = (e_1, \ldots, e_n, -(e_1 + \cdots + e_n))$, i.e. σ_0 is generated by $(e_1, \ldots, e_n)$ and, for $i = 1, \ldots, n$, the cone σ_i is generated by $(e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n, -(e_1 + \cdots + e_n)$. For $i = 0, \ldots, n$, the affine toric varieties X_{σ_i} are copies of $\mathbb{C}^n$ giving the classical charts of $\mathbb{P}^n$.

Example 17. Let $d_0, \ldots, d_n$ be positive integers. Consider the same fan than the previous one (example 16) but consider the lattice N' generated by the vectors $(1/d_i) \cdot v_i$, for $i = 0, \ldots, n$. Then the resulting toric variety is

$$\mathbb{P}(d_0, \ldots, d_n) = \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$$

where $\mathbb{C}^*$ acts by $\zeta \cdot (x_0, \ldots, x_n) = (\zeta^{d_0} x_0, \ldots, \zeta^{d_n} x_n)$. It is called twisted or weighted projective space.

Definition. A cone σ defined by the set of vectors $x_1, \ldots x_r$ is a *simplex* if all the vectors x_i are linearly independent. A fan Δ is *simplicial* if all cones of Δ are simplices.

Definition. A vector $x \in \mathbb{Z}^n$ is *primitive* if its coordinates are coprime. A cone is *regular* if the vectors $(x_1, \ldots, x_r)$ spanning the cone are primitive and there exists primitive vectors $x_{r+1}, \ldots, x_n$ such that

$\det(x_1,\ldots,x_n) = \pm 1$. In another words, the vectors $(x_1,\ldots,x_r)$ can be completed in a basis of the lattice N. A fan is *regular* if all its cones are regular ones.

Definitions 1. A fan Δ is *complete* if its cones cover $\mathbb{R}^n$, i.e. $|\Delta| = \mathbb{R}^n$.

2. A fan is *polytopal* if there exists a polytope P such that $0 \in P$ and Δ is spanned by the faces of P (let us recall that a polytope is the convex hull of a finite number of points).

Remarks 1. Every complete fan in $\mathbb{R}^2$ is polytopal,

2. Not every complete fan is isomorphic to a polytopal one. For example take the cube in $\mathbb{R}^3$ with all coordinates ± 1. The faces of the cube provide a polytopal fan. Now replace the point $(1,1,1)$ by $(1,2,3)$ and consider the corresponding fan. It is clearly not isomorphic to a polytopal one : there exists 4 points not lying in the same affine plane.

Theorem. 1. The fan Δ is complete if and only if X_Δ is compact.

2. The fan Δ is regular if and only if X_Δ is smooth.

3. The fan Δ is polytopal if and only if X_Δ is projective.

Let us give some precisions about these results :

An affine toric variety X_σ is smooth if and only if $X_\sigma = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ where $k = \dim \sigma$.

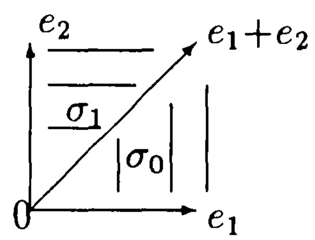
If the fan Δ is simplicial, then X_Δ is an orbifold (or V-manifold) : it has only quotient singularities.

Also we remark that if Δ is complete, then X_Δ is a compactification of $\mathbf{T} = (\mathbb{C}^*)^n$.

Resolution of singularities ([5], §2.6)

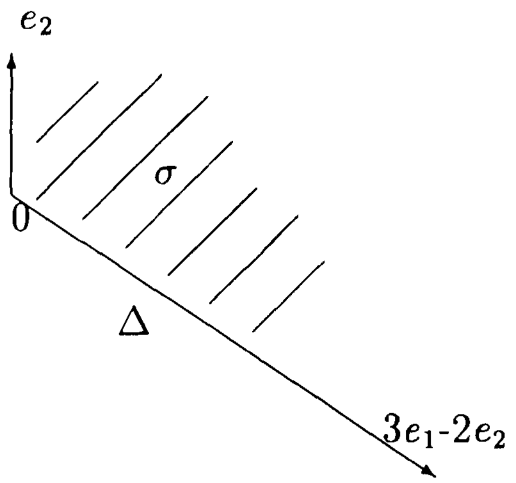
We give only examples. The general way of resolving singularities follows these examples (see for example [5], §2.6)

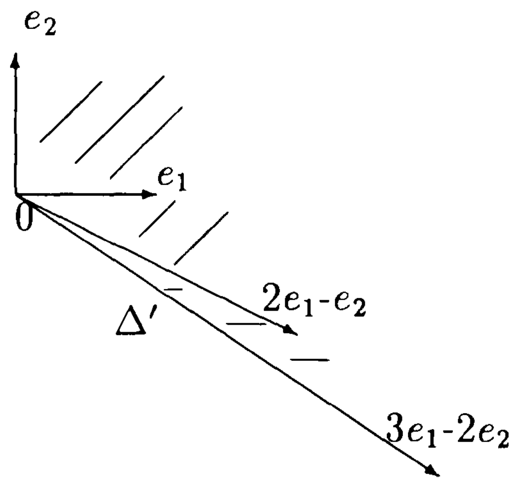
Example 18. Consider the following fan



The corresponding toric variety is a blow-up of a point in $\mathbb{C}^2$.

Example 19. Consider the following fan (cone) Δ and its subdivision Δ' :





The fan Δ' is a regular fan, hence $X_{\Delta'}$ is a smooth toric variety. The identity map of N provides a map $X_{\Delta'} \to X_{\Delta}$ which is birational proper. It is an isomorphism on the open torus $\mathbf{T}$ contained in each. This is the first example (and standard one) of resolution of singularities.

The procedure is the following : beginning with the cone σ generated by the two vectors $v = e_2$ and $v' = 3e_1 - 2e_2$, we add primitive vectors (here $v_1 = e_1$ and $v_2 = 2e_1 - e_2$) such that, with $v_0 = v$ and $v_3 = v'$, we have

$$\lambda_i v_i = v_{i-1} + v_{i+1} \qquad i = 1, 2$$

For $i = 1, 2$, the vectors v_i correspond to exceptional divisors $E_i \cong \mathbb{P}^1$ in $X_{\Delta'}$ and their self-intersection is $(E_i, E_i) = -\lambda_i$. In this particular case, we obtain two exceptional divisors with self-intersection -2.

5 The torus action and the orbits.

The torus $\mathbf{T} = (\mathbb{C}^*)^n$ is a group operating on itself by multiplication. The action of the torus on each affine toric variety X_σ is described as follows :

Let $(a_1, \dots, a_k)$ be a system of generators for the monoid S_σ. With the previous coordinates of $\mathbb{R}^n$, each a_i is written $a_i = (\alpha_i^1, \dots \alpha_i^n)$ with $\alpha_i^j \in \mathbb{Z}$ and $t \in \mathbf{T}$ is written $t = (t_1, \dots, t_n)$ with $t_j \in \mathbb{C}^*$. A point $x \in X_\sigma$ is written $x = (x_1, \dots, x_k) \in \mathbb{C}^k$. The action of $\mathbf{T}$ on X_σ is given by :

$$\begin{array}{rcl} \mathbf{T} \times X_\sigma & \to & X_\sigma \\ (t, x) & \mapsto & t \cdot x = (t^{a_1} x_1, \dots, t^{a_k} x_k) \end{array}$$

where $t^{a_i} = t_1^{\alpha_i^1} \cdots t_n^{\alpha_i^n} \in \mathbb{C}^*$.

Now let Δ be a fan in $\mathbb{R}^n$ and let τ be a face of a cone $\sigma \in \Delta$. The identification $X_\tau \cong X_\sigma \setminus (u_k = 0)$ is compatible with the torus action, which implies that the gluing maps $\psi_{\sigma,\sigma'}$ respect also this torus action. We obtain the :

Theorem. Let Δ be a fan in $\mathbb{R}^n$, the torus action on the affine toric varieties X_σ, for $\sigma \in \Delta$, provide a torus action on the toric variety X_Δ.

It is clear that the embedded tori in each X_σ correspond each other by the gluing maps. We obtain an open embedding of the torus $\mathbf{T} = (\mathbb{C}^*)^n$ in the toric variety X_Δ.

Let $\Delta = \{0\}$, we seen that $X_\Delta = (\mathbb{C}^*)^n$ is the torus. There is only one orbit which is the entire space and is the orbit of the point whose coordinates u_i are $(1, \dots, 1)$ in $\mathbb{C}^n$. In fact, for every toric variety, the apex $\sigma = \{0\}$ of Δ provides an open dense orbit which is the embedded torus $\mathbf{T} = (\mathbb{C}^*)^n$. Let us describe the other orbits.

Theorem. (see [5], §2.1 and 3.1) Let Δ be a fan in $\mathbb{R}^n$, to each $\sigma \in \Delta$, we can associate a distinguished point $x_\sigma \in X_\sigma \subset X_\Delta$ and the orbit $O_\sigma \subset X_\sigma$ of x_σ satisfying :

1) $X_\sigma = \coprod_{\tau<\sigma} O_\tau$,

2) if V_τ denotes the closure of the orbit O_τ, then $V_\tau = \coprod_{\tau<\sigma} O_\sigma$,

3) $O_\tau = V_\tau \setminus \bigcup_{\substack{\tau<\sigma \\ \tau\neq\sigma}} V_\sigma$.

Let τ be a face of a cone σ, then $O_\sigma \subset \overline{O_\tau}$. The image of $V_\tau = \overline{O_\tau}$ in a representation of X_σ can be determined in the following way :

Consider a system of generators $(a_1, \dots a_k)$ of the monoid S_σ, denote I the set of indices $1 \le i \le k$ such that $a_i \notin \tau^\perp$. In other words, if $(x_1, \dots, x_s)$ denote the vectors that span τ, we have

$$i \in I \iff \forall j, \quad 1 \le j \le s \quad \langle a_i, x_j \rangle \neq 0$$

In X_σ with coordinates $u_i = z^{a_i}$, then V_τ is defined by $u_i = 0$ if $i \in I$.

Let us give two examples :

Example 20. In the case of example 5, X_σ has coordinates $(u_1, u_2, u_3) = (z_1, z_1 z_2, z_1 z_2^2)$ and S_σ is generated by $a_1 = e_1^*$, $a_2 = e_1^* + e_2^*$ and $a_3 = e_1^* + 2e_2^*$. Let us consider the edge τ_1 generated by e_2, then

$$i \in I \iff \langle a_i, e_2 \rangle \neq 0$$

hence $I = \{2, 3\}$. In X_σ the set V_{τ_1} is defined by $u_2 = 0$, $u_3 = 0$. In $\mathbb{C}^3 = \mathbb{C}^3_{(\xi_1,\xi_2,\xi_3)}$ it is $V_{\tau_1} = \mathbb{C}_{\xi_1} \times \{0\} \times \{0\}$.

Consider the edge τ_2 generated by $2e_1 - e_2$, then

$$i \in I \iff \langle a_i, 2e_1 - e_2 \rangle \neq 0$$

hence $I = \{1, 2\}$. In X_σ, the set V_{τ_2} is defined by $u_1 = 0$, $u_2 = 0$. In $\mathbb{C}^3_{(\xi_1,\xi_2,\xi_3)}$, we have $V_{\tau_2} = \{0\} \times \{0\} \times \mathbb{C}_{\xi_3}$.

The cone σ is a face of itself. For this face, $I = \{1, 2, 3\}$ and, in X_σ, the set V_σ is defined by $u_1 = 0$, $u_2 = 0$, $u_3 = 0$. Hence $V_\sigma = O_\sigma$ is the origin $(0, 0, 0) \in \mathbb{C}^3$.

We can conclude by the list of orbits in this example :

$$\begin{aligned}
O_\sigma &= \{(0,0,0)\} \\
O_{\tau_1} &= \mathbb{C}^*_{\xi_1} \times \{0\} \times \{0\}, \text{ orbit of the distinguished point } x_{\tau_1} = (1,0,0) \\
O_{\tau_2} &= \{0\} \times \{0\} \times \mathbb{C}^*_{\xi_3}, \text{ orbit of the distinguished point } x_{\tau_2} = (0,0,1) \\
O_{\{0\}} \cdot &= (\mathbb{C}^*)^2, \text{orbit of the distinguished point} x_{\{0\}} = (1,1,1)
\end{aligned}$$

Example 21. Orbits in $\mathbb{P}^2$ (see the pictures in §4).

With the notations and the pictures of §4, let us consider the image of $V_\tau = \overline{O_\tau}$ in X_{σ_0} and U_{σ_1}. The monoid S_{σ_0} is generated by $a_1 = e_1^*$ and $a_2 = e_2^*$. In X_{σ_0} with coordinates $(u_1, u_2) = (z_1, z_2)$, we have :

$$i \in I \iff \langle a_i, e_2 \rangle \neq 0$$

hence $I = \{2\}$ and, in $X_{\sigma_0} = \mathbb{C}^2_{(u_1,u_2)}$, V_τ is defined by $u_2 = 0$. Hence V_τ is $\mathbb{C}_{\xi_1} \times \{0\}$ and $O_\tau = \mathbb{C}^*_{z_1} \times \{0\}$ is the orbit of $\{x_\tau\} = (1,0)$. This point is a representation of the point $(1:1:0)$ of $\mathbb{P}^2$.

The monoid S_{σ_1} is generated by $a_1 = -e_1^*$ and $a_2 = -e_1^* + e_2^*$. In X_{σ_1} with coordinates $(u_1, u_2) = (z_1^{-1}, z_1^{-1} z_2)$, we have :

$$i \in I \iff \langle a_i, e_2 \rangle \neq 0$$

hence $I = \{2\}$ and, in $X_{\sigma_1} = \mathbb{C}^2_{(u_1,u_2)}$, V_τ is defined by $u_2 = z_1^{-1} z_2 = 0$. Hence V_τ is $\mathbb{C}_{(z_1^{-1})} \times \{0\}$. The orbit $O_\tau = \mathbb{C}^*_{(z_1^{-1})} \times \{0\}$ is the same than before, i.e. the orbit of $\{x_\tau\}$.

The projective space is the union of 7 orbits of the torus action :

- $O_{\{0\}} = (\mathbb{C}^*)^2$,

- 3 orbits homeomorphic to $\mathbb{C}^*$ corresponding to the three edges and whose images in each X_{σ_i} are described in the same way than O_τ. They are the orbits of the points $(1:1:0)$, $(1:0:1)$ and $(0:1:1)$ of $\mathbb{P}^2$.

- 3 fixed points $\{x_{\sigma_i}\}$, $i = 1,2,3$ corresponding to the 2-dimensional cones σ_i. They are fixed points of the torus action and are the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$.

These examples provide examples of general situations :

Properties. 1. If $\dim_{\mathbf{R}} \sigma = n$, then O_σ is a fixed point $\{x_\sigma\}$. Consider a representation of X_σ in $\mathbb{C}^k$, then $O_\sigma = \{x_\sigma\}$ corresponds to the origin of $\mathbb{C}^k$.

2. If $\dim_{\mathbf{R}} \sigma = k$, then $O_\sigma \cong (\mathbb{C}^*)^{n-k}$.

3. Let τ_i be an edge (1-dimensional cone) in Δ, then $O_{\tau_i} \cong (\mathbb{C}^*)^{n-1}$. If $\dim_{\mathbf{R}} \Delta = n$, then V_{τ_i} is a codimension one variety in X_Δ. We will see that V_{τ_i} is a Weil divisor.

4. The distinguished point $\{x_\sigma\}$ corresponding to each cone σ can be defined in a direct way (cf [5]).

6 Divisors.

In this section, we will denote by X a complex algebraic variety.

A *Weil divisor* is an element of the free abelian group $W(X)$ generated by the irreducible closed subvarieties of (complex) codimension 1 in X. Such a divisor can be written :

$$\sum n_i A_i - \sum m_j B_j \qquad \text{with } n_i, m_j > 0$$

where the A_i and B_j are subvarieties of codimension 1 in X.

For example, in the space $\mathbf{C}^2$ with coordinates (z_1, z_2), let us consider the axis $z_1 = 0$ denoted by A, and the axis $z_2 = 0$ denoted by B. An example of Weil divisor is given by $2A - B$.

Let us denote by $\mathcal{R}(U)$ the set of rational functions in the open set U in X. A *Cartier divisor* is given by a covering $X = \bigcup U_\alpha$ of X and by nonzero rational functions $f_\alpha \in \mathcal{R}(U_\alpha)$ such that for any α and β, we have $f_\alpha / f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ (nowhere zero holomorphic function). We denote by $D = (U_\alpha, f_\alpha)$ such a divisor. The set of Cartier divisors is a group denoted by $C(X)$.

Let us consider the example of $X = \mathbf{C}^2$ covered by only one open set $U = \mathbf{C}^2$ and consider, in U, the rational function $f(z_1, z_2) = z_1^2/z_2$, we obtain a Cartier divisor $D = (U, f)$.

Recall that a an algebraic complex variety is *normal* if the local ring at every point is an integrally closed ring. In particular, toric varieties are normal, each ring R_σ is integrally closed.

Proposition. For a normal variety X, there is an inclusion

$$C(X) \hookrightarrow W(X)$$

Let us explicit this inclusion in the previous example : If $A = \{f = 0\}$ is the set of zeroes of f counted with multiplicities and $B = \{1/f = 0\}$ is the set of poles of f counted with multiplicities, then the previous Weil divisor $2A - B$ corresponds to the previous Cartier divisor $D = (U, f)$.

In general this inclusion is defined by

$$[D] = \sum_{\mathrm{codim}(V,X)=1} \mathrm{ord}_V(D) \cdot V$$

where $\mathrm{ord}_V(D)$ is the vanishing order of an equation for D in the local ring along the subvariety V. If X is normal, then local rings are discrete valuation rings and the order is the naive one.

In fact, the previous example is an example of *principal divisor* : The subgroup of principal divisors, denoted by $P(X)$, is the subgroup of Cartier divisors corresponding to the nonzero rational functions. Let us consider the quotients :

$$\mathcal{C}(X) = C(X)/P(X) \qquad \text{and} \qquad \mathcal{W}(X) = W(X)/P(X)$$

There is an inclusion $\mathcal{C}(X) \hookrightarrow \mathcal{W}(X)$, which is not an equality as shown by the example of the toric variety of example 10 (with q=1) : let X be the quotient variety of $\mathbf{C}^2$ by the subgroup G of p-th roots of unity. Then, we have :

$$\mathcal{C}(X) = \{0\} \hookrightarrow \mathcal{W}(X) = \mathbf{Z}_p \ .$$

Let $X = X_\Delta$ be a toric variety. The Weil and Cartier divisor classes, invariant by the action of the torus $\mathbf{T}$ will be denoted respectively $C^{\mathbf{T}}(X)$ and $W^{\mathbf{T}}(X)$. In the same way, the subgroup of the invariant principal divisors will be denoted $P^{\mathbf{T}}(X)$. We define $\mathcal{C}^{\mathbf{T}}(X) = C^{\mathbf{T}}(X)/P^{\mathbf{T}}(X)$ et $\mathcal{W}^{\mathbf{T}}(X) = W^{\mathbf{T}}(X)/P^{\mathbf{T}}(X)$. There is still an inclusion

$$\mathcal{C}^{\mathbf{T}}(X) \hookrightarrow \mathcal{W}^{\mathbf{T}}(X)$$

Let Δ be a fan containing q edges and let X_Δ be the associated toric variety. Let τ_i be a edge of Δ and denote by $D_i = V_{\tau_i}$ the closure of the orbit O_{τ_i} associated to τ_i, then D_i is an invariant Weil divisor and all such divisors are on the form

$$\sum_{i=1}^{q} \lambda_i D_i \qquad \lambda_i \in \mathbb{Z} .$$

We obtain :

Lemma. The group of invariant Weil divisors is homeomorphic to :

$$W^{\mathbf{T}}(X) \cong \bigoplus_{i=1}^{q} \mathbb{Z}[D_i]$$

If $u \in M$, then there is a surjective homomorphism

$$\begin{array}{rccl} \text{div}: & M & \longrightarrow & C^{\mathbf{T}}(X) \\ & u & \longmapsto & \text{div}(u) = \sum_{i=1}^{q} \langle u, v_i \rangle D_i \end{array}$$

where v_i is the first lattice point on the edge τ_i. This implies :

Lemma. Let $u \in M$ and v_i the first lattice point of the edge τ_i, then

$$\text{ord}_{V_{\tau_i}}(\text{div}(u)) = \langle u, v_i \rangle$$

Example 22. In the case of example 5, there are two invariant Weil divisors corresponding to the two edges of the cone σ : D_1 corresponding to the edge τ_1 of e_2 and D_2 corresponding to the edge

τ_2 spanned by $2e_1 - e_2$. If $u \in M$ has coordinates (a, b) in $(\mathbb{C}^*)^2$, then $\mathrm{div}(u) = bD_1 + (2a - b)D_2$ and all invariant Cartier divisors are on this form. For example, $2D_1$ and $2D_2$ are such Cartier divisors but D_1 and D_2 are not.

The two divisors $2D_1$ and $2D_2$ are principal divisors, so we obtain : $\mathcal{C}^{\mathbf{T}}(X) = 0$ and $\mathcal{W}^{\mathbf{T}}(X) = \mathbb{Z}_2$.

Example 23. Let σ be the cone spanned by $x_1 = 2e_1 - e_2$ and $x_2 = -e_1 + 2e_2$. Each of these two vectors span a edge τ_i and the two corresponding Weil divisors are denoted D_1 and D_2. Then $\lambda_1 D_1 + \lambda_2 D_2$ is a Cartier divisor if and only if $\lambda_1 = \lambda_2 \bmod 3$ (Exercise).

7 Divisors, homology and cohomology.

In this section we will consider the general case of a complex algebraic variety.

Let n denote the (complex) dimension of X. A Weil divisor is a cycle in X. The application which associates, to each Weil divisor, its homology class defines in an evident way an homomorphism κ : $W(X) \longrightarrow H_{2n-2}(X)$. The image of a principal divisor is zero, so we obtain an homomorphism, still denoted

$$\kappa : \mathcal{W}(X) \longrightarrow H_{2n-2}(X) .$$

In other hand, for a normal variety, there is an isomorphism (cf. [6], II, Prop. 6.15)

$$\alpha : \mathcal{C}(X) \xrightarrow{\cong} \mathrm{Pic}(X)$$

between the group of classes of Cartier divisors and the Picard group of X, denoted $\mathrm{Pic}(X)$. This one is the group of isomorphy classes of line bundles (or isomorphy classes of invertible sheaves) on X. The isomorphism α is given by the map which associates, to the divisor $D = (U_\alpha, f_\alpha)$, the line bundle $\mathcal{O}(D)$ whose transition functions $U_\alpha \rightarrow U_\beta$ are given by f_α / f_β. Reciprocally, given an invertible sheaf, we associate the class of the divisor of a global rational and non trivial section.

By composition of α with the morphism $\text{Pic}(X) \to H^2(X)$ which associates to each line bundle ξ on X, its Chern class $c^1(\xi)$, we obtain a morphism denoted

$$c^1 : \mathcal{C}(X) \longrightarrow H^2(X) .$$

8 Poincaré homomorphism.

The toric varieties are examples of pseudovarieties of (real) even dimension. By definition, a pseudovariety X of (real) dimension $2n$ is a connected topological space such that there is a closed subspace Σ such that :

(a) $X - \Sigma$ is an oriented smooth variety, of dimension $2n$, dense in X,

(b) $\dim \Sigma \leq 2n - 2$.

A $2n$-pseudovariety admits a fundamental class in integer homology $[X] \in H_{2n}(X)$. The Poincaré morphism

$$H^i(X) \longrightarrow H_{2n-i}(X)$$

is the cap-product by the fundamental class. If X is smooth, it is an isomorphism.

An example of pseudovariety for which the Poincaré homomorphism is not an isomorphism is given by the toric variety of example 10 (with q=1). We have $H^2(X) = 0$ and $H_2(X) = \mathbf{Z}_p$.

Theorem. Let X be a normal compact pseudovariety, there is a commutative diagram :

$$\begin{array}{ccc} \mathcal{C}(X) & \hookrightarrow & \mathcal{W}(X) \\ \downarrow c^1 & & \downarrow \kappa \\ H^2(X) & \stackrel{\cap[X]}{\longrightarrow} & H_{2n-2}(X) \end{array}$$

where the horizontal down arrow is the Poincaré morphism of the pseudovariety X.

If X is a compact toric variety, we can prove the following result :

Theorem. [2] Let $X = X_\Delta$ be a compact toric variety, there is a commutative diagram :

$$\begin{array}{ccc} \mathcal{C}^{\mathbf{T}}(X) & \hookrightarrow & \mathcal{W}^{\mathbf{T}}(X) \\ \downarrow \cong & & \downarrow \cong \\ H^2(X) & \stackrel{\cap[X]}{\longrightarrow} & H_{2n-2}(X) \end{array}$$

where the vertical isomorphisms correspond to the morphisms c_1 and κ of the previous theorem.

We obtain, in an evident way, an interpretation of the Poincaré morphism in terms of divisors, for the compact toric varieties. In particular, the Poincaré morphism $H^2(X) \longrightarrow H_{2n-2}(X)$ is injective.

This theorem can be generalized to any toric variety (see [2]) and the Poincaré homomorphism can be described in terms of Cartier and Weil divisors : Let X be a degenerated toric variety, i.e. containing a subtorus $\mathbf{T}''$ of $\mathbf{T}$ in factor : $X = Y \times \mathbf{T}''$ where Y is a toric variety relatively to the torus $\mathbf{T}'$ such that $\mathbf{T} = \mathbf{T}' \times \mathbf{T}''$. We have the following result :

Theorem. [2] Let $X = X_\Delta$ be a n-dimensional toric variety containing a toric factor $\mathbf{T}''$ of dimension $n-d$, then we have the following isomorphisms :

i) $H^1(X) \cong H^{\mathrm{cld}}_{2n-1}(X) \cong H^1(\mathbf{T}'') \cong H^{\mathrm{cld}}_{2n-2d-1}(\mathbf{T}'') \cong \mathbb{Z}^{n-d}$;

the homomorphisms c^1 and κ are injective and there are isomorphisms

ii) $$H^2(X) \cong \mathcal{C}^{\mathbf{T}}(X) \oplus H^2(\mathbf{T}'') \cong \mathcal{C}^{\mathbf{T}}(X) \oplus \mathbb{Z}^b\,;$$

iii) $$H^{\mathrm{cld}}_{2n-2}(X) \cong \mathcal{W}^{\mathbf{T}}(X) \oplus H^{\mathrm{cld}}_{2n-2d-2}(\mathbf{T}'') \cong \mathcal{W}^{\mathbf{T}}(X) \oplus \mathbb{Z}^b$$

with $b := \binom{n-d}{2}$, such that the following diagram commutes :

$$\begin{array}{ccc} \mathcal{C}^{\mathbf{T}}(X) \oplus H^2(\mathbf{T}'') & \longrightarrow & \mathcal{W}^{\mathbf{T}}(X) \oplus H^{\mathrm{cld}}_{2n-2d-2}(\mathbf{T}'') \\ {\scriptstyle c^1 \oplus \mathrm{pr}^*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \kappa \oplus \mathrm{pr}^*}\downarrow{\scriptstyle \cong} \\ H^2(X) & \xrightarrow{\cap[X]} & H^{\mathrm{cld}}_{2n-2}(X) \end{array} .$$

This diagram can be completed by the intersection homology of X_Δ which admits also an interpretation in terms of divisors (see [3]). Let us give the (simpler) compact case :

Let $\bar{p}$ be a perversity, we define $V_{\bar{p}}$ as the open invariant subset of X_Δ, union of orbits B such that $\mathrm{codim}_{\mathbb{C}} B \leq \max\{i\,;\ \bar{p}(2i) \leq 1\}$. Let

$$\mathcal{C}^{\mathbf{T}}_{\bar{p}}(X) := \{[D] \in \mathcal{W}^{\mathbf{T}}(X)\,;\ D|_{V_{\bar{p}}} \in \mathcal{C}^{\mathbf{T}}(V_{\bar{p}})\}$$

be the group of invariant Weil divisors on X whose restriction to $V_{\bar{p}}$ is a Cartier divisor. This group is clearly isomorphic to $\mathcal{C}^{\mathbf{T}}(V_{\bar{p}})$.

Theorem. Let X be a n-dimensional compact toric variety, then :

i) We have :
$$IH^{\bar{p},\mathrm{cld}}_{2n-1}(X) = 0\,.$$

ii) there is a natural isomorphism $\mathcal{C}^{\mathbf{T}}_{\bar{p}}(X) \xrightarrow{\cong} IH^{\bar{p},\mathrm{cld}}_{2n-2}(X)$ such that the following diagram commutes :

$$\begin{array}{ccccc} \mathcal{C}^{\mathbf{T}}(X) & \xrightarrow{\subset} & \mathcal{C}^{\mathbf{T}}_{\bar{p}}(X) & \xrightarrow{\subset} & \mathcal{W}^{\mathbf{T}}(X) \\ {\scriptstyle c^1}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & {\scriptstyle \kappa}\downarrow{\scriptstyle \cong} \\ H^2(X) & \xrightarrow{\alpha} & IH^{\bar{p},\mathrm{cld}}_{2n-2}(X) & \xrightarrow{\omega} & H^{\mathrm{cld}}_{2n-2}(X) \end{array} .$$

where the composition of the two lower horizontal arrows is the Poincaré homomorphism.

9 Characteristic classes.

Let X_Δ be a smooth toric variety. The Poincaré homomorphism is an isomorphism between $H^k(X_\Delta)$ and $H_{2n-k}(X_\Delta)$ for every k. The Chern characteristic classes of X_Δ are usually defined in cohomology but their image in homology can be easily described in terms of the orbits. In fact, the total homology Chern class of X_Δ is :

$$\begin{aligned} c(X_\Delta) &= \Pi_{i=1}^{q}(1+D_i) \\ &= \Sigma_{\sigma\in\Delta}[V_\sigma] \end{aligned}$$

where $D_i = V_{\tau_i}$ are the divisors corresponding to the edges of Δ. The intersection product is given by

$$D_i \cdot V_\sigma = \begin{cases} V_\gamma & \text{if } \sigma \text{ and } \tau_i \text{ span a cone } \gamma \text{ in } \Delta \\ 0 & \text{in the other case.} \end{cases}$$

In [1] it is shown that this result is also true for singular toric varieties. More precisely, it is well known that there is no cohomology Chern classes for a singular algebraic variety. In homology we can define the Schwartz-MacPherson classes which generalize homology Chern classes and we obtain the following result :

Theorem. [1] Let X_Δ be any toric variety, the total Schwartz-MacPherson class of X_Δ is given by :

$$c(X_\Delta) = \sum_{\sigma\in\Delta}[V_\sigma]$$

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