Integration of Algebraic Functions and the Riemann-Kempf Singularity Theorem

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Introduction

These notes begin with the problem of integrating algebraic functions like

$$\int \frac{1}{\sqrt{1-x^6}}dx.$$

By extending the domain of definition from the real to the complex numbers, the problem becomes the integration of a multivalued algebraic function defined on the Riemann sphere, which we then transform to the integration of a rational differential 1-form on a compact Riemann surface C. The integration of such a differential form is a multivalued function due to the presence of poles with non-zero residues that incorporate a logarithmic term to the function and the non-trivial topology of C, that is measured by the genus $g \geq 0$. In order to concentrate on the serious part of the multivaluedness of these functions, one restricts to integrate Abelian differentials, which means that no poles are allowed. These differentials form a vector space of dimension g, and by choosing a basis one wants to understand the integrals of them together

$$\int_p^* (\omega_1, \ldots, \omega_g) : C \to \mathbb{C}^g$$

Of course the above function is multivalued, but it has a 'mild' multivaluedness, since the difference between two branches form an additive subgroup Λ of $\mathbb{C}^g$, called the periods of C, which is discrete with a compact fundamental region. The quotient group $\mathbb{C}^g/\Lambda$ is called the **Jacobian variety** $J(C)$ and the map

$$\int_p^* (\omega_1, \ldots, \omega_g) : C \to J(C) = \mathbb{C}^g/\Lambda$$

is called the **universal Abelian integral**. This is an algebraic map. The Abelian differential forms on $J(C)$ may be written as $\sum_{j=1}^g a_j dz_j$ where a_j are constants and its integral correspond to the multivalued functions on $J(C)$ induced by the linear function $\sum_{j=1}^g a_j z_j$ on $\mathbb{C}^g$. These 'constant' differential forms pull back via the universal Abelian integral to the Abelian differentials on C. Hence the integrals of Abelian differentials on C may be decomposed as the composition of an algebraic map and a 'multivalued linear function' on $J(C)$. The fibres of the integral of an Abelian differential

$$\int \sum_{j=1}^g a_j dz_j$$

on $J(C)$ corresponds to parallel hyperplanes in $\mathbb{C}^g$ projected to $J(C)$, and may be understood as the interplay of the hyperplane with the periods. We then want to intersect this codimension 1 foliation in $J(C)$ with the image of the universal Abelian integral, which as we mentioned, is an algebraic map. This reduces the problem of understanding the integrals of Abelian differentials to understanding an algebraic map and linear foliations of codimension 1 in $J(C)$ and how they intersect.

We then explain the connection found by Abel between the integration of Abelian differentials and function theory of C. Since the Jacobian variety $J(C)$ is a group, we can construct by addition on the image the n^{th}-Abelian map

$$\int_{np} : C^{(n)} \to J(C) = \mathbb{C}^g/\Lambda$$

$$\int_{np}(c_1 + \ldots + c_n) := \int_p^{c_1}(\omega_1, \ldots, \omega_g) + \ldots + \int_p^{c_n}(\omega_1, \ldots, \omega_g)$$

defined on the symmetric product $C^{(n)} := C^n/Sym(n)$ of C. Abel's Theorem relates the fibers of these maps to rational functions on C. Where the n^{th} Abelian map is injective, it means that there is no rational function associated, but where it is not injective, then the fibre may be identified with a projective space of a finite dimensional family of rational functions on C of degree n. If we denote the image variety by

$$\mathcal{W}_n := \int_{np}(C^{(n)}) \subset J(C)$$

then there is a relationship between the singularities of $\mathcal{W}_n$ and the fibers of $\int_{np}$, which is the content of the Riemann-Kempf singularity theorem. From the degree of the singularity of $\mathcal{W}_n$ at v, we can read off the dimension of the fibre $\int_{np}^{-1}(v)$ and hence of the associated family of rational functions. Furthermore, one can give a geometric description of the tangent cone to $\mathcal{W}_n$ at v in terms of the linear geometry of the canonical embedding of C in $\mathbb{P}^{g-1}$.

This theorem was first noted by Bernhard Riemann (1826-1866) who analyzed $\mathcal{W}_{g-1}$. In this case, $\mathcal{W}_{g-1}$ is a hypersurface, and hence defined by one equation in $J(C)$. This equation may be written as a holomorphic function on $\mathbb{C}^g$ and it is Riemann's famous Theta function.

The jump in the level of discussion, from the explicit integration of algebraic functions to the analysis of the singularities of Riemann's Theta function, has always impressed me. The ability of Riemann of expressing his result with the modest machinery at hand, is also remarkable. The modern algebraic proof by George Kempf [4], using Grothendieck's variational machinery, is also a significant contribution to understanding the Riemann-Kempf theorem. I recommend [4] for deeper reading.

There are 3 sections. In section 1 we show how to pass from an algebraic function to the Riemann surface that it defines. In section 2 we explain Abel's theorem, and finally in section 3 we state the Riemann-Kempf singularity theorem.

These notes were written for the Algebraic Geometry Summer School in Bilkent University, Ankara, Turkey, from the 7 till the 18 of August of 1995. I would like to thank my friend professor Sinan Sertöz for the excellent environment during the school, as well as Professor İhsan Doğramacı for being an extraordinary benefactor of the Arts and Sciences.

1 Integration of Algebraic Functions

1.1 Integration of Elementary Algebraic Functions

The integral of a rational function of 1 variable can always be carried out:

Example 1.1 *If $X(x)$ is a rational function of $x \in \mathbb{R}$ then*

$$\int X dx = Rat(x) + \beta \sum Log(Rat(x)) + \beta \sum ArcTan(Rat(x))$$

for some rational functions $Rat(x)$ and real numbers β.

Proof: Let the rational function X be

$$X(x) = \frac{a_m x^m + \ldots + a_0}{b_n x^n + \ldots + b_0} = Pol(x) + \frac{c_{n-1}x^{n-1} + \ldots + c_0}{b_n x^n + \ldots + b_0}$$

where we have used the Euclidean Algorithm of division. If

$$b_n x^n + \ldots + b_0 = \Pi_j b_n (x - r_j)^{n_j} \Pi_k [(x - \alpha_k)(x - \bar{\alpha}_k)]^{m_k},\ r_j \in \mathbb{R},\ \alpha_k \in \mathbb{C} - \mathbb{R}$$

is the factorization of the denominator of X, then we can further expand X into partial fractions as

$$X(x) = Pol(x) + \sum_{j, 0 < j_1 \leq n_j} \frac{d_{j,j_1}}{(x - r_j)^{j_1}} + \sum_{k, 0 < k_1 \leq m_k} [\frac{e_{k,k_1}}{(x - \alpha_k)^{k_1}} + \frac{\bar{e}_{k,k_1}}{(x - \bar{\alpha}_k)^{k_1}}]$$

with $d_{j,j_1} \in \mathbb{R}$ and $e_{k,k_1} \in \mathbb{C}$. So

$$\begin{aligned}\int X(x)dx &= Rat(x) + \sum_j \int \frac{d_{j,1}}{(x - r_j)} dx + \sum_k \int [\frac{e_{k,1}}{(x - \alpha_k)} + \frac{\bar{e}_{k,1}}{(x - \bar{\alpha}_k)}] dx \\ &= Rat(x) + \sum_j d_{j,1} Log(x - r_j) + \sum_k Re(e_{k,1}) Log(|x - \alpha_k|^2) \\ &\quad + 2\sum_k Im(e_{k,1}) Arg(\alpha_k - x)\end{aligned}$$

And finally

$$Arg(\alpha_k - x) = ArcTan(\frac{Im(\alpha_k)}{Re(\alpha_k) - x})$$

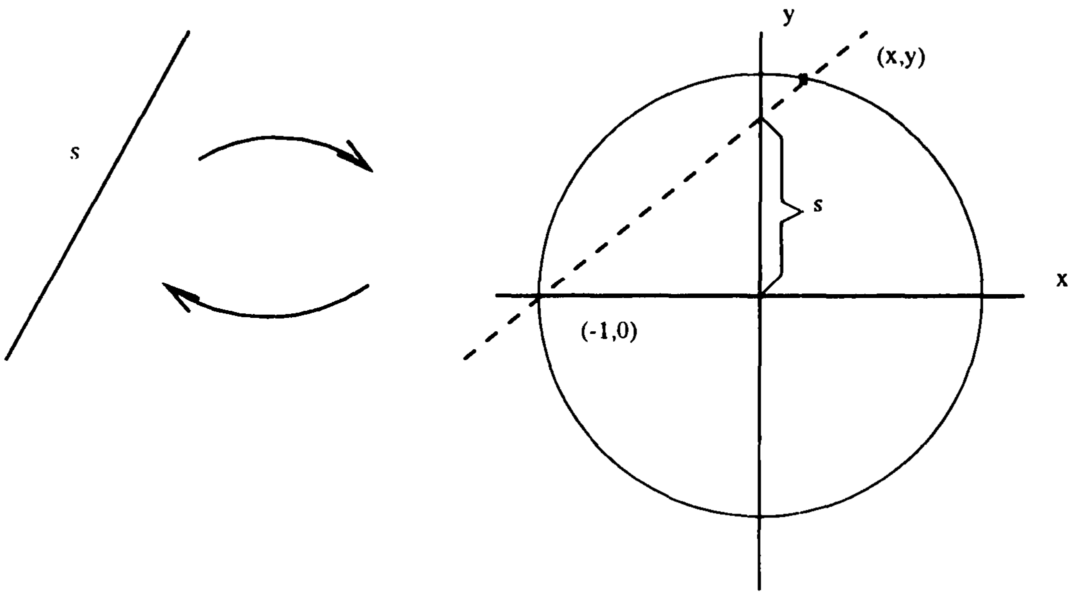
□

More generally we have:

Example 1.2 *If $X(x)$ is a rational function of x and $\sqrt{1 - x^2}$ then*

$$\begin{aligned}\int X(x)dx &= Rat(x, \sqrt{1 - x^2}) + \sum \beta Log(Rat(x, \sqrt{1 - x^2})) \\ &\quad + \sum \beta ArcTan(Rat(x, \sqrt{1 - x^2}))\end{aligned}$$

Figure 1: Rational parametrization of $x^2 + y^2 - 1 = 0$



Proof: $y = \sqrt{1-x^2}$ if and only if $x^2 + y^2 = 1$. Let C be the unit circle. There is a rational parametrization of C by rational functions:

$$s \in \mathbb{R} \to (x,y) = (\frac{-s^2+1}{s^2+1}, \frac{2s}{s^2+1}) \in C \qquad (x,y) \in C \to s = \frac{y}{x+1}$$

Doing a change of variables in the integral and applying Example 1.1 we obtain:

$$\int X(x, \sqrt{1-x^2})dx = \int X(x,y)dx = \int X(\frac{-s^2+1}{s^2+1}, \frac{2s}{s^2+1}))(\frac{-4s}{(s^2+1)^2})ds$$

$$= Rat(s) + \sum \beta Log(Rat(s)) + \sum \beta ArcTan(Rat(s))$$

$$= Rat(\frac{y}{x+1}) + \sum \beta Log(Rat(\frac{y}{x+1})) + \sum \beta ArcTan(Rat(\frac{y}{x+1}))$$

$$= Rat(\frac{\sqrt{1-x^2}}{x+1}) + \sum \beta Log(Rat(\frac{\sqrt{1-x^2}}{x+1})) + \sum \beta ArcTan(Rat(\frac{\sqrt{1-x^2}}{x+1}))$$

□

Example 1.3 *If $X(x)$ is a rational function of x and $\sqrt{x^2+bx+c}$ then*

$$\int X(x)dx = Rat(x, \sqrt{x^2+bx+c}) + \sum \beta Log(Rat(x, \sqrt{x^2+bx+c}))$$

$$+ \sum \beta ArcTan(Rat(x, \sqrt{x^2+bx+c}))$$

Proof: Parametrize $y^2 = x^2 + bx + c$ with a rational function of $s \in \mathbb{R}$ and repeat the argument in Exercise 1.2. □

Trying to integrate more complicated algebraic functions, we almost never get such simple answers, because $\int X(x)dx$ is a much too complicated function to admit a representation in terms of elementary expressions. Classically the elliptic integrals

$$\int \frac{1}{\sqrt{1-x^4}}dx$$

were found not to be integrable in terms of elementary functions. To see this, it was convenient to extend x to a complex variable.

1.2 Basic Facts of Functions of a Complex Variable and Their Integrals

$\mathbb{R}^2$ and $\mathbb{C}$ are naturally and canonically isomorphic, via $(x, y) \to x + iy$.

The $\mathbb{C}$-linear map $\mathbb{C} \to \mathbb{C}$ obtained by multiplying with the complex number $a+ib$, corresponds to the $\mathbb{R}$-linear map $\mathbb{R}^2 \to \mathbb{R}^2$ with matrix representation $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

The $\mathbb{R}$-linear map $\mathbb{R}^2 \to \mathbb{R}^2$ with matrix representation $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ corresponds to a $\mathbb{C}$-linear map $\mathbb{C} \to \mathbb{C}$ if and only if $a = d$ and $c = -b$.

A C^1-function $f = (f^1, f^2) : U \to \mathbb{R}^2$, U open in R^2 is **holomorphic** if and only if its linear approximations at every point of U correspond to $\mathbb{C}$ linear map, i.e. if and only if it satisfies the Cauchy-Riemann differential equations

$$\frac{\partial f^1}{\partial x} = \frac{\partial f^2}{\partial y} \qquad \frac{\partial f^1}{\partial y} = -\frac{\partial f^2}{\partial x}$$

Let U be an open set in $\mathbb{R}^2$ and $A, B : U \to \mathbb{R}$ be C^1-functions on U. A **real 1-form** is an expression of the form $\omega = Adx + Bdy$. If $\gamma = (\gamma^1, \gamma^2) : [a, b] \to U$ is a piecewise smooth curve in U, the **integral of the 1-form** ω **over the curve** γ is by definition:

$$\int_\gamma \omega = \int_a^b [A(\gamma(t))\frac{d\gamma^1}{dt}(t) + B(\gamma(t))\frac{d\gamma^2}{dt}(t)]dt \qquad (1.1)$$

If $A, B : U \to \mathbb{C}$ are C^1-functions on U, then $\omega = Adx + Bdy$ is a **(complex valued) 1-form**, and the integral of ω over the curve γ is defined again by (1.1).

We can write complex valued differential forms in a different way using a new basis:

$$\mathbb{C}dx \oplus \mathbb{C}dy = \mathbb{C}dz \oplus \mathbb{C}d\bar{z}$$

where we have the linear relations between the basis:

$$dz = dx + idy \qquad d\bar{z} = dx - idy$$

$$dx = \frac{dz + d\bar{z}}{2} \qquad dy = \frac{dz - d\bar{z}}{2i}$$

By definition, we then have

$$\int_\gamma Adz + Bd\bar{z} = \int_\gamma A(dx + idy) + B(dx - idy)$$

The **differential** $d\omega$ **of a** C^1 **1-form** ω is the 2-form

$$d\omega = [\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}]dx \wedge dy$$

Recall that 2 forms may be integrated over open subsets of $\mathbb{R}^2$.

Stokes' Theorem *If ω is a C^1-form on the open set U in $\mathbb{R}^2$ and if V is an open set with compact closure in U and having a boundary ∂V formed by a finite number of smooth curves, then*

$$\int_{\partial V} \omega = \iint_V d\omega$$

Lemma 1.4 *Let $f = f^1 + if^2$ be a complex valued C^1-function on $U \subset \mathbb{R}^2$, then $d(fdz) = 0$ if and only if f is holomorphic in U.*

Proof:

$$d(fdz) = d((f^1 + if^2)(dx + idy)) = d((f^1 + if^2)dx + (-f^2 + if^1)dy)$$

$$= [\frac{\partial}{\partial x}(-f^2 + if^1) - \frac{\partial}{\partial y}(f^1 + if^2)]dx \wedge dy =$$
$$[(-\frac{\partial f^2}{\partial x} - \frac{\partial f^1}{\partial y}) + i(\frac{\partial f^1}{\partial x} - \frac{\partial f^2}{\partial y})]dx \wedge dy$$

Hence the result follows from the Cauchy-Riemann equations. □

A **holomorphic 1-form** fdz on $U \subset \mathbb{C}$ is a differential form with f a holomorphic function on U. The Cauchy-Riemann differential equations (embedded in Lemma 1.4) and Stokes' Theorem blend to give:

Cauchy's Lemma *If ω is a holomorphic 1-form on U and V is an open set with compact closure in U and boundary a finite number of C^1-curves we have*

$$\int_{\partial V} \omega = 0$$

Corollary 1.5 *If fdz is a holomorphic 1-form on a disc $D \subset \mathbb{C}$ and $\gamma : [a, b] \to D$ is a C^1 curve in D then $\int_\gamma fdz$ depends only on the extreme points $\gamma(a), \gamma(b)$ of γ.*

We say that 2 curves $\gamma_0, \gamma_1 : [a, b] \to U$ are **homotopic** if:

1) They have the same extreme points $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$;

2) There is a continuous deformation of γ_0 to γ_1 keeping the extreme points fixed: i.e. there exists a continuous map $\Gamma : [a, b] \times [0, 1] \to U$ with $\Gamma(\ ,0) = \gamma_0$, $\Gamma(\ ,1) = \gamma_1$ and for every $s \in [0, 1]$ we have $\Gamma(a, s) = \gamma_0(a)$ and $\Gamma(b, s) = \gamma_0(b)$.

Corollary 1.6 *If fdz is a holomorphic 1-form on an open set $U \subset \mathbb{C}$ and $\gamma_0, \gamma_1 : [a, b] \to D$ are C^1 curves in U which are homotopic then*

$$\int_{\gamma_0} fdz = \int_{\gamma_1} fdz$$

We may define the **integral of a holomorphic 1-form** $f(z)dz$ as the function

$$z \to \int_p^z f(w)dw$$

where we carry out the integration over a path from p to a point z_0 and then make small extensions of this path to nearby point z. We have

$$\frac{d}{dz}\int_p^z f(w)dw = f(z)$$

If we choose another path of integration from p to z_0 the function we obtain differs from the preceding one by adding a constant.

Cauchy's integral formula : *Let $f : U \to \mathbb{C}$ be a holomorphic function, $\{|z - z_0| \leq r\} \subset U$, then for $|z - z_0| < r$ we have*

$$f(z) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(w)}{w - z}dw$$

Corollary 1.7 *Under the above hypothesis we have*

*1)***Power series expansion:** *For $|z - z_0| < r$ we have*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \qquad a_n = \frac{1}{n!}f^n(z_0) := \frac{1}{n!}\frac{d^n f}{dz^n}(z_0)$$

2) **Mean Value Property:**

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

Maximum Modulus Principle *If $f : U \to \mathbb{C}$ is holomorphic and W is a compact subset of V, then*

$$max\{|f(z)| \mid z \in W\} = max\{|f(z)| \mid z \in \partial W\}$$

Corollary 1.8 (Laurent series expansion) *If $f : D^* \to \mathbb{C}$ is a holomorphic function defined in the punctured disc $D^* = \{0 < |z - z_0| < r\}$ then for $z \in D^*$ we have*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Under the above assumptions, we say that z_0 is a **removable singularity of f** if $a_n = 0$ for all $n < 0$. In this case by defining $f(z_0) = a_0$ we have a holomorphic function on $D = \{|z - z_0| < r\}$. z_0 is a **pole** of f if $a_n = 0$ for $n < -N < 0$ and $a_{-N} \neq 0$. We say in this case that $N = ord(f, z_0)$ is the **order of the pole** of f at z_0. If $a_n \neq 0$ for an infinite number of negative n, then we say that f has an **essential singularity** at z_0 (i.e. an infinite number of non-zero terms in the negative Laurent series expansion). A function that is holomorphic except for a discrete set of poles is a **meromorphic function**.

The term a_{-1} is the **residue of f at** z_0, and it is denoted by $Res(f, z_0)$.

Example 1.9

The exponential function $e : \mathbb{C} \to \mathbb{C}^* := \mathbb{C} - \{0\}$ is defined by

$$e^z := \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^{Re(z)}[Cos(Im(z)) + iSin(Im(z))]$$

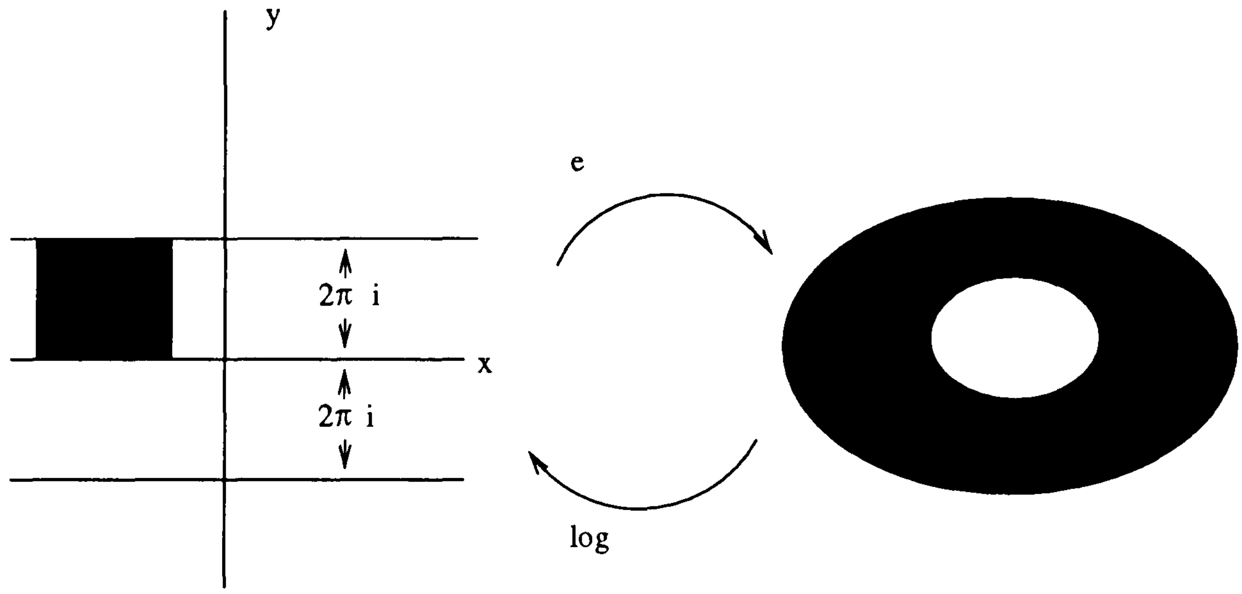
Example 1.10

The Logarithmic 'function' is the multivalued function defined by the formula

$$Log(z) = \int_1^z \frac{1}{w} dw$$

$Log : \mathbb{C}^* \to \mathbb{C}$. It is easy to see that the logarithmic function is the

Figure 2: The exponential and Logarithmic Functions



inverse of the exponential function (i.e. $\frac{d}{dz}e^{Log(z)} = 1$), and that the difference between 2 branches is an integer multiple of $2\pi i$.

We also have the related multivalued holomorphic function $Log(z - z_0) : \mathbb{C} - \{z_0\} \to \mathbb{C}$ satisfying

$$Log(z - z_0) = \int_1^z \frac{1}{w - z_0} dw$$

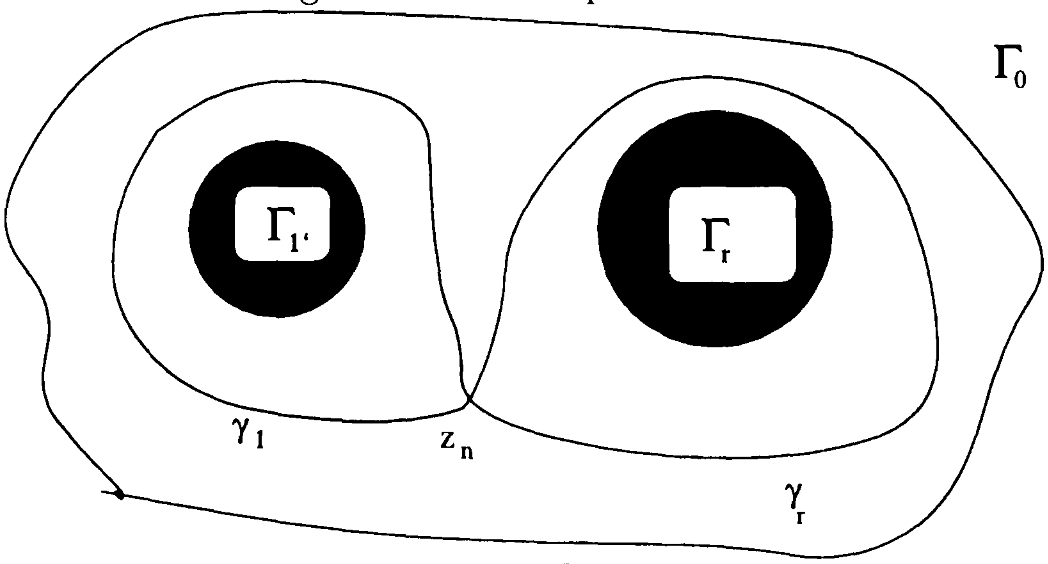
Its inverse is the function $e^w + z_0$.

Corollary 1.11 *Let $f : D^* \to \mathbb{C}$ be a holomorphic function defined in the punctured disc $D^* = \{0 < |z - z_0| < r\}$ with Laurent expansion $f(z) = \sum_{-\infty}^{\infty} a_n(z - z_0)^n$ then $\int f(z)dz$ is a multivalued holomorphic function admitting a representation*

$$\int f(z)dz = \sum_{n\in\mathbb{Z}-\{-1\}} \frac{1}{n+1} a_n(z-z_0)^{n+1} + a_{-1}Log(z-z_0) + Constant$$

Two different branches of $\int f(z)dz$ differ by $2\pi i a_{-1}$(integer), where a_{-1} is the residue of f at z_0. If this residue is 0, then $\int f(z)dz$ is a holomorphic function on D^*.

Figure 3: A multiple connected domain



Proof: One may permute the summation of the Laurent series with the integral, due to uniform convergence of the series. Then one integrates the monomials one by one, giving another monomial except for $n = -1$ where one gets a logarithm. □

Let $U \subset \mathbb{C}$ be a multiple connected domain in the complex plane, that is, such that its boundary ∂U is the union of a finite number of simple closed curves $\Gamma_0, \ldots, \Gamma_r$, where Γ_0 corresponds to the exterior boundary. A closed loop is a continuous map from the circle S^1 to U that usually takes a marked point $* \in S^1$ to some marked point $z_0 \in U$. Two closed loops based at z_0 are **homotopic** if we may deform one to the other by means of a continuous 1-parameter family of closed loops based at z_0. The **fundamental group** $\pi_1(U, z_0)$ of U is the group of homotopy classes of closed loops in U based at z_0 and it is a free group generated by loops $\gamma_1, \ldots, \gamma_r$ which go once around only one component Γ_k. The function $\int f(z)dz$ is again a multivalued function. The difference between any two branches is a number $\sum_{k=1}^{r}(integer)\beta_k$ where $\beta_k = \int_{\gamma_k} f(z)dz$ are the **periods** of the integral around the holes.

So we see that $\int f(z)dz$ is multivalued because of 2 different reasons: either because f has a pole with a non-zero residue, or because of the non-trivial topology of U.

We will end this subsection with

Residue Theorem *Let $f : U - \{z_1, \ldots, z_r\} \to \mathbb{C}$ be a holomorphic function and $W \subset U$ be an open set containing $\{z_1, \ldots, z_r\}$, with compact closure in U and whose boundary consists of a finite number of piecewise smooth curves, then*

$$\int_{\partial W} f(z)dz = 2\pi i \sum_{k=1}^{r} Res(f, z_k)$$

Proof: Apply Cauchy's Lemma to $W - \cup_{k=1}^{r}\{|z - z_k| < \epsilon\}$ and Corollary 1.11 to each punctured disc $\{0 < |z - z_0| < \epsilon\}$. □

1.3 Geometric Facts of Functions of a Complex Variable

Non-zero $\mathbb{C}$ linear maps on $\mathbb{C}$ may be determined as those $\mathbb{R}$-linear maps that are orientation preserving and conformal, in the sense that they preserve angles. Since holomorphic functions may be approximated by $\mathbb{C}$-linear maps, holomorphic maps preserve angles at those points where the derivative is non-vanishing.

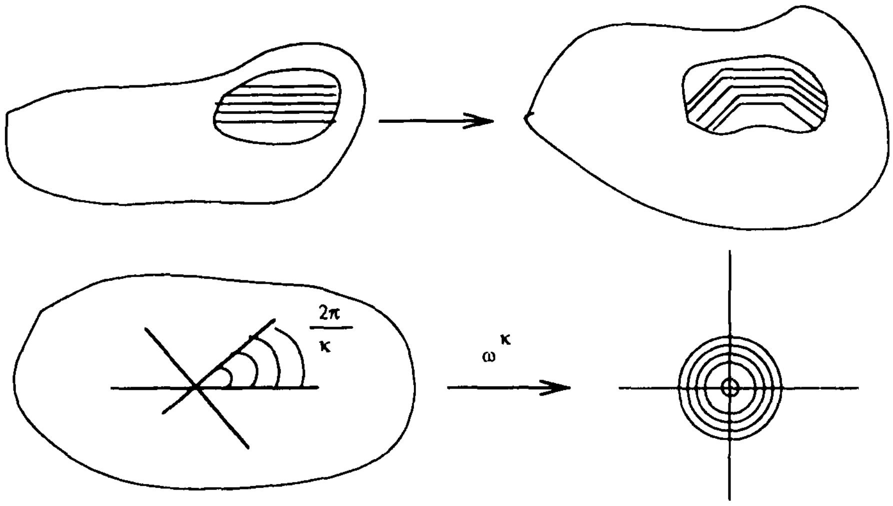
Proposition 1.12 *Let $f : U \to \mathbb{C}$ be a holomorphic map and $z_0 \in U$:*

1) If $f'(z_0) \neq 0$, then there exists a neighborhood U_0 of z_0 and U_1 of $f(z_0)$ such that the restriction of f induces a homeomorphism $f : U_0 \to U_1$ with a holomorphic inverse (i.e. f is a local biholomorphism).

2) If $f'(z_0) = \ldots = f^{k-1}(z_0) = 0$ and $f^k(z_0) \neq 0$, then there exist neighborhoods U_0 of z_0, U_1 of $f(z_0)$ and a change of variable w in U_0 such that the restriction of f to these neighborhoods and variables has the form $f(w) = a_0 + w^k$.

Proof: 1) Since $det \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2 + b^2$, we may apply the Implicit

Figure 4: Local behaviour of holomorphic maps



Function Theorem in $\mathbb{R}^2$ to obtain the result.

2) Expanding f in power series

$$f(z) = a_0 + \sum_{n \geq k} a_n (z - z_0)^n = a_0 + (z - z_0)^k [\sum_{n \geq k} a_n (z - z_0)^{n-k}]$$

Let $h(z) = [\sum_{n \geq k} a_n (z - z_0)^{n-k}]^{1/k} = e^{Log[\sum_{n \geq k} a_n (z - z_0)^{n-k}]^{1/k}}$ and set the new variable $w = (z - z_0) h(z)$. Then we have $f(w) = a_0 + (z - z_0)^k h(z)^k = a_0 + w^k$ □

Geometrically this means that locally holomorphic maps are very simple:

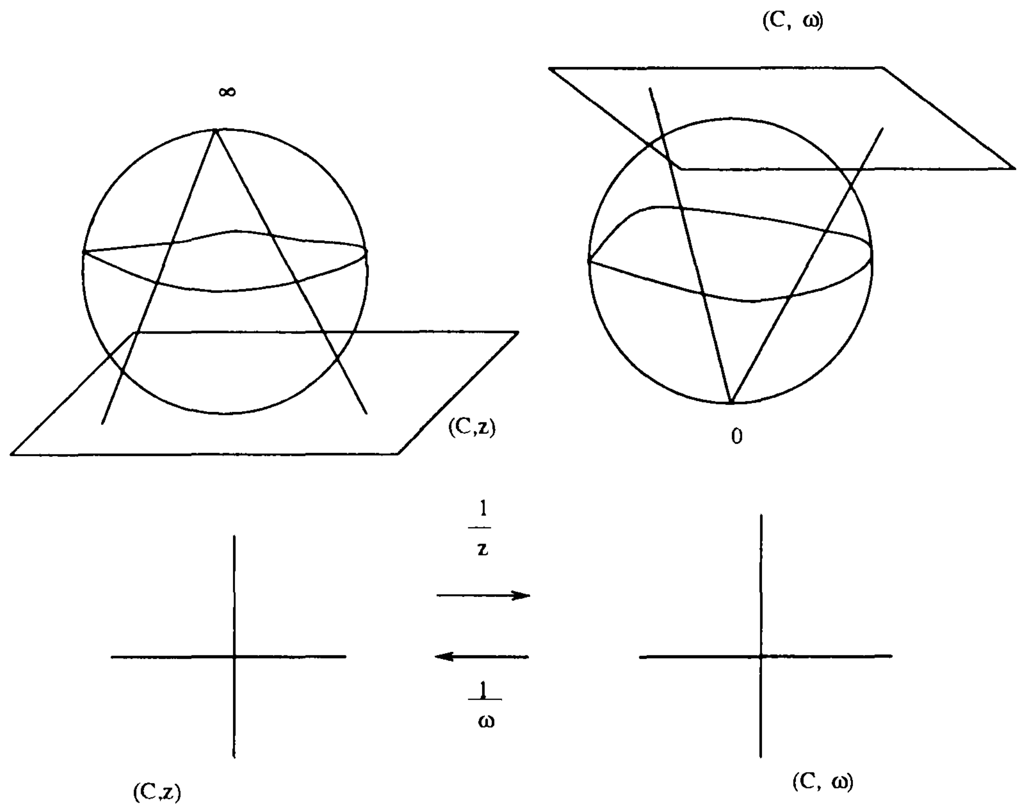
A point z_0 of type 2) will be called a **branch point with ramification index** $k := \nu(f, z_0)$.

Example 1.13

The Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is obtained by gluing a point to the complex plane, as in stereographic projection:

Mathematically, it is simpler to consider 2 copies of the complex plane $(\mathbb{C}, z)$ and $(\mathbb{C}, w)$ each with its own variable and glue $(\mathbb{C}^*, z) \to (\mathbb{C}^*, w)$ with the maps $z \to w = \frac{1}{z}$ and $w \to z = \frac{1}{w}$.

Figure 5: The Riemann Sphere



Let U be an open set in $\mathbb{C}$ and $f : U \to \bar{\mathbb{C}}$. f is said to be a **holomorphic map to the Riemann sphere** if:

1) $f : U - f^{-1}(\infty) \to (\mathbb{C}, z)$ is a holomorphic function;

2) $\frac{1}{f} : U - f^{-1}(0) \to (\mathbb{C}, w)$ is a holomorphic function.

Recall that a **meromorphic function** f on the open set U of the complex plane is a holomorphic function $f : U - \Lambda \to \mathbb{C}$, where Λ is a discrete set of points of U, such that f has a pole or a removable singularity at each point of Λ.

Theorem 1.14 *There is a one to one correspondence between meromorphic functions on $U \subset \mathbb{C}$ and holomorphic maps $U \to \bar{\mathbb{C}}$.*

Proof: 1) Assume f is meromorphic on U and let Λ be the poles of f. If $z_0 \in \Lambda$ then

$$f(z) = \sum_{n=-k}^{\infty} a_n(z-z_0)^n = (z-z_0)^{-k} \sum_{n=-k}^{\infty} a_n(z-z_0)^{n+k} = (z-z_0)^{-k} h(z)$$

In the other coordinate of $\bar{\mathbb{C}}$, f is represented by

$$\frac{1}{f(z)} = (z - z_0)^k \frac{1}{h(z)}$$

which is holomorphic since $h(z_0) = a_{-k} \neq 0$

2) Let $f : U \to \bar{\mathbb{C}}$ be a holomorphic map to the Riemann sphere, and let $\Lambda = f^{-1}(\infty)$. By definition, $f : U - \Lambda \to \mathbb{C}$ is holomorphic. For $z \in \Lambda$ we have $\frac{1}{f(z)}$ is holomorphic:

$$\frac{1}{f(z)} = b_k(z - z_0)^k + \ldots = (z - z_0)^k [b_k + \ldots]$$

Hence $f(z) = (z - z_0)^{-k}$[holomorphic]. Hence f has a pole at z_0. □

A **Riemann surface** is a Hausdorff topological space S together with an open covering $\{U_j\}$ of S with homeomorphisms $\phi : U_k \to V_k \subset \mathbb{C}$ onto open sets V_k of $\mathbb{C}$ such that the transition coordinates

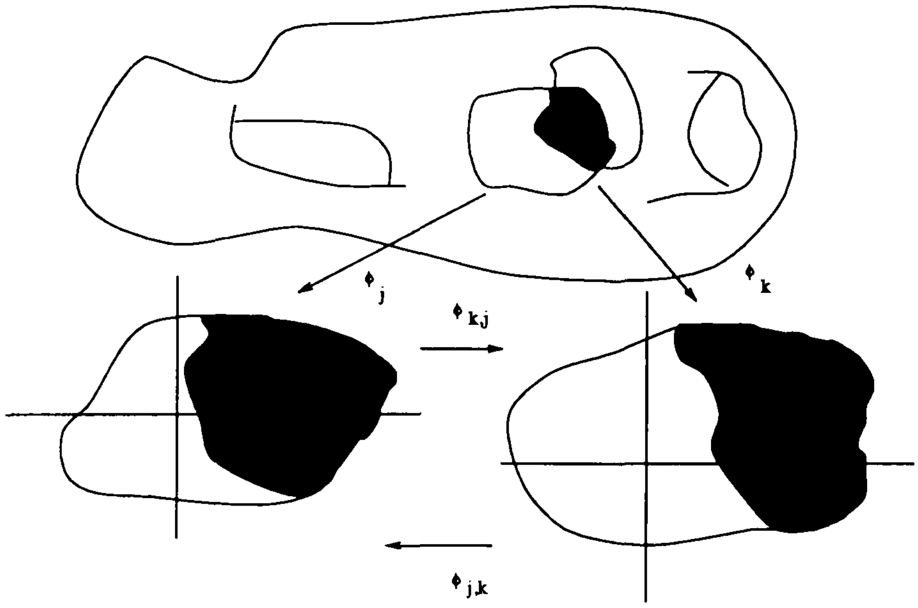
$$\phi_{k,j} : \phi_j(U_j \cap U_k) \to \phi_k(U_j \cap U_k)$$

are bijective holomorphic maps.

An example of a Riemann surface is the Riemann sphere $\bar{\mathbb{C}}$.

Concepts which are invariant under biholomorphic maps can be introduced on a Riemann surface by precomposing with the local coordinate charts. For example a **meromorphic function** f on the Riemann surface C is a function $f : C - \Lambda \to \mathbb{C}$, where Λ is a discrete set of points on C such that $f \circ \phi_\alpha^{-1}(z_\alpha)$ is a meromorphic function for $\{(U_\alpha, z_\alpha)\}$ a covering of C by coordinates charts. A **holomorphic map** between Riemann surfaces $f : C \to C'$ is a continuous map such that in local coordinates of domain and codomain it is a holomorphic function. Theorem 1.14 extends to Riemann surfaces, gives a 1-1 correspondence between meromorphic functions on C

Figure 6: Coordinate charts of a Riemann surface



and holomorphic maps from C to the Riemann sphere $\bar{\mathbb{C}}$. A **meromorphic 1-form** ω on C is a collection of meromorphic 1-forms $\omega_\alpha = A_\alpha(z_\alpha)dz_\alpha$ on a covering $\{(U_\alpha, z_\alpha)\}$ such that

$$A_\alpha(z_\alpha) = A_\beta(z_\beta)\frac{dz_\beta}{dz_\alpha}$$

Meromorphic objects defined on compact Riemann surfaces will be called **rational**. Implicitly in this notation, we are using that there is a 1 to 1 correspondence between compact Riemann surfaces and complete smooth complex curves, and this correspondence extends from meromorphic objects defined on the Riemann surface to rational objects defined on the complex curve.

An oriented compact topological surface has a unique topological invariant, the genus g, that can be any non-negative integer $0, 1, , \ldots$. Every such surface is equivalent to a torus with g-handles. Its fundamental group is a free group with 2g generators and such that they satisfy the unique relation

$$[\alpha_1, \alpha_{g+1}] \ldots [\alpha_g, \alpha_{2g}] = id \quad ; \quad [\alpha, \beta] := \alpha\beta\alpha^{-1}\beta^{-1}$$

Recall that the first homology group $H_1(C, \mathbb{Z})$ of the compact surface C of genus g is the abelianization of the fundamental group $\pi_1(C,p)$, and so it is a free abelian group with the same 2g generators as $\pi_1(C,p)$. These generators may be chosen as $\alpha_1, \ldots, \alpha_{2g}$ with the property that α_i intersects α_{g+i} once positively, and do not intersect any other α_k, This basis will be called a **canonical basis**.

Proposition 1.15 *Let $f : C \to \mathbb{C}$ be a holomorphic function defined on the compact and connected Riemann surface C, then f is a constant function.*

Proof: The real valued function $|f| : C \to \mathbb{R}$ attains its maximum value at some point p_0. Take coordinates (U, z) around p_0. By the Maximum Modulus Principle, f is constant on U, but then by analytic continuation, it will be constant on C. □

Recall a classical **Theorem of Euler** that says that if one triangulates a compact orientable surface of genus g, and e_0, e_1, e_2 denotes respectively the number of 0, 1 and 2 cells then $e_0 - e_1 + e_2 = 2 - 2g$.

Riemann-Hurwitz Formula *Let $f : C \to C'$ be a non-constant holomorphic map between 2 compact and connected Riemann surfaces of genuses g and g', and let B be the ramification points of f. Then $f : C - f^{-1}(f(B)) \to C' - f(B)$ is a finite covering map, say with n sheets, and we have*

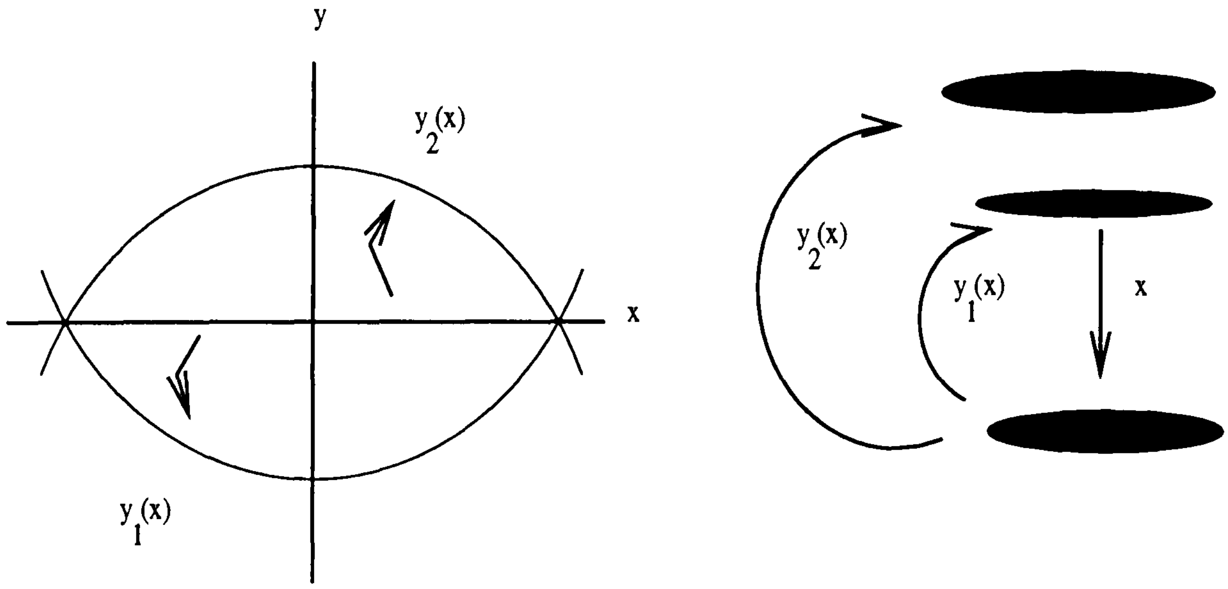
$$2 - 2g = n(2 - 2g') - \sum_{q \in B} (\nu(f, q) - 1)$$

where $\nu(f, q)$ is the branching order of f at q.

Proof: Outside the critical values $f(B)$ it follows from Proposition 1.12 that f is a covering map, since it is locally covering at each point and there are only a finite number of points in the inverse image by the compactness hypothesis. The connectedness hypothesis implies that the number of inverse images remains constant, so that f outside of $f^{-1}(B)$ is a covering map with n sheets.

Triangulate C' so that the singular values of f are 0-cells in the

Figure 7: $x^2 + y^2 - 1 = 0, \quad y = \sqrt{1 - x^2}$



triangulation, and pull back the triangulation to C'. It is immediate that $e_2' = ne_2$, $e_1' = ne_1$ and $e_0' = ne_1 - \sum_{q \in B}(\nu(q,f) - 1)$. By taking the alternating sum and using Euler's Theorem we obtain the formula. □.

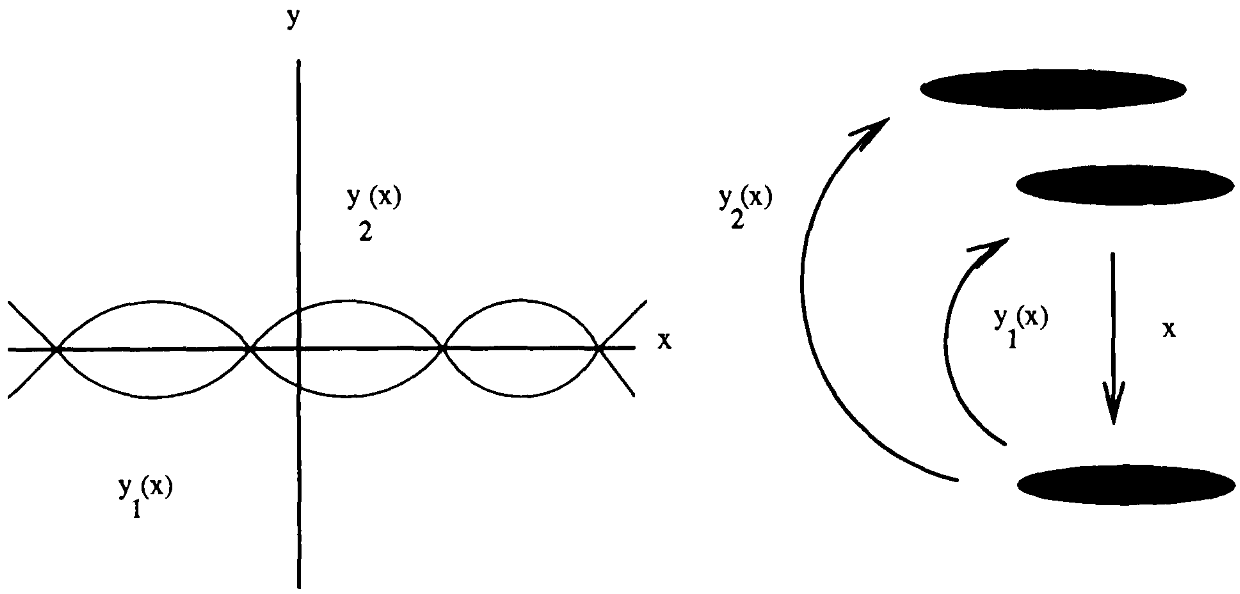
1.4 Algebraic Functions

Let $f(x,y) = \sum_{j=1}^{n} a_j(x)y^j$ be an irreducible polynomial in 2 variables x and y with complex coefficients and with positive degree n in y. The height y of a point (x,y) on the Riemann surface C' is by definition an **algebraic function** of the horizontal position.

Algebraic functions are multivalued holomorphic functions, but acquiring only a finite number of distinct values (that is, they have only 'mild' multivaluedness).

Theorem 1.16 *Given an irreducible polynomial $f(x,y) = \sum_{j=1}^{n} a_j(x)y^j$, $n > 0$, and $C' \subset \mathbb{C}^2$ the complex algebraic curve defined by $f = 0$, then there is a unique way to construct a connected compact Riemann surface C with 2 rational functions x and y such that $(x,y) : C \to C' \subset \mathbb{C}^2$ is a birational isomorphism; that is, such that any rational function on C' may be expressed as a rational*

Figure 8: $x^4 + y^2 - 1 = 0, \quad y = \sqrt{1 - x^4}$



function of x and y.

Proof: 1) There are a finite number of points $B \subset C'$ that satisfy $f = \frac{\partial f}{\partial x} = 0$. Outside of $x(B)$, C' looks locally like the graph of n holomorphic functions by the Implicit Function Theorem (of functions from $\mathbb{R}^4 = \mathbb{C}^2$ to $\mathbb{R}^2 = \mathbb{C}$).

2) If $(x_1, y_1) \in B$ is such that $\frac{\partial f}{\partial y}(x_1, y_1) \neq 0$, we may use y as a local variable of C' in a neighborhood of (x_1, y_1). C' is locally the graph of a function $x(y)$. The map $y \to (x(y), y) \to x(y)$ has a branch point of index $k := \nu(y_1, x(y)) > 1$, so that C' is around (x_1, y_1) the graph of a multivalued function of type $x^{\frac{1}{k}}$.

3) If (x_1, y_1) satisfies $f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, then (x_1, y_1) is a singular point of C'. If we intersect C' with a small sphere around (x_1, y_1) we will obtain a finite number of closed loops in S^3 which may be knotted. This implies that locally $C' - \{(x_1, y_1)\}$ has the analytic type of several punctured disc, embedded in a non-trivial way in $\mathbb{C}^2$. We may complete 'abstractly' each punctured disc with a disc to obtain a desingularization of C' at these points.

4) Connectedness follows directly from the irreducibility of f.

5) Carry out the above procedure for the other 3 canonical charts of

$\bar{\mathbb{C}} \times \bar{\mathbb{C}}$ by taking coordinates $x_1 = \frac{1}{x}$ and/or $y_1 = \frac{1}{y}$. Since $\bar{\mathbb{C}} \times \bar{\mathbb{C}}$ is compact and the condition defined by $f = 0$ is closed, we obtain a compact Riemann surface C. □

Given an irreducible polynomial $f(x, y)$ and a rational function $X(x, y)$, let $y(x)$ be the algebraic function defined by $f(x, y) = 0$. The expression $X(x, y(x))$ is a rational function of x and $y(x)$ and it corresponds to a rational function defined on the compact Riemann surface C associated to f in Theorem 1.16. **The integral of a rational function of x and the algebraic function defined by** $f(x, y) = 0$

$$\int X(x, y(x))dx \qquad\qquad f(x, y(x)) = 0$$

is the (multivalued) function obtained by integrating a rational 1-form on the Riemann surface C associated to f.

The procedure of computing the integral $\int X(x, y(x))dx$ of an algebraic function (hence finite multivalued) on $\bar{\mathbb{C}}$ can then be replaced by the integral of a rational 1-form on a compact Riemann surface. In this way we have removed the obstruction of integrating a multivalued function and we can now concentrate our attention on the serious part of the multivaluedness of $\int X(x, y(x))dx$.

2 Abel's Theorem

2.1 The Universal Abelian Integral

Let C be a compact and connected Riemann surface of genus g and ω a meromorphic 1-form on C. The 'function' $\int_p^c \omega : C \to \mathbb{C}$ is in general multivalued, and as we have seen in the previous section, this can happen for 2 different types of reasons:

1) ω has a pole with a non-zero residue (Corollary 1.11).

2) Due to the non-trivial topology of C: i.e. The value of the integral $\int_p^c \omega$ depends on the homotopy class of the path from p to c that we use to carry out the integration, and for any closed curve $\alpha \subset C$ with base point in p, the integral $\int_\alpha \omega$ is the difference between 2 branches of $\int_p^c \omega$, differing one from the other by precomposing the path of integration with α.

In order to concentrate our attention on the second kind of multivaluedness, we will only integrate special kinds of 1-forms on C:

Definition *An* **Abelian Differential** ω *on the compact and connected Riemann surface C is a holomorphic 1-form on C. Namely, ω may be written locally as $\omega_j dz_j$, with ω_j a holomorphic function, and we have the compatibility relationship $\omega_j = \frac{dz_k}{dz_j}\omega_k$.*

Theorem 2.1 *The space $H^0(C,\Omega_C)$ of Abelian Differentials on a Riemann surface of genus g has dimension g. The De'Rham 1-cohomology $H^1_{Dr}(C) := \frac{closed\ 1-forms}{exact\ -forms}$ is isomorphic to*

$$H^0(C,\Omega_C) \oplus \bar{H}^0(C,\Omega_C)$$

and there is a perfect pairing induced by integration

$$\int : H_1(C,\mathbb{C}) \times H^1_{Dr}(C) \to \mathbb{C}$$

We will accept this Theorem without proof (see [3]). We will just point out its content. The dimension of the space of Abelian differentials is a computation. The description of the De'Rham cohomology group as holomorphic plus antiholomorphic forms is choosing a canonical representative on each cohomology class. This is a consequence of Hodge Theory. The perfect pairing is a consequence of the duality between (singular) homology and cohomology (Poincaré duality). The last assertion uses the above and the isomorphism between singular cohomology and De'Rham cohomology (Differential Topology).

Example 2.2

We want to analyze the integral

$$\int_p^c \frac{dy}{\sqrt{1-x^4}} dx$$

Let $C' \subset \mathbb{C}^2$ be the curve defined by $f(x,y) = x^4+y^2-1 = 0$. The set of points defined by $f = \frac{\partial f}{\partial y} = 0$ are $B := \{(1,0),(i,0),(-1,0),(-i,0)\}$. Outside of these points C' may be defined as the graph of a function $y(x)$. On B we have $\frac{\partial f}{\partial x} \neq 0$, so that C' is a smooth curve. Changing coordinates $x_1 = \frac{1}{x}$ and $y_1 = \frac{1}{y}$ we obtain an ordinary double point on $\bar{\mathbb{C}}' \subset \bar{C} \times \bar{\mathbb{C}}$. This means that the Riemann surface C constructed in Theorem 1.10 is C' union 2 points (over $x = \infty$).

The projection to the x coordinate gives a holomorphic map $x : C \to \bar{\mathbb{C}}$ which is 2 to 1, being branched on B. We claim that C has genus 1. To see this, we apply the Riemann-Hurwitz formula to the above map x: It is 2 to 1, and has 4 branch points of index 2. Since the genus of $\bar{\mathbb{C}}$ is 0 we have $2 - 2g = 2(2) - 4 = 0$, so $g = 1$.

We claim that $\omega = \frac{dx}{y}$ is a holomorphic differential on C (and hence it is a basis for all holomorphic differentials on C). To see this, observe that dx is a meromorphic differential on $\bar{\mathbb{C}}$ that has at ∞ a pole of order 2 (since $d(\frac{1}{x}) = -\frac{1}{x^2}$). Hence $x^*(dx)$ on C is a meromorphic differential on C that has 2 poles of order 2 (at the points over $x = \infty$) and vanishes at the 4 points B of order 1 (the branch points of x). Now y is a meromorphic function on C that vanishes on B and has also poles of order 2 at the 2 points over $x = \infty$. Hence ω is a non-vanishing holomorphic differential on C.

We can write out generators of the fundamental group of C as follows. Consider the arc of the unit circle in the x-plane parametrized by $e^{2\pi i t}$, $t \in [0, \frac{1}{4}]$, and its 2 liftings to C': $\tau_1 := (e^{2\pi i t}, +\sqrt{1-e^{4\pi i t}})$ and $\tau_2 := (e^{2\pi i t}, -\sqrt{1-e^{4\pi i t}})$. Let α_1 be the loop on C obtained by going first with τ_1 and then returning with τ_2^{-1}. Define α_2 in a similar manner, but with respect to the parameter $t \in [\frac{1}{4}, \frac{1}{2}]$. One checks that this is a basis of $H_1(C, \mathbb{Z})$.

The map $M : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $M(x,y) = (ix, y)$ induces an automorphism of C of order 4, that sends the loop α_1 to α_2 and $M^*\omega = i\omega$. If $\lambda_j := \int_{\alpha_j} \omega$ then by the above we have $\lambda_2 = i\lambda_1$. One can see that $\lambda_j \neq 0$, since otherwise $\int \omega$ would define a holomorphic function on C (uni-valued, since it would be independent of the homotopy class of the curve on which we integrate), but such functions are constant by Proposition 1.15 and so $0 = d(\int \omega) = \omega$, a contradiction.

Denote by $\Gamma = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2 \subset \mathbb{C}$ the additive subgroup generated by λ_1 and $\lambda_2 := i\lambda_1$, which is a discrete subgroup (in the sense of having no accumulation points). The quotient group $\mathbb{C}/\Gamma$ is a compact topological group, which also has the structure of a Riemann surface, with the group structure being holomorphic. Since the integral $\int_p^c \omega$ is well defined modulo the addition of an element of Γ, we obtain a well defined holomorphic map

$$\int_p^c : C \to \mathbb{C}/\Gamma$$

One can show that this map is a biholomorphism. □

A key idea in generalizing the above example to $g > 1$ is to compute the Abelian integrals $\int \omega$ simultaneously for all Abelian differentials.

Let $\omega_1, \ldots, \omega_g$ be a basis for the Abelian differentials on the connected and compact Riemann surface of genus g. Consider the multivalued map

$$\int_p^c (\omega_1, \ldots, \omega_g) : C \to \mathbb{C}^g$$

Let $\alpha_1, \ldots, \alpha_{2g}$ be a canonical basis of generators of $\pi_1(C, p)$ and Γ be the additive subgroup of $\mathbb{C}^g$ defined by

$$\Gamma := \{\sum_{j=1}^{2g} m_j \Lambda_j \ / \ m_j \in \mathbb{Z}\} \quad , \quad \Lambda_j := \int_{\alpha_j} (\omega_1, \ldots, \omega_g)$$

The elements of Γ are the **periods of the Abelian Integrals on** C.

Theorem 2.3 *The additive subgroup $\Gamma \subset \mathbb{C}^g$ is a discrete subgroup, the quotient $\mathbb{C}^g/\Gamma$ is a compact complex manifold of dimension g with an Abelian holomorphic group structure and the map*

$$\int_p^c (\omega_1, \ldots, \omega_g) : C \to \mathbb{C}^g/\Gamma \tag{2.1}$$

is a holomorphic map (univalued).

Proof: It will suffice to show that the $2g$ periods Λ_j of the Abelian integrals are $\mathbb{R}$-linearly independent in $\mathbb{C}^g$. So assume that they satisfy a linear relation:

$$\sum_{j=1}^{2g} r_j \Lambda_j = 0$$

By conjugating we have that for $k = 1, \ldots, g$:

$$\sum_{j=1}^{2g} r_j \int_{\alpha_j} \omega_k = 0 \quad , \quad \sum_{j=1}^{2g} r_j \int_{\alpha_j} \bar{\omega_k} = 0$$

Since by Theorem 2.1 the holomorphic and antiholomorphic 1-forms are a basis of the de'Rham 1-cohomology group and the 1-homology and the de'Rham 1-cohomology form a perfect pairing, we have that the above equalities imply that $\sum_{j=1}^{2g} r_j[\alpha_j] = 0 \in H_1(C, \mathbb{C})$, which implies that $r_j = 0$ since α_j are basis for $H_1(C, \mathbb{Z})$. □

Definition The Jacobian of the compact and connected Riemann surface *C is the compact complex manifold $J(C) := \mathbb{C}^g/\Gamma$ and the map (2.1) is called the* **universal Abelian Integral**

$$\int_p : C \to J(C)$$

As we mentioned in the introduction, the Abelian differential 1-forms on $J(C)$ may be written as $\sum_{j=1}^{g} a_j dz_j$ where a_j are constants (since the only holomorphic functions on $J(C)$ are constants, and the cotangent bundle of $J(C)$ is a direct sum of trivial bundles). Its integral correspond to the multivalued functions on $J(C)$ induced by the linear function $\sum_{j=1}^{g} a_j z_j$ on $\mathbb{C}^g$. These 'constant' differential forms pull back via the universal Abelian integral to the Abelian differentials on C. Hence the integrals of Abelian differentials on C may be decomposed as the composition of an algebraic map and a 'multivalued linear function' on $J(C)$. The fibres of the integral of an Abelian differential

$$\int \sum_{j=1}^{g} a_j dz_j$$

on $J(C)$ corresponds to parallel hyperplanes in $\mathbb{C}^g$ projected to $J(C)$, and may be understood as the interplay of the hyperplane with the periods. We then want to intersect this codimension 1 foliation in $J(C)$ with the image of the universal Abelian integral, which as we mentioned, is an algebraic map. This reduces the problem of understanding the integrals of Abelian differentials to understanding an algebraic map and a linear foliation of codimension 1 in $J(C)$ and how they intersect.

Another image that one has is to consider the 'lift' of the universal Abelian integral to the universal covers

$$\tilde{\int}_{n\bar{p}}^{\tilde{*}} (\omega_1, \ldots, \omega_g) : \tilde{C} \to \mathbb{C}^g$$

The image curve is an analytic curve in $\mathbb{C}^g$, defined by the vanishing of a finite number of entire functions. This map is not injective, since it factors through the 'maximal Abelian cover' of C, which is the covering of C corresponding to the subgroup of commutators of the fundamental group of C. The image curve is actually biholomorphic to the maximal Abelian cover of C.

2.2 Abel's Theorem

Abel discovered in 1826 a remarkable property of Abelian integrals.

Let C be the compact Riemann surface associated to the polynomial $f(x,y)$ as in Theorem 1.10, and let $\omega = X(x,y)dx$ be a rational 1-form defined on C. Let $y_1(x),\ldots,y_n(x)$ be different branches of the algebraic function defined by $f = 0$ on a small open set in $U \subset \bar{\mathbb{C}}$, and consider the meromorphic 1-form on U:

$$[X(x,y_1(x)) + \ldots + X(x,y_n(x))]dx \qquad (2.2)$$

If by analytic continuation of $\{y_j(x)\}$ in $\bar{\mathbb{C}}$ we return back to U we will obtain that the $\{y_j(x)\}$ are reshuffled, but in any case the meromorphic 1-form in (2.2) on U remains the same, since addition in $\mathbb{C}$ is commutative ($\mathbb{C}$ after all is an 'Abelian' group). This is also true at the ramification points of $y_j(x)$. Hence the expression (2.2) gives rise to a meromorphic (i.e. rational) 1-form in $\bar{\mathbb{C}}$, but by example 1.1 we know all about the integrals of such differential forms:

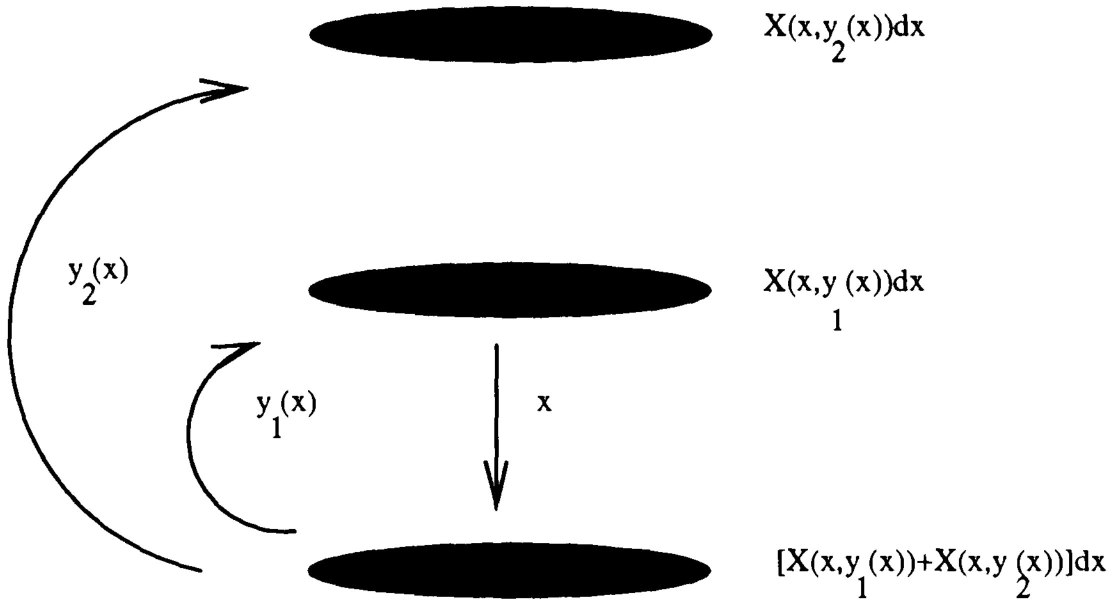
$$\int_p^{(x,y_1(x))} X(x,y)dx + \ldots + \int_p^{(x,y_n(x))} X(x,y)dx = Easy\ Function(x)$$

The most dramatic case is when $X(x,y)dx$ is an Abelian differential, since there are no Abelian differentials in $\bar{\mathbb{C}}$ (it's genus is 0), we obtain:

Abel's Theorem *If $X(x,y)dx$ is an Abelian differential on the compact Riemann surface C associated to the irreducible polynomial $f(x,y)$ of degree n in y, and $y_1(x),\ldots,y_n(x)$ are the branches of the algebraic function defined by f, then*

$$\int_p^{(x,y_1(x))} X(x,y)dx + \ldots + \int_p^{(x,y_n(x))} X(x,y)dx = Constant$$

Figure 9: Sums of local 1-forms



We change slightly our point of view. Let C be a compact and connected Riemann surface and let $f : C \to \mathbb{C}$ be a rational function (or a holomorphic map to $\bar{\mathbb{C}}$). For every $x \in \bar{\mathbb{C}}$ let $\{c_1(x), \ldots, c_n(x)\} = f^{-1}(x)$ be the inverse images of x under f, counting multiplicities. Since we do not have a preferred order for these n-points of C, we use the additive notation $c_1(x) + \ldots + c_n(x)$, where with this notation we are just meaning n-points of C with no preferred order.

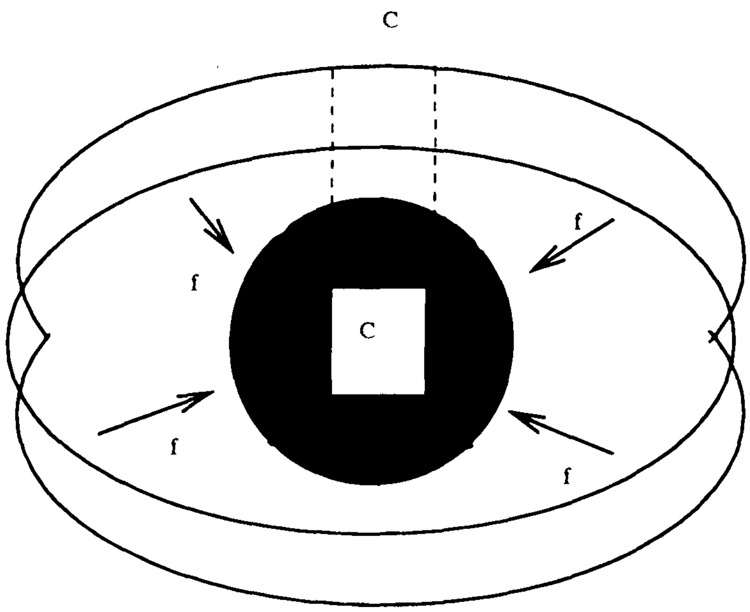
Abel's Theorem: *Let ω be an Abelian differential on the compact and connected Riemann surface and f a non-constant rational function on C, with $f^{-1}(x) = c_1(x) + \ldots + c_n(x)$, then*

$$\int_p^{c_1(x)} \omega + \ldots + \int_p^{c_n(x)} \omega = Constant$$

Let's introduce a better language to express Abel's Theorem. For $n \geq 1$ we can define a function

$$\int_{np} (\omega_1, \ldots, \omega_g) : C \times \ldots \times C \to J(C)$$

Figure 10: An illustration of Abelian sums



$$(c_1, \ldots, c_n) \mapsto \int_p^{c_1} (\omega_1, \ldots, \omega_g) + \ldots \int_p^{c_n} (\omega_1, \ldots, \omega_g)$$

Actually, if $i : X \to G$ is a function from a set X to an abelian group G, we can define a function $i^n : X \times \ldots \times X \to G$ by $i^n(x_1, \ldots, x_n) = i(x_1) + \ldots + i(x_n)$ by using the group structure in the image. If we denote by Sym_n the group of permutations on n letters, the map i^n is invariant under the action of Sym_n, and hence gives rise to a map defined on the orbit space $i^{(n)} : \frac{X^n}{Sym_n} \to G$. The set $\frac{X^n}{Sym_n}$ is called the symmetric product of X and is denoted by $X^{(n)}$.

For our case of interest, let $C^{(n)}$ be the n^{th}-**symmetric product of the Riemann surface** C.

Lemma 2.4 *Let C be a compact and connected Riemann surface and f a non-constant rational function on C then:*

1) $C^{(n)}$ is a compact complex manifold of dimension n,

2) There is an integer n (called the degree of the rational function f) and a holomorphic map $f^{-1} : \bar{\mathbb{C}} \to C^{(n)}$ defined by $x \in \bar{\mathbb{C}} \to f^{-1}(x)$.

Proof: : 1) We will begin by observing that $\mathbb{C}^{(n)}$ is biholomorphic to the projective space $\mathbb{P}^n$. This is true, since given a polynomial in t with complex coefficients of degree less than or equal to n we can associate to it its roots (with multiplicities), that will be an 'unordered' set of complex numbers. If the degree is less than n, we just complete putting ∞ with multiplicity so that the total number is n. Such polynomials have $n+1$ complex coefficients and they determine the same roots if and only if they differ by a non-zero constant. In all, we see that $\bar{\mathbb{C}}^{(n)}$ may be identified with the projective space of polynomials of degree less than or equal to n. Taking coordinate charts of the Riemann surface C, we can use the above computation locally to conclude that $C^{(n)}$ is a complex manifold of dimension n.

2) It follows from Theorem 1.14 that rational functions are holomorphic maps to the Riemann sphere $\bar{\mathbb{C}}$. In the proof of the Riemann-Hurwitz formula we saw that a holomorphic map may be interpreted as a branched covering, with a finite number of sheets n. Outside of the critical values $f(B)$, one obtains n holomorphic functions $c_1(x), \ldots, c_n(x)$ describing the different branches. So we have a holomorphic map $c_1(x)+\ldots+c_n(x)$. From the description in Proposition 1.12 of the behaviour of f at the branch points (i.e. coordinates were it takes the form w^k) we see that it extends to a holomorphic function at the critical values of f. □

The n^{th} Abelian map is the holomorphic map:

$$\int_{np} : C^{(n)} \to Jac(C)$$

defined by

$$\int_{np}(c_1 + \ldots + c_n) = \int_p^{c_1} (\omega_1, \ldots, \omega_g) + \ldots + \int_p^{c_n} (\omega_1, \ldots, \omega_g)$$

where as before $\omega_1, \ldots, \omega_g$ is a basis of the Abelian differentials.

Abel's Theorem *Let C be a compact and connected Riemann surface of genus g, f a non-constant rational function of C then the map*

$$\int_{np} \circ f^{-1} : \bar{\mathbb{C}} \to C^{(n)} \to Jac(C) \tag{2.3}$$

is a constant map.

Proof: The 1-forms $\sum_{j=1}^{g} a_j dz_j$ on $\mathbb{C}^g$ descend to holomorphic 1-forms on $Jac(C)$ for any value of the constants a_j. Pull back these holomorphic 1-forms to $\bar{\mathbb{C}}$ via the map (2.3). In local coordinates we may parametrize C by $(h_1(w), \ldots, h_g(w))$, to obtain holomorphic 1-forms $\sum_{j=1}^{g} a_j \frac{\partial h_j}{\partial w} dw$ on $\bar{\mathbb{C}}$. But there are no non-zero holomorphic 1-forms on $\bar{\mathbb{C}}$. Since this is true for any values of a_j, we obtain that $\frac{\partial h_j}{\partial w} = 0$, and hence h is constant. □

It follows also from the above proof of Abel's Theorem that if we have a holomorphic map from a projective space $\mathbb{P}^r \to C^{(n)}$ then it is also transformed by the n^{th} Abelian map $\int_{np}$ to a constant: $\mathbb{P}^r$ is full of lines, since through any pair of points there is a line and lines are sent to a point. So $\mathbb{P}^r$ is contracted to a point by the Abelian map $\int_{np}$. In the next section we will organize in a more efficient way the rational functions on C, we will find some projective spaces in $C^{(n)}$ and we will be able to describe the fibers $\int_{np}^{-1}$ of the n^{th} Abelian map.

2.3 Meromorphic Functions on Compact Riemann Surfaces

Let C be a compact and connected Riemann surface. We will call the points of the symmetric product $C^{(n)}$ **positive divisors of degree n**, and we will write them as $D = n_1p_1 + \ldots + n_rp_r$, with $n_j \in \mathbb{Z}_+$ and $p_j \in C$, with $n_1 + \ldots + n_r = n$ to reflect the multiplicity of the point p_j in the divisor D (i.e. the number of times that the point appears repeated in the divisor D). More generally, the **group of divisor** $Div(C)$ **on** C is the free abelian group generated by the points of C and its elements, called **divisors**, are expressions of the form $D = n_1p_1 + \ldots + n_rp_r$, with $n_j \in \mathbb{Z}$ and $p_j \in C$, where $n_1 + \ldots + n_r$ is called the **degree of the divisor**.

Let f be a rational function on C of degree n (recall from Lemma 2.4 that n is the number of sheets of the associated holomorphic map $C \to \bar{\mathbb{C}}$). The **divisor of zeroes** $(f)_0$ of f is the positive divisor $f^{-1}(0)$, the **divisor of poles** $(f)_\infty$ of f is the positive divisor $f^{-1}(\infty)$ and the **divisor** of f is $(f) = (f)_0 - (f)_\infty$. Divisors of rational functions on C are called **principal divisors.**

We say that the positive divisors $D_1 = m_1 p_1 + \ldots + m_r p_r$ and $D_2 = n_1 q_1 + \ldots + n_s q_s$ are **linearly equivalent** if and only if there is a rational function f on C and a positive divisor D such that

$$(f)_0 + D = m_1 p_1 + \ldots + m_r p_r \qquad (f)_\infty + D = n_1 q_1 + \ldots + n_s q_s$$

That is, two positive divisors are equivalent if after cancelling the common factors there is a rational function on C that has poles in one and zeroes on the other. Or better, if $D_1 - D_2 = (f)$ is a principal divisor (i.e. we let the cancelling be automatic in $Div(C)$).

Example 2.5

All divisors of degree n on the Riemann sphere are linearly equivalent, since given $D_1 = m_1 p_1 + \ldots + m_r p_r$ and $D_2 = n_1 q_1 + \ldots + n_s q_s$ two positive divisors of $\bar{\mathbb{C}}$ of degree n we can construct the rational function

$$\frac{\Pi_{p_j \neq \infty}(t - p_j)^{m_j}}{\Pi_{q_j \neq \infty}(t - q_j)^{n_j}}$$

on $\bar{\mathbb{C}}$ that vanishes on D_1 and has poles on D_2. We had already observed in the proof of Lemma 2.4 that $\bar{\mathbb{C}}^{(n)} = \mathbb{P}^n$ and that there are no Abelian differentials on $\bar{\mathbb{C}}$, so that the n^{th} Abelian map degenerates to a constant map. So its fiber is all of $\bar{\mathbb{C}}^{(n)}$: That is, the fibers of the n^{th} Abelian map are the equivalence classes of divisors (one class in this case) and this fiber is a projective space $\mathbb{P}^n$.

Example 2.6

Let C be now a Riemann surface of genus $g > 1$, and we want to analyze when two divisors of degree 1 are linearly equivalent. Let $p, q \in C$, $p \neq q$ be two distinct points in C and assume that they are linearly equivalent. Hence there is a rational function on C that has zeroes only at p with multiplicity 1 and poles only at q with multiplicity 1. Interpreting f as a holomorphic map $f : C \to \bar{\mathbb{C}}$ to the Riemann sphere, then it has degree 1. Applying Lemma 2.4 we see that f is injective, and also surjective; hence a biholomorphism, which contradicts the hypothesis that the genus $g > 0$. Hence, on Riemann surfaces of $g > 0$ divisors of degree one are not linearly equivalent (except to itself).

Example 2.7

Let now C be a curve of genus $g > 0$ and assume that $D_1 = p_1 + p_2$ is linearly equivalent to $D_2 = q_1 + q_2$. By Example 2.6, we have that $p_j \neq q_k$, and let f be the rational function on C that vanishes on D_1 and has poles on D_2. Viewing $f : C \to \bar{\mathbb{C}}$ as a holomorphic map to the Riemann sphere of degree 2, we can apply the Riemann-Hurwitz formula to obtain

$$2 - 2g = 2(2 - 0) - \sum(\nu(q, f) - 1) = 4 - d$$

where f has $d = 2g + 2$ branch points of order 2. Let $\{\alpha_j\}$ be the critical values of f (i.e the images of the branch points) and assume for simplicity that they are all finite (otherwise, postcompose with a holomorphic automorphism of $\bar{\mathbb{C}}$ moving ∞ to a finite value). One checks that C is biholomorphic to the Riemann surface associated to the polynomial

$$F(x, y) = y^2 - \Pi_{j=0}^{2g+2}(x - \alpha_j)$$

So, there are Riemann surfaces of arbitrary genus $g > 0$ such that the 2^{nd} Abelian map $\int_{2p} : C^{(2)} \to Jac(C)$ defined on the complex 2-dimensional manifold $C^{(2)}$ contracts a line. These Riemann surfaces are called **hyperelliptic**, and it is more difficult to see that only a 'few' Riemann surfaces of $g \geq 3$ are hyperelliptic. Hence, the nature of the Abelian maps can depend on the complex structure of the Riemann Surface C. For non-hyperelliptic Riemann surfaces the 2^{nd} Abelian map is injective and $C^{(2)}$ does not have any copy of $\bar{\mathbb{C}}$ (even though the topological structure of $C^{(2)}$ is independent of the complex structure on C).

Given a positive divisor $D = m_1 p_1 + \ldots + m_r p_r$ on the compact and connected Riemann surface C define $Rat(D)$ as the set of rational functions on c that have poles at most on the points $p_1, \ldots, p_r$ with order at p_j at most m_j: $\nu(f, p_j) \leq m_j$.

Proposition 2.8 *1) Linear Equivalence of positive divisors of degree n is an equivalence relation on $C^{(n)}$.*

2) For a positive divisor D of degree n, $Rat(D)$ is a vector space of dimension at most $n+1$.

3) For a positive divisor D of degree n, there is a holomorphic map from the projective space of lines in $Rat(D)$ to $C^{(n)}$

$$Proj(Rat(D)) \to C^{(n)} \qquad f \to (f)_0 - (f)_\infty + D$$

whose image is the set $|D|$ of positive divisors linearly equivalent to D.

Proof: 1) D is linearly equivalent to itself through a constant function, and if f reflects the linear equivalence between D and E, then the rational function $\frac{1}{f}$ reflects the equivalence between E and D.

Assume that D_1 is linearly equivalent to D_2 and that D_2 is linearly equivalent to D_3. Let this be expressed by

$$D_1 - D_2 = (f) \qquad D_2 - D_3 = (g)$$

for suitable rational functions f and g on C. Let F_1 be the positive divisor representing the zeroes of (f) which are also poles of (g), and F_2 the positive divisor representing the zeroes of (g) which are also poles of (f), then

$$\begin{aligned}(fg)_0 &= (f)_0 + (g)_0 - (F_1 + F_2)\\ (fg)_\infty &= (f)_\infty + (g)_\infty - (F_1 + F_2)\\ (fg) &= (f) + (g)\end{aligned}$$

Hence

$$D_1 - D_3 = (D_1 - D_2) + (D_2 - D_3) = (f) + (g) = (fg)$$

2) Let $D = \sum_{j=1}^{r} m_j p_j$ and $V(D)$ be the vector space of dimension n formed by the partial Laurent expansions:

$$V(D) = \oplus_{j=1}^{r}[\oplus_{k_j=1}^{m_j}[\frac{\mathbb{C}}{(z_j - p_j)^{k_j}}]],$$

where z_j are local variables around p_j. Consider the map which associates to a rational function in D the negative terms of its Laurent series expansion at its poles:

$$\rho : Rat(D) \to V(D)$$

An element in the kernel of ρ is a rational function with poles at most on D, but such that its Laurent series expansion at these points is actually a power series expansion. That is, a holomorphic function on C. But by Proposition 1.15 it is a constant function. Hence, by linear Algebra, the dimension of $Rat(D)$ is at most $n + 1$.

3) Let $f_1, \ldots, f_s$ be a basis for $Rat(D)$ and consider the holomorphic function

$$F : [\mathbb{C}^s - \{0\}] \times [C - \cup_{j=1}^r \{p_j\}] \to \mathbb{C}, \quad F(t_1, \ldots, t_s, z) = \sum_{j=1}^{s} t_j f(z)$$

and let Z be the algebraic subvariety of $[\mathbb{C}^s - \{0\}] \times [C - \cup j = 1^r \{p_j\}]$ defined by $F = 0$. The projection to the first factor $[\mathbb{C}^s - \{0\}]$ is a finite map, so it will have a behaviour of a 'branched covering'. Using local coordinates and properties of the behaviour of the roots of polynomials with respect to parameters (see [3]) one obtains local holomorphic maps to $C^{(m)}$. To understand the behaviour of Z near the points of D, one considers coordinates around p_j. The family of maps F has a 'general' order of a pole at a point p_j and this drops precisely when there is a point of Z approaching. Analyzing this situation carefully gives the result. (Actually it is easier to prove this part using holomorphic sections of a line bundle by cancelling the poles by multiplying by a holomorphic section of the line bundle. The result will then follow since the similar variety Z will be proper and finite over $\mathbb{C}^s - \{0\}$). Since the holomorphic map to $C^{(n)}$ is independent of multiplying by a non-zero constant, we obtain that it actually is defined on $Proj(Rat(D))$. □.

Our previous observations then gives the final form of:

Abel's Theorem *Let D be a positive divisor of degree n, and let $|D|$ be the projective space of divisors linearly equivalent to D in $C^{(n)}$, then the n^{th} Abelian map $\int_{np}$ sends $|D|$ to a point.*

A proof of the converse of Abel's Theorem may be found in [3] p.235. It asserts that the fibers of the n^{th} Abelian map $\int_{np}$ are the different complete linear series.

3 The Riemann-Kempf Singularity Theorem

3.1 The Canonical Curve

Let C be a compact and connected Riemann surface of genus $g \geq 2$ and let $\omega_1, \ldots, \omega_g$ be a basis of the Abelian differentials on C. The rational functions obtained by dividing the Abelian differentials with ω_1 defines a holomorphic map

$$(\frac{\omega_2}{\omega_1}, \ldots, \frac{\omega_g}{\omega_1}) : C - \{\omega_1 = 0\} \to \mathbb{C}^{g-1}$$

This map extends to a holomorphic map from C to projective space $\mathbb{P}^{g-1}$. In local coordinates (U_α, z_α) the differentials may be written as $\omega_j = A_j^\alpha(z_\alpha)dz_\alpha$ and in the intersection of coordinates we obtain the relations $A_j^\beta = A_j^\alpha \frac{dz_\alpha}{dz_\beta}$. Hence, if we define maps to projective space

$$U_\alpha \to \mathbb{P}^{g-1} \qquad z_\alpha \to (A_1^\alpha(z_\alpha) : \ldots : A_g^\alpha(z_\alpha))$$

then they coincide in the intersection. The map obtained by gluing these local maps

$$i_C : C \to \mathbb{P}^{g-1}$$

is called the **canonical map of** C and the image curve is called the **canonical curve**.

The geometry of the canonical curve $i_C(C)$ contains many secrets of the (rational) function theory of the Riemann surface C. We begin to describe **the linear geometry of the canonical curve**. The canonical map is injective if C is not hyperelliptic, and we identify C with its image $i_C(C)$. The canonical curve C is not contained in any hyperplane in $\mathbb{P}^{g-1}$, since $\omega_1, \ldots, \omega_g$ is a basis of the space of Abelian differentials on C. Let H be a hyperplane in $\mathbb{P}^{g-1}$. The intersection of H with the canonical curve is a divisor on C of degree $2g-2$. The divisors that one obtains by intersecting with all hyperplanes is the complete linear series called the **canonical linear series** $|\kappa|$ and it correspond to the divisors of degree $2g-2$ that are zero-sets of Abelian differentials on C.

Let $D = p_1 + \ldots + p_n$ be a positive divisor in C with n distinct points. The **span** $< D >$ of D is the smallest linear subspace of $\mathbb{P}^{g-1}$ that contains the points $p_1, \ldots, p_n$. If the points are independent, then the dimension of the span is $n-1$, otherwise it is $n-1-r$ for some positive integer r. r measures the number of independent linear relations of the points $\{p_i\}$.

Geometric Riemann-Roch Theorem *If C is a non-hyperelliptic curve and $D = p_1 + \ldots + p_n$ is a positive divisor whose span $< D >$ has dimension $n-1-r$, then the complete linear system $|D|$ has dimension r.*

For a proof see [3], p.248.

Corollary 3.1 *If C is a non-hyperelliptic curve and $D = p_1 + \ldots + p_n$ is a positive divisor then:*

1) If the span $< D >$ of D is $\mathbb{P}^{g-1}$, then the dimension of the complete linear series $|D|$ is $n-g$.

2) If the span $< D >$ of D has dimension less than $g-1$, let E be a positive divisor with $D + E \in |\kappa|$, then

$$dim(|D|) = codim(< E >, \mathbb{P}^{g-1}) - 1$$

3) If $D' \in |D|$ then $dim < D > = dim < D' >$.

Proof: 1) If D spans all $\mathbb{P}^{g-1}$ (necessarily $n \geq g$), the Riemann-Roch Theorem says exactly that the complete linear system $|D|$ has dimension $n-g$.

2) If on the other hand the span of D is not all $\mathbb{P}^{g-1}$ let H be a hyperplane containing D. The intersection of H with the curve C is $C \cap H = D + E$, where E is a positive divisor of degree $2g - 2 - n$. We have to prove that the span of E has codimension $r + 1$, where $r = \dim|D|$. To see this, let H' be a hyperplane containing $< E >$, then. $H' \cap C = D' + E$. By construction, $D' \in |D|$. Conversely, given an element $D' \in |D|$ then $E + D' \in |\kappa|$ and so there is a hyperplane containing E that intersects C on $D' + E$. So there is a one to one correspondence between hyperplanes containing E and elements in $|D|$.

3) Apply 2). □

As 3) shows, if the span of D has dimension $n - 1 - r$, then for any other divisor $D' \in |D|$ we will also have that the span $< D' >$ has dimension $n - 1 - r$. In this manner, we obtain an r-dimensional family of $n-1-r$-linear subspaces of $\mathbb{P}^{g-1}$. The union of these linear subspaces

$$Cone(|D|) = \cup_{D' \in |D|} < D' >$$

gives rise to a variety of $\mathbb{P}^{g-1}$ that can be shown to have dimension $n - 1$ and that we will call the cone of $|D|$.

3.2 The Infinitesimal Geometry of the Abelian maps

Now we turn to show that the canonical curve is related to the infinitesimal geometry of the universal Abelian integral.

Lemma 3.2 *The derivative of the universal Abelian Integral* $\int_p$: $C \to Jac(C)$ *is the canonical map* i_C.

Proof: Recall first of all that we have chosen (and fixed) a basis $\omega_1, \ldots, \omega_g$ of the Abelian differentials and that $Jac(C) = \mathbb{C}^g/\Lambda$ is a compact Abelian group. This implies that the tangent space of $Jac(C)$ at any point p can be identified with the tangent space $T_0 Jac(C)$ at 0 via the derivative of the translation in $Jac(C)$ that sends p to 0. That is, the natural identification that one has of the tangent vector in $\mathbb{C}^g$ descends to give canonical isomorphisms of tangent spaces in $Jac(C)$.

Using local coordinates, then

$$\frac{d}{dz}\int_p^z (A_1(z), \ldots, A_g(z))dz|_{z=z_0} = (A_1(z_0), \ldots, A_g(z_0))$$

The tangent to the curve $\int_p(C)$ at the point $\int_p^c$ can be identified (after translating it to the origin in $Jac(C)$ and taking the point in $Proj(T_0 Jac(C))$ which it represents) with $i_C(c)$. □

We can generalize this computation from the universal Abelian integral to any Abelian map $\int_{np} : C^{(n)} \longrightarrow Jac(C)$

Lemma 3.3 *The derivative of the Abelian map* $\int_{np} : C^{(n)} \longrightarrow Jac(C)$ *at the point* $D = p_1 + \ldots + p_n$ *is a linear map* $T_D C^{(n)} \longrightarrow T_{\int_{np} D} Jac(C)$ *whose image may be identified to the linear span* $< D >$ *of* D *in* $\mathbb{P}^{g-1} = Proj T_{\int_{np} D} Jac(C)$.

Proof: Assuming for simplicity that the points are distinct, then the derivative we want to calculate coincides with the derivative of the map $\int_{np} : C^n \longrightarrow Jac(C)$ defined on the product of C, instead of the symmetric product, at D. Here, we can take partial derivatives with respect to each factor of C. By Lemma 3.2 each of this partial derivatives corresponds to a point $i_C(p_i)$, and hence the image of the linear map correspond to the span of D. □

3.3 The Riemann-Kempf Singularity Theorem

Define

$$\mathcal{W}_n = \int_{np}(C^{(n)}) \subset Jac(C)$$

as the image variety of the n^{th} Abelian map. It is a compact algebraic variety, possibly with singularities. $\mathcal{W}_n$ has in the structure of its singularities information about the function theory of the curve C:

Riemann-Kempf Singularity Theorem *Let C be a compact Riemann surface of genus $g \geq 2$. For $|D|$ a linear system of degree n and dimension r, the tangent cone to $\mathcal{W}_n$ at $\int_{np}(D)$ is the previously defined cone:*

$$T_{\int_{np}(D)}\mathcal{W}_n = Cone(|D|) = \cup_{D' \in |D|} < D' > .$$

It has degree $\binom{g-n+r}{r}$, *and is swept out once by the planes $|D'|$.*

There are 2 kinds of proofs:

1) Analytic ([1,5]): Lemma 3.3 gives a description of the the derivative of the n^{th} Abelian map, and one continues with this idea.

2) Algebraic: Uses Grothendieck's variational machinery. It is described in full in [4] and we will finish these notes by giving a brief description of its method of proof.

The role of the Jacobian $J(C)$ is taken by the Picard varieties $Pic_n(C)$ which parametrize classes of invertible sheaves on C of degree n (see [2] or [3] for assumed background in what follows). There is a map

$$\int : C^{(n)} \rightarrow Pic_n(C) \tag{3.1}$$

which associates to each divisor D the invertible sheaf $\mathcal{O}_C(D)$ of meromorphic functions with poles bounded by D. The image varieties again are denoted by $\mathcal{W}_n$. The fibres of the map (3.1) over $\mathcal{O}_C(D)$ may be identified with the projective space of lines in the space of global sections $H^0(C, \mathcal{O}_C(D))$ of $\mathcal{O}_C(D)$. Hence, the understanding of how the space of global sections $H^0(C, \xi)$ of the invertible sheaf ξ vary with $\xi \in Pic_n(C)$ is related to the problem of the structure of the varieties $\mathcal{W}_n$.

The main reduction is to convert the problem to a problem of 'matrices with parameters'. That is, to show that given $\xi_0 \in Pic_n(C)$ there is an affine neighborhood U of ξ_0 in $Pic_n(C)$ and an $s \times t$ matrix $A = (A_{i,j})$ with coefficients in the ring of regular functions $H^0(U, \mathcal{O}_U)$ such that for every $\xi \in U$ we have

$$dim H^0(C, \xi) = dim Ker A(\xi)$$

$$dim H^1(C, \xi) = dim Coker A(\xi)$$

This in particular shows that the varieties $\mathcal{W}_n$ are determinental varieties, since the condition defining a point in $\mathcal{W}_n$ is that the rank of $A(\xi)$ is one less than maximal, which may be written as the vanishing of all the maximal minors of $A(\xi)$.

The case $n = g - 1$ is the simplest, in the sense that $\mathcal{W}_{g-1}$ has codimension 1, and so is defined by the vanishing of a single function. The above reduction says in this case that A is an $s \times s$ matrix and $det(A)$ is the equation defining $\mathcal{W}_{g-1}$. One can actually find an A such that $s = dim H^0(C, \xi_0)$ and $A(\xi_0) = 0$. Expanding A in terms of regular parameters at ξ_0 we have that the entries of the matrix A begin at least with linear terms, and hence the determinant will begin with terms of degree at least s.

If $s = 1$ then one has to show that A, which is now just 1 function, has a non-trivial linear term in its expansion around ξ_0. The linear terms of A at ξ_0

$$A'(\xi_0) : Tan_{\xi_0}(Pic_{g-1}C) = H^1(C, \mathcal{O}_C) \to Hom_{\mathbb{C}}(H^0(C, \xi_0), H^1(C, \xi_0)) \tag{3.2}$$

can be interpreted as the first order variation of the connected pair of functors (H^0, H^1) ([4] p.81). (3.2) is seen to be the cup product. Now one uses:

*) For any non-zero section $\sigma \in H^0(C, \xi)$, the cup product

$$\sigma \cup \; : H^1(C, \mathcal{O}_C) \to H^1(C, \xi)$$

is surjective.

This is enough to show that the 'first order variation' of A is non-zero, and hence $\mathcal{W}_{g-1}$ is smooth at ξ. And conversely, at any smooth point we must have that $s = dimH^0(C,\xi) = 1$, since we just saw that the degree is at least s. This proves that the non-singular points of $\mathcal{W}_{g-1}$ correspond exactly to invertible sheaves with global sections of dimension 1.

For $s > 1$ one has to show that for the corresponding map (3.2) the subset

$$\{v \in Tan_{\xi_0} Pic_{g-1} C \quad | \quad A'(0)v : H^0(C,\xi_0) \to H^1(C,\xi_0) \text{ has nontrivial kernel}\}$$

is not the entire tangent space. This again is carried out using *) above. This shows that $\mathcal{W}_{g-1}$ is defined at ξ_0 by a function whose first non-zero term has degree exactly $s = dimH^0(C,\xi_0)$. This sketches a proof of Riemann's Theorem.

Homological Algebra provide methods to compute 'intrinsic derivatives' using higher order operations, as is shown in (3.2). Note that by using Serre-duality, (3.2) is equivalent to the multiplication map:

$$H^0(C,\xi_0) \otimes H^0(C, \kappa \otimes \xi_0^*) \to H^0(C,\kappa)$$

One of the main advantages of this method, is that it is valid independently if the points of a divisor D are distinct or not.

We finish by writing Riemann's Theta function. Let $\delta_1, \ldots, \delta_{2g}$ be normalized generators of the fundamental group and choose a basis $\omega_1, \ldots, \omega_g$ of the Abelian differentials such that $\int_{\delta_i} \omega_j = \delta_{i,j}$ for $1 \leq$

$i, j \le g$. If Z denotes $\int_{\delta_i} \omega_j = \delta_{i,j}$ for $g+1 \le j \le 2g$ and $1 \le j \le g$, then Z is a symmetric matrix with positive definite imaginary part ([3], 232). The Theta function is the function on $\mathbb{C}^g$ defined by the convergent power series

$$\Theta(w) = \sum_{n \in \mathbb{Z}^g} e^{\pi i < n, Zn>} e^{2\pi i < n, w>} \tag{3.3}$$

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