MacPherson's Graph Construction

Michał Kwieciński

Uniwersytet Jagielloński,
Instytut Matematyki,
ul. Reymonta 4,
30-059 Kraków, Poland

e-mail: kwiecins@im.uj.edu.pl

1 Definition of the Graph Construction.

MacPherson's Grassmannian graph construction is a tool for studying the singularities of a morphism of vector bundles over an algebraic variety. The aim of this lecture is to discuss selected topics concerning this construction and its applications, nb. results contained in the author's thesis [10] (where detailed proofs can be found). We will focus on the definition of the graph construction, its equivalent description using Fitting ideals and some applications to characteristic classes. We will also relate it to the Nash transform using a "modified fibred product".

Consider a morphism $d : E_2 \to E_1$ of algebraic vector bundles over a complex algebraic variety M. Locally d can be represented by a matrix whose entries are regular functions on M. If, at a point of M, the rank of that matrix is smaller than its generic value, we say that that point is in the singular locus of that morphism. Morphisms with non empty singular locus often occur in the study of characteristic classes of singular varieties and of coherent sheaves. The following construction ([11],[1],[5]), has been introduced by MacPherson. Its avatars appear already in [12],[13].

Let e_2 be the rank of E_2 and let us write $G = \mathrm{Grass}_{e_2}(E_2 \oplus E_1)$ for the Grassmann bundle of e_2-dimensional subspaces of the vector bundle $E_2 \oplus E_1$. Consider the embedding ($\mathbf{C}$ is the complex line)

$$\Gamma : M \times \mathbf{C} \to G \times \mathbf{P}^1\mathbf{C},$$

$$\Gamma(x,\lambda) = (graph(\lambda d_x) , (1:\lambda)).$$

Let W be the closure of $\Gamma(M \times \mathbf{C})$. Now W is an algebraic variety, which comes equipped with two natural projections : $\pi : W \to M$, induced by that of the bundle G and $p : W \to \mathbf{P}^1\mathbf{C}$, the restriction of the projection of $G \times \mathbf{P}^1\mathbf{C}$ on its second factor. The scheme $Z_\infty = p^{-1}((0:1)) \subset W$ is canonically embedded in G, by

$$Z_\infty \hookrightarrow G \times \{(0:1)\} \cong G$$

and the restriction of π to Z_∞ coincides with the restriction of the natural projection $G \to M$. The pullback of the tautological bundle on G by the projection $\eta : G \times \mathbf{P}^1\mathbf{C} \to G$ is a bundle on $G \times \mathbf{P}^1\mathbf{C}$ whose restriction to W is denoted Ξ. Set $\xi = \Xi|_{Z_\infty}$.

Definition 1 *The collection of space, maps and bundle (W, π, p, Ξ) will be called the graph construction for the morphism d. The triple $(Z_\infty, \pi|_{Z_\infty}, \xi)$ will be called the scheme at infinity for the morphism d.*

The idea is that the graph construction deforms $M = p^{-1}((1:0))$ to Z_∞, also improving the behaviour of the bundles. We will mainly be interested not in the Z_∞ scheme itself, but in the cycle it induces in the Chow group of M. (See [5], for a description of how a scheme gives rise to a cycle. Its irreducible components are given multiplicities which reflect how "non reduced" they are.)

The data of the graph construction are summarized by the following diagramme.

$$\begin{array}{ccccc} \xi & & & & \Xi \\ \downarrow & & & & \downarrow \\ Z_\infty & & \hookrightarrow & & W \\ \downarrow & & & & \downarrow \\ M & \cong & M \times \{(0:1)\} & \hookrightarrow & M \times \mathbf{P}^1\mathbf{C} \end{array}$$

First, what does the graph construction do to morphisms which do not have any singularity ? This is easily answered (see the proof of lemma 1.1 in Chapter I of [1]). If $d : E_2 \to E_1$ is a morphism of vector bundles, of constant rank, then :

- The embedding $\Gamma : M \times \mathbf{C} \to G \times \mathbf{P}^1\mathbf{C}$ extends to an isomorphism
$$\overline{\Gamma} : M \times \mathbf{P}^1\mathbf{C} \xrightarrow{\cong} W .$$
- In particular, the restriction $\overline{\Gamma}|_{M\times\{(0:1)\}}$ gives an isomorphism
$$M \times \{(0:1)\} \xrightarrow{\cong} Z_\infty .$$
- Identifying M and $M \times \{(0:1)\}$ one gets
$$(\overline{\Gamma}|_{M\times\{(0:1)\}})^*\xi \cong (\ker d \oplus \operatorname{Im} d).$$

Before examining an example of a singular morphism of vector bundles, we need to be able to calculate the graph construction more explicitly. The following section shows how to do this.

2 Fitting Ideals.

It turns out that the graph construction for a morphism of vector bundles is isomorphic to a blowup of an ideal constructed from the Fitting ideals of the cokernel sheaf of the morphism. This ideal was introduced independently in [16] and [10].

Let $\mathcal{G}$ be a coherent sheaf on M. Its j-th Fitting ideal - $\mathrm{Fitt}_j(\mathcal{G})$ (see [3]) is a coherent sheaf of ideals in $\mathcal{O}_M$, defined locally using a finite presentation ϕ of the sheaf $\mathcal{G}$ on an open subset U of M :

$$\mathcal{O}_U^q \xrightarrow{\phi} \mathcal{O}_U^p \to \mathcal{G}|_U \to 0 .$$

For $j \geq p$ one defines $\mathrm{Fitt}_j(\mathcal{G})|_U = \mathcal{O}_U$ and if $q < p$ then for $0 \leq j < p-q$, on sets $\mathrm{Fitt}_j(\mathcal{G})|_U = (0)$. For $\max\{0, p-q\} \leq j < p$, $\mathrm{Fitt}_j(\mathcal{G})|_U$ is defined as the sheaf of ideals generated by the $(p-j)$-minors of the matrix of ϕ in some chosen bases of the two free modules $\mathcal{O}_U^q$ and $\mathcal{O}_U^p$. It is well known that this definition is independent of the choice of the finite presentation and of the bases.

Let us now consider the variety $M \times \mathbf{P}^1\mathbf{C}$. We will define an ideal $\mathcal{J}(\mathcal{G}) \subset \mathcal{O}_{M\times\mathbf{P}^1\mathbf{C}}$, using the Fitting ideals of $\mathcal{G}$. Let $(\mu : \nu)$ be homogeneous coordinates on $\mathbf{P}^1\mathbf{C}$. Denote $U_0 = \mathbf{P}^1\mathbf{C} - \{(0 : 1)\}$ and $U_\infty = \mathbf{P}^1\mathbf{C} - \{(1 : 0)\}$.

Definition 2 *Let $\mathcal{G}$ be a coherent sheaf on M. The ideal $\mathcal{J}(\mathcal{G}) \subset \mathcal{O}_{M\times\mathbf{P}^1\mathbf{C}}$ is defined by*

$$\begin{aligned} \mathcal{J}(\mathcal{G})|_{M\times U_0} &= \mathcal{O}_{M\times U_0} \\ \mathcal{J}(\mathcal{G})|_{M\times U_\infty} &= \sum_{j=0}^{\infty} \left(\left(\frac{\mu}{\nu}\right)^j\right) \mathrm{Fitt}_{j+\mathrm{rk}\,\mathcal{G}}(\mathcal{G})\mathcal{O}_{M\times U_\infty} \end{aligned}$$

The sum above is just the ideal of sums of elements of the summand ideals. Remark that in a neighbourhood of $x \in M$, the Fitting ideals of index greater than $\dim_{\mathbf{C}}(\mathcal{G}_x/\mathrm{m}_x\mathcal{G}_x) - 1$ are trivial. One deduces, that the sum defining $\mathcal{J}(\mathcal{G})$ is in fact locally finite and that the two expressions of the above definition coincide on $M \times (U_0 \cap U_\infty)$.

Theorem 3 *Let $d : E_2 \to E_1$ be a morphism of vector bundles on M, $\mathcal{G} =$ CoKer d - the cokernel sheaf of this morphism. Then the graph construction W for the morphism d is canonically isomorphic to the blowup of $M \times \mathbf{P}^1\mathbf{C}$ in the ideal $\mathcal{J}(\mathcal{G})$ defined above :*

$$W \cong \mathrm{Bl}_{\mathcal{J}(\mathcal{G})}\left(M \times \mathbf{P}^1\mathbf{C}\right) .$$

By this we understand that also the natural projections of these two spaces on $M \times \mathbf{P}^1\mathbf{C}$ coincide. In particular W depends only on CoKer d.

The proof is done essentially by computing the graph construction in Plücker coordinates and using standard facts about blowups and the vanishing of Fitting ideals.

The above result gives us a straightforward way of computing the scheme at infinity :

Corollary 4 *Under the same assumptions as before, Z_∞ is isomorphic to the total transform of $M \times \{\infty\}$ in the blowup of $M \times \mathbf{P}^1\mathbf{C}$ in the ideal $\mathcal{J}(\mathcal{G})$.*

Describing the graph construction by a blowup has several advantages. First of all, in any concrete example, it allows us to calculate the graph construction effectively. Indeed, the computation of a blowup is the computation of a closure of an image of an algebraic set in a regular map, or equivalently the calculation of a kernel of a morphism of algebras. It can thus be done using Gröbner basis based algorithms.

3 An Example.

The results of the preceding section allow us to study the following example of a singular map of (trivial) vector bundles over $\mathbf{C}$. In fact this example is quite general in the sense that the study of the graph construction for any morphism of vector bundles over a smooth curve can be easily brought down to the study of this example.

Example 5 *We will study a morphism of trivial bundles of the same rank over the affine 1-space* $\mathbf{C}$*, with isolated singularity:*

$$\mathbf{C} \times \mathbf{C}^k \to \mathbf{C} \times \mathbf{C}^k$$

defined by the matrix

$$A = \begin{pmatrix} x^{n_k} & 0 & \cdots & 0 \\ 0 & x^{n_{k-1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x^{n_1} \end{pmatrix}$$

Here x *is the variable of the base* $\mathbf{C}$ *and the* n_i *are positive integers, such that* $n_k > \cdots > n_1 > 0$*. In this case, the cycle* $[Z_\infty]$ *can be expressed as follows :*

$$[Z_\infty] = [V_{k+1}] + n_k[V_k] + \cdots + n_1[V_1] ,$$

where V_{k+1} *projects isomorphically to the base* $\mathbf{C}$ *and* $V_i \cong \mathbf{P}^1\mathbf{C}$ *for* $i = 1, \ldots, k$*. For all* i, j *the intersection* $V_i \cap V_j$ *is empty if* $|i-j| > 1$ *and is transversal in one single isolated point if* $|i-j| = 1$*. In particular* $V_{k+1} \cong \mathbf{C}$ *intersects* V_k *in* 0*. All copies of* $\mathbf{P}^1\mathbf{C}$ *project to* 0*.*

Here is a sketch of the computation. Use Theorem 3 to calculate the Grassmannian graph construction as a blowup. The blowup in question is (over $\mathbf{C} \times U_\infty$) the Proj of the Rees algebra :

$$\begin{aligned} \mathcal{A} &= \mathbf{C}[x, \mu, Tx^{n_k+\cdots+n_1}, T\mu x^{n_{k-1}+\cdots+n_1}, \ldots, T\mu^{k-1}x^{n_1}, T\mu^k] \\ &\subset \mathbf{C}[x, \mu, T] \end{aligned}$$

(graded by the formal variable T). The affine open subschemes of that Proj are the Spec of the algebras $\mathcal{A}_j$, each of which is defined as the subalgebra of elements of degree 0 of the localization of $\mathcal{A}$ in the element $T\mu^j x^{n_{k-j}+\cdots+n_1}$. One easily shows that A_j is isomorphic to a sub-algebra of $\mathbb{C}[x, \mu, \frac{1}{x}, \frac{1}{\mu}]$:

$$A_j \cong \mathbb{C}\left[x, \mu, \frac{x^{n_{k-j+1}}}{\mu}, \frac{\mu}{x^{n_{k-j}}}\right] .$$

Expressing A_j as a quotient of a polynomial ring will allow us to see the components of Z_∞ and their multiplicities. This is done as follows : if n, m are positive integers with $n > m \geq 1$, then there is an isomorphism of $\mathbb{C}[x, \mu]$ - algebras :

$$\mathbb{C}[x, \mu, a, b]\big/(\mu - bx^m, ab - x^{n-m}) \xrightarrow{\cong} B = \mathbb{C}\left[x, \mu, \frac{x^n}{\mu}, \frac{\mu}{x^m}\right] \quad (*)$$

which sends a to $\frac{x^n}{\mu}$ and b to $\frac{\mu}{x^m}$. Hint:

$$B \cong \mathbb{C}[x, \mu, a, b]\big/(J \cap \mathbb{C}[x, \mu, a, b]) ,$$

where $J = (x\tilde{x} - 1,\ \mu\tilde{\mu} - 1,\ a - \tilde{\mu}x^n,\ b - \mu\tilde{x}^m) \subset \mathbb{C}[x, \mu, a, b, \tilde{x}, \tilde{\mu}]$ and

$$\left\{\tilde{\mu} - \tilde{x}^n a,\ \tilde{x}x - 1,\ \mu - bx^m,\ ab - x^{n-m}\right\}$$

is a reduced Gröbner basis of J in the pure lexicographic order such that

$$\tilde{\mu} > \tilde{x} > \mu > a > b > x$$

(it generates J and all the S-polynomials of Buchberger vanish).

Clearly one can apply the isomorphism $(*)$ to $B = A_j$ and in our case the corresponding affine part of Z_∞ is computable by letting $\mu = 0$ in the quotient algebra in $(*)$. A primary decomposition and simple computation of lengths ends the calculation.

4 Analyticity.

Another consequence of using Fitting ideals is the analyticity of the graph construction in the case of an analytic morphism of analytic vector bundles over an analytic space. The blowup of an analytic coherent sheaf of ideals being analytic, from theorem 3 (from its proof to be precise) we deduce the following result

Theorem 6 *For an analytic morphism $d : E_2 \to E_1$, of analytic vector bundles on an analytic space M,* the set *"graph construction" - W defined as at the beginning of this paragraph is an analytic subset of* $\mathrm{Grass}_{e_2}(E_2 \oplus E_1) \times \mathbf{P}^1\mathbf{C}$. *In particular Z_∞, defined also as at the beginning, is a well defined analytic space.*

This generalizes results of Sinan Sertöz ([20], 4.Theorem 1.), ([21], Theorem III.1.) who proved the analyticity of the graph construction in the case, where the base space is compact kählerian, using the Białynicki-Birula decomposition for a $\mathbf{C}^*$ action. Other results of Sertöz about residues of singular foliations [21] are thus immediately generalized.

5 The Modified Fibred Product.

To pursue our study of the graph construction, we need yet another tool - the "modified fibred product". A blowup is an example of a modification. A modification is a proper morphism for which there exists a nowhere dense subset of the target space (called the discriminant set) such that this morphism is an isomorphism outside this set and its inverse image. The term modification is used more frequently in analytic than in algebraic geometry. The results developed in this section are valid in both categories. We will also say that a space is a modification of another space X, if it comes equipped with a morphism to X, which is a modification. The modifications of a fixed variety X form a category (morphisms between them must commute with the modifications).

It is well known, that the fibred product of two modifications is usually not a modification (consider the product of two copies of the same blowup of an algebraic variety). The remedy is to pick some of the components of the fibred product and form a new space. The theorem below explains the precise meaning of this statement.

Theorem 7 *Let Y_1 and Y_2 be two reduced modifications of an algebraic variety X, with discriminant sets D_1 and D_2. Denote by*

π, π_1, π_2 the canonical projections of $Y_1 \times_X Y_2$ on X, Y_1, Y_2 respectively. Define $Y_1 \overset{m}{\times}_X Y_2$ by

$$Y_1 \overset{m}{\times}_X Y_2 = \overline{\left(Y_1 \times_X Y_2 - \pi^{-1}(D_1 \cup D_2)\right)}$$

and the maps ρ, ρ_1, ρ_2 as the restrictions to $Y_1 \overset{m}{\times}_X Y_2$ of π, π_1, π_2 respectively. Then $\left((Y_1 \overset{m}{\times}_X Y_2, \rho), \rho_1, \rho_2\right)$ is the (categorical) product of Y_1 and Y_2 in the category of reduced modifications of X. We will call it the **modified fibred product** *of Y_1 and Y_2 over X.*

Several similar constructions exist in the literature: the *junction* of two morphisms which are compositions of blowups [8],2.9, the *join* of two morphisms of which one is a modification [11], the *strict transform* of a morphism by a blowup [9] and the *meromorphic junction* [22] (applied to "inverses" of modifications).

The two properties below of the modified fibred product, show that it is essentially different from the usual fibred product and allow it to be computed in certain important cases.

- The modified fibred product is idempotent, ie. $Y \overset{m}{\times}_X Y \cong Y$.

- More generally, if there exists a morphism $Y_1 \to Y_2$ of modifications of X, then $Y_1 \overset{m}{\times}_X Y_2 \cong Y_1$.

Theorem 7 would not be true if we had not limited ourselves to considering reduced spaces. Indeed one can show that in the category of all modifications of $\mathbf{C}^2$ the product (as a categorical product) of two copies of the blowup of $\mathbf{C}^2$ at the origin does not exist (show that it would be a subscheme of the usual fibred product and would have to contain a "double", "triple", "quadruple" etc. exceptional divisor).

6 The Nash Transform.

An earlier notion akin to the graph construction needs to be mentioned here. It is the so called Nash transform with respect to a coherent sheaf. For a coherent sheaf $\mathcal{F}$ on X, we will denote by $\mathrm{T}(\mathcal{F})$ its $\mathcal{O}_X^{\cdot}$ - torsion. For a morphism $f : Y \to X$, (after [17]) we will denote by $\mathcal{F} \circ f$ the "torsion free analytic inverse image" of $\mathcal{F}$ by the morphism f (the usual analytic inverse image modulo its torsion over $\mathcal{O}_Y$).

In the sequel, all the spaces considered will be irreducible and reduced. Consider all the reduced modifications $\phi : Y \to X$ of X such that $\mathcal{F} \circ \phi$ is a locally free sheaf (it is then of the same rank as $\mathcal{F}$). By [17] or [15], there is a minimal such modification (i.e. every other such modification is a modification of this minimal one), which is called **the Nash transform** of X with respect to $\mathcal{F}$.

Locally, the Nash transform is constructed in [17] as follows. One represents $\mathcal{F}$ as the cokernel of a morphism of vector bundles $d : E_1 \to E_0$. Then d is of constant rank over an open dense set U of the base space. One defines $X_{\mathcal{F}}$ as the closure, in the total space of the the bundle $H = \mathrm{Grass}_{\mathrm{rk}\,\mathcal{F}}(E_0^{\vee})$, of the image of the section "kernel of the dual morphism ". In [17] it is shown that $\mathcal{F} \circ \phi_{\mathcal{F}}$ corresponds to the restriction of the dual bundle to the tautological bundle on H.

7 The Distinguished Component.

We now take a closer look at the scheme at infinity. It turns out that it has a distinguished irreducible component. We will describe this component using the Nash transform and the modified fibred product.

One of the components of Z_∞ for a morphism $d : E_2 \to E_1$ is a modification of the base space M. In [1] this distinguished component, which will be denoted $\widehat{M}$ is described as the closure of the section "kernel $\oplus$ image" of the Grassmann bundle:

$$\widehat{M} \hookrightarrow \mathrm{Grass}_{e_2-s}E_2 \times_{\overline{M}} \mathrm{Grass}_s E_1 \hookrightarrow \mathrm{Grass}_{e_2}(E_2 \oplus E_1) ,$$

$$\widehat{M} = \overline{\ker d|_U \oplus \operatorname{Im} d|_U} .$$

Above, s is the generic rank of the morphism d and U is the open dense subset of M where the rank of the morphism is equal to the generic rank. Below, we prove different characterizations of the distinguished component, establishing in particular that it is a modification of the Nash transform of the cokernel sheaf considered. In [1] it is also proved that the union of the irreducible components of Z_∞ other than $\widehat{M}$ projects by $\pi|_{Z_\infty}$ to the singular locus of the map d. In fact, a stronger scheme-theoretic result (below) is true.

Theorem 8 *Let $d : E_2 \to E_1$ be a morphism of vector bundles. Consider the Z_∞ scheme for d. Then*

- *Z_∞ has a distinguished primary component $\widehat{M}$, which is isomorphic to the blowup of M in the ideal* $\mathrm{Fitt}_{\mathbf{rk}\,(\mathrm{CoKer}\,d)}(\mathrm{CoKer}\,d)$,
- *the restriction of the projection $\pi|_{Z_\infty}$ to an isolated primary component of Z_∞, other than $\widehat{M}$, factorizes through the inclusion in M of the zero - subscheme of the ideal* $\mathrm{Fitt}_{\mathbf{rk}\,(\mathrm{CoKer}\,d)}(\mathrm{CoKer}\,d)$,
- *$\widehat{M}$ is isomorphic to the modified fibred product of Nash transforms :*

$$M_{\mathrm{CoIm}\,d} \overset{m}{\times}_M M_{\mathrm{CoKer}\,d} ,$$

and the restriction to $\widehat{M}$ of the tautological bundle ξ is linked to the tautological bundles on the Nash transforms by the formula:

$$(\pi|_{\widehat{M}})^*(E_2 \oplus E_1)\big/(\xi|_{\widehat{M}}) \cong (\mathrm{CoIm}\,d \oplus \mathrm{CoKer}\,d) \circ \pi|_{\widehat{M}} .$$

The first two statements of this theorem are proved by examining the ideal in theorem 3. This theorem implies in particular that the Z_∞ scheme has a distinguished primary component $\widehat{M}$, which is the strict transform (in the sense of [7],II.7.15) of $M \times \{\infty\}$ in the blowup of $M \times \mathbf{P}^1\mathbf{C}$ in the ideal $\mathcal{J}(\mathrm{CoKer}\, d)$. Remark that the restriction of the ideal $\mathcal{J}(\mathrm{CoKer}\, d)$ to $M \times \{\infty\}$ is equal to $\mathrm{Fitt}_{\mathrm{rk}\,(\mathrm{CoKer}\, d)}(\mathrm{CoKer}\, d)$ (and in particular is non zero). To prove the third statement, one would have to delve more deeply into the construction of the Nash transform.

8 The Graph Construction for Sheaves.

A natural habitat of singular morphisms of vector bundles is the locally free resolution of a coherent sheaf. Suppose that M is smooth and let $\mathcal{F}$ be a coherent algebraic sheaf on M. After [2] or [7] (exercise III.6.9), $\mathcal{F}$ has a finite locally free resolution:

$$0 \to \mathcal{E}_k \xrightarrow{d_k} \mathcal{E}_{k-1} \xrightarrow{d_{k-1}} \dots \xrightarrow{d_2} \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{d_0} \mathcal{E}_{-1} = \mathcal{F} \to 0 \ . \ (*)$$

This resolution gives a complex of vector bundles (corresponding to the locally free sheaves in the resolution).

$$0 \to E_k \xrightarrow{d_k} E_{k-1} \xrightarrow{d_{k-1}} \dots \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} = 0 \ , \ (**)$$

This complex is exact over the open dense subset of M on which the sheaf $\mathcal{F}$ is locally free. We will keep the symbols $d_k, \dots, d_1$ for the induced morphisms of vector bundles. Furthermore, taking some liberties with the notation, $d_0 : E_0 \to E_{-1} = 0$ will denote the zero morphism. Any time an ambiguity is possible we will state precisely whether we have in mind a morphism of sheafs or of bundles. For example there is no ambiguity in the definition of $\mathrm{CoKer}\, d_i$.

Consider the graph constructions (W_i, π_i, p_i, Ξ_i) and the schemes at infinity $(Z_{\infty,i},\ \pi_i|_{Z_{\infty,i}},\ \xi_i)$ for each one of the morphisms d_i of the complex $(**)$.

Definition 9 *The* **graph construction for the sheaf** $\mathcal{F}$ *will be the collection* (W, π, p, Ξ) *defined as follows*

$$W \ = \ W_k \overset{m}{\times}_{M\times\mathbf{P}^1\mathbf{C}} \cdots \overset{m}{\times}_{M\times\mathbf{P}^1\mathbf{C}} W_0 \ ,$$

the map (π, p) will be the natural projection of the modified fibred product on $M \times \mathbf{P}^1\mathbf{C}$. Further, Ξ will be a virtual bundle defined as the formal alternating sum

$$\Xi = \sum_{i=0}^{k}(-1)^i \rho_i^* \Xi_i ,$$

where ρ_i is the projection of the modified fibred product on its i-th factor. The **scheme at infinity for the sheaf** $\mathcal{F}$ *is the triple $(Z_\infty, \pi|_{Z_\infty}, \xi)$, defined by $Z_\infty = p^{-1}((0:1))$ and $\xi = \Xi|_{Z_\infty}$.*

Most interesting for us will be the class of the virtual bundle in the Grothendieck group of algebraic vector bundles (locally free sheaves) K^0. Over a manifold, this is isomorphic to the Grothendieck group of coherent sheaves K_0 (see [2] or [7] exercise III.6.9).

The use of Fitting ideals allow us to see that in fact the graph construction does not depend on the chosen resolution.

Theorem 10 *For a coherent sheaf $\mathcal{F}$ on an algebraic manifold M, the following objects of the graph construction for $\mathcal{F}$ do not depend on the choice of a locally free resolution :*

- *the spaces (W_i, π_i, p_i) of the graph construction for the i-th boundary morphism,*
- *the space (W, π, p) of the graph construction for the sheaf $\mathcal{F}$,*
- *the scheme $(Z_\infty, \pi|_{Z_\infty})$ and the class of the virtual tautological bundle ξ in coherent K-theory $K^0(Z_\infty)$.*

The original definition of the graph construction for a sheaf did not use the modified fibred product, but a simultaneous construction for all morphisms concerned. The above independence theorem first appeared [6] where a slight modification of the graph construction is presented (the bundles of the resolution are twisted). That paper too, uses neither modified fibred products nor Fitting ideals, the use of which, except for the part concerning bundles, allows us to sketch a quick proof of the above theorem :

By theorem 3, W_i is the blowup of $M \times \mathbf{P}^1\mathbf{C}$ in the ideal $\mathcal{J}(\mathrm{CoKer}\, d_i)$ of the definition 2. Now, by ([3], theorem 20.2), a resolution is locally uniquely determined up to a trivial component. Hence, $\mathcal{J}(\mathrm{CoKer}\, d_i)$ does not depend on the locally free resolution chosen. This ends the proof.

The Z_∞ scheme for a sheaf also has a distinguished irreducible component $\widehat{M}$ which is a modification of M. The union of the other components of Z_∞ projects by $\pi|_{Z_\infty}$ to the singular locus of the sheaf. In [1] it is described as the closure of a section of the Grassmann bundle made up of simultaneous sums of the kernel and image sections of all morphisms involved in the resolution of the sheaf. The proposition below gives a different description of this component, similar to the one given in the preceding section for the single morphism case. In particular it compares the distinguished component to the Nash transform $M_{\mathcal{F}}$ of the sheaf and the Nash transforms of its syzygies, comparing also the tautological bundles. This part generalizes the result of [1] (proof of II.2.1), which says that if the support of a sheaf $\mathcal{F}$ is nowhere dense in M, then $\xi_{|\widehat{M}} = 0$.

Proposition 11 *Let $\widehat{M_i}$ be the distinguished component of $Z_{\infty,i}$, for a coherent sheaf $\mathcal{F}$, $\widehat{M}$ the distinguished component of Z_∞. Then*

- $\widehat{M} \;\cong\; \widehat{M_k} \overset{m}{\times}_M \cdots \overset{m}{\times}_M \widehat{M_0}$,
- $\widehat{M} \;\cong\; M_{\mathrm{CoKer}\, d_k} \overset{m}{\times}_M \cdots \overset{m}{\times}_M M_{\mathrm{CoKer}\, d_0}$,
- *There exists a canonical modification $\Phi : \widehat{M} \to M_{\mathcal{F}}$ and in $K^0(\widehat{M})$ we have :*

$$\xi|_{\widehat{M}} \;=\; \Phi^*(\mathcal{F} \circ \Phi_{\mathcal{F}}) \; .$$

- *The sheaf $\mathcal{F} \circ (\pi|_{\widehat{M}})$ is locally free and, in $K^0(\widehat{M})$, we have :* $\xi|_{\widehat{M}} = \mathcal{F} \circ (\pi|_{\widehat{M}})$.

9 Mather Classes.

As is usual in the study of this kind of problems, by "cohomology", we shall understand one of the classical functorial theories, together with the corresponding homology theory, equipped with the usual formal properties: cup and cap product, projection formula, fundamental class (in homology) of an irreducible algebraic set (such that for a modification, the direct image of the fundamental class of the source space is the fundamental class of the target space). We shall suppose further, that there is a homomorphism from the Chow group (of algebraic cycles modulo rational equivalence) to homology. To a subscheme of an algebraic variety, we associate a fundamental algebraic cycle, as in ([5],1.5) and so a fundamental class in homology, which coincides with the above mentioned fundamental class in the case of an irreducible and reduced subvariety. For example, if we restrict our attention to compact varieties, then singular homology and cohomology will do.

A characteristic class in full generality is a "way to associate" a cohomology class cl^* to a vector bundle, such that for a morphism $f : X \to Y$ and for a vector bundle E on Y, we have the naturality formula $\mathrm{cl}^* f^* E = f^* \mathrm{cl}^* E$. We will also suppose, that this natural transformation takes its values only in even dimensional cohomology. For example the total Chern class in singular cohomology with integer or rational coefficients is a characteristic class in this sense. Likewise, any polynomial in the Chern classes is a characteristic class. Another important example of a characteristic class is the Chern character, see for example ([5], 3.2) or ([14], Problem 16B). which appears in different Riemann-Roch - type theorems [2],[1].

Having a characteristic class defined on vector bundles, we would like to obtain one defined on all coherent sheaves. One way of achieving that is the Mather class construction, so called because it resembles the construction of Chern - Mather classes for singular varieties. This construction works over all varieties (smooth or singular), but has the disadvantage of producing a homology class instead of a cohomology class.

Definition 12 *Let* cl^* *be any characteristic class. It induces a (homology)* **Mather class** cl_*, *defined for coherent sheaf* $\mathcal{F}$ *on a variety* X, *by:*

$$\mathrm{cl}_*\mathcal{F} \;=\; (\phi_\mathcal{F})_*(\mathrm{cl}^*(\mathcal{F}\circ\phi_\mathcal{F})\cap[X_\mathcal{F}])\ .$$

Here $(X_\mathcal{F},\phi_\mathcal{F})$ is the Nash transform of X relative to $\mathcal{F}$ and hence $\mathcal{F}\circ\phi_\mathcal{F}$ is a locally free sheaf (so it makes sense to take its characteristic class). Further $[X_\mathcal{F}]$ is the fundamental (homology) class of $X_\mathcal{F}$ and $(\phi_\mathcal{F})_*$ is the morphism induced in homology by the map $\phi_\mathcal{F}$.

The universal property of the Nash transform allows us to characterize the Mather class in an axiomatic way. For a coherent sheaf $\mathcal{F}$ on a variety X, we have :

- if $\mathcal{F}$ is locally free, then $\mathrm{cl}_*\mathcal{F}=\mathrm{cl}^*\mathcal{F}\cap[X]$,
- $\mathrm{cl}_*\mathcal{F}=\mathrm{cl}_*(\mathcal{F}/\mathrm{T}(\mathcal{F}))$,
- for a modification $f:Y\to X$ $\mathrm{cl}_*\mathcal{F}=f_*\mathrm{cl}_*f^*\mathcal{F}$.

These properties characterize the Mather class cl_* completely.

The above definition concerning Chern - Mather classes applies in particular to the Chern character ch^* and Chern class c^*. inducing a *Chern-Mather character* ch_* and a (total) *Chern-Mather class* c_* - both in homology. These objects have some nice natural properties, which one can prove using the universal property of the Nash transform and the modified fibred product. The key to most proofs is the observation that

$$X_{\mathcal{F}_1\oplus\mathcal{F}_2}\cong X_{\mathcal{F}_1}\overset{m}{\times}_X X_{\mathcal{F}_2}\ .$$

Here is a sample of properties of Mather classes : For two coherent sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on X we have:

$$\mathrm{ch}_*(\mathcal{F}_1\oplus\mathcal{F}_2) \;=\; \mathrm{ch}_*\mathcal{F}_1+\mathrm{ch}_*\mathcal{F}_2.$$

If $\mathcal{F}$ is a coherent sheaf on X and $\mathcal{E}$ a locally free sheaf on X, then

$$\mathrm{ch}_*(\mathcal{E}\otimes\mathcal{F})=\mathrm{ch}^*\mathcal{E}\cap\mathrm{ch}_*\mathcal{F}\ .$$

And if we have a short exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{F}^{L} \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow 0 \ ,$$

the sheaf $\mathcal{E}$ being locally free, then

- $\mathrm{ch}_*\mathcal{F} \ = \ \mathrm{ch}_*\mathcal{F}' + \mathrm{ch}_*\mathcal{E}$,
- $\mathrm{c}_*\mathcal{F} = \mathrm{c}^*\mathcal{E} \cap \mathrm{c}_*\mathcal{F}'$.

The big drawback of Chern - Mather classes is that they are not well defined in K-theory. For example the Chern - Mather character of the middle term of a short exact sequence of coherent sheaves need not be equal to the sum of the Chern characters of the lateral terms.

Chern - Mather classes and characters were studied in a slightly different setup by Marie - Hélène Schwartz in [18] and [19]. There, a Nash transform relative to a *linear space* (see [4]) is constructed and is subsequently used to develop the theory. This Nash transform corresponds to the one mentioned above by the anti-equivalence of coherent sheaves and linear spaces and so also these classes also correspond to those defined here.

10 Mather Classes and K-theory Classes.

On a manifold, the Chern class and the Chern character can be extended to natural transformations of K_0 to cohomology, considered as functors to the category of sets. For the Chern character, this transformation can be considered also as a transformation of functors from K-theory to the category of abelian (additive) groups. The Chern class (resp. Chern character) of a sheaf is defined as the alternating product (resp. alternating sum) of the Chern classes (resp. Chern characters) of locally free sheaves appearing in a resolution of the sheaf concerned. We will show how to compare a part of them with corresponding Mather classes.

In fact, any characteristic class cl^* defined on K_0, is in particular a characteristic class on vector bundles only, which we shall also denote cl^*. Hence it induces a homological Mather class cl_*. The theorem below shows that parts of the two classes are equal. It generalizes theorem II.2.1.b) of [1], which states that the dual of the Chern character of a coherent sheaf on an algebraic manifold has a representative with support included in the support of the sheaf (if this support is nowhere dense in the base manifold, then on one hand it is equal to the singular locus of the sheaf and on the other hand its Chern Mather character is zero).

Theorem 13 *Let M be a compact complex manifold. Let* cl^* *be a characteristic class on the Grothendieck group and* cl_* *- the corresponding homological Mather class. Then for a coherent sheaf $\mathcal{F}$ on M, the homology class*

$$\mathrm{cl}^*(\mathcal{F}) \cap [M] \;-\; \mathrm{cl}_*(\mathcal{F})$$

has a representative supported in the singular locus of the sheaf $\mathcal{F}$. In particular, if $n = \dim_{\mathbf{C}} M$ and s is the dimension of the singular locus of the sheaf $\mathcal{F}$ then, for $i = s+1, \ldots, n$ we have

$$\mathrm{cl}_i(\mathcal{F}) \;=\; \mathrm{cl}^{(n-i)}(\mathcal{F}) \cap [M].$$

The proof of this theorem makes use of the graph construction (W, π, p, Ξ) for the sheaf $\mathcal{F}$. Like in [1], one remarks that the cycles associated to the schemes $Z_0 = p^{-1}((1:0))$ and Z_∞ are rationally equivalent and so, that

$$\mathrm{cl}^*\Xi \cap [Z_0] \;=\; \mathrm{cl}^*\Xi \cap [Z_\infty]\ ,$$

Then, one studies the two terms of the above equality, after composing with π_*. The left hand side becomes $\mathrm{cl}^*(\mathcal{F}) \cap [M]$ - this can be proved directly. The right hand side becomes $\mathrm{cl}_*(\mathcal{F})$ plus a cycle supported in the singular locus. Essential in the proof of this last statement is the description of the distinguished component of Z_∞ in proposition 11. The distinguished component provides $\mathrm{cl}_*(\mathcal{F})$ and the other components only contribute a cycle supported in the singular locus. Apart from that, the proof makes repetitive use of all the naturality properties involved, projection formulas and universal properties of the Nash transform and modified fibred products.

11 A Vanishing Theorem.

The Chern classes of a vector bundle, of index greater than the rank of the vector bundle vanish. This is no longer true for coherent sheaves. However, Chern - Mather classes are pushforwards of Chern classes of vector bundles and so they do vanish for certain indices. This remark, coupled with Poincaré duality and theorem 13 implies the following theorem on the vanishing of the usual Chern classes of sheaves.

Theorem 14 *Let $\mathcal{F}$ be a coherent sheaf on a compact complex manifold M of (complex) dimension n. Let s be the dimension of the singular locus of $\mathcal{F}$. Suppose $i > \operatorname{rk}\mathcal{F}$ and $i < n - s$. Then the i-th Chern class of $\mathcal{F}$ vanishes :*

$$c^i\mathcal{F} = 0 \ .$$

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