Computing Invariants of Hypersurface Singularities

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1 Introduction

This set of notes is intended for distribution to the participants of the Algebraic Geometry Summer School in Bilkent University, Ankara, Turkey. The notes provide a brief overview of some invariants associated to hypersurface singularities and indicate how one goes about computing them. They are idiosyncratic, dealing with matters that my students and I have found interesting. I have tried to supply references to more complete (and balanced) treatments throughout the text. But the road to hell is paved with good intentions. Caveat lector.

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These notes are divided into five parts as follows, roughly corresponding to the contents of each lecture.

1. Brief overview of singularities
2. Limits of tangent spaces
3. Computational methods in polynomial rings
4. Milnor numbers
5. Computational methods in local rings

Good general references on singularity theory include [2], [10], [19] and [21]. Good general references on computation in polynomial rings include [3], [8], and [4].

2 Singularities: a brief overview

We begin with some basic definitions that we shall use throughout. Let $\mathbf{C}$ be the field of complex numbers.

Definition 2.1 *A subset* $V \subset \mathbf{C}^n$ *is an* **algebraic set** *if* V *is the set of common zeroes of polynomials in* $\mathbf{C}[x_1, \ldots, x_n]$.

Definition 2.2 *An algebraic set* V *is called a* **variety** *or an* **irreducible** *algebraic set if it cannot be expressed as the union of two proper algebraic subsets.*

The following proposition characterizes the property of being a variety algebraically.

Proposition 2.3 *Let $V \subset \mathbf{C}^n$ be an algebraic set and let $\mathbf{I}(V)$ be the ideal of $\mathbf{C}[x_1, \ldots, x_n]$ consisting of all polynomials which vanish at every point of V. Then*

$$V \text{ a variety} \iff \mathbf{I}(V) \text{ prime} \iff \mathbf{C}[x_1, \ldots, x_n]/\mathbf{I}(V) \text{ an integral domain.}$$

This characterization ensures that the following definition makes sense.

Definition 2.4 *If V is a variety, then the* **dimension** *of V, denoted* $\dim(V)$, *is the transcendence degree over $\mathbf{C}$ of the quotient field of* $\mathbf{C}[x_1, \ldots, x_n]/\mathbf{I}(V)$.

We are now ready to define a singular point.

Definition 2.5 *Let $V \subset \mathbf{C}^n$ be a nonempty algebraic set and suppose that $f_1, \ldots, f_k \in \mathbf{C}[x_1, \ldots, x_n]$, $k < \infty$ are such that $\mathbf{I}(V)$ is generated by $f_1 \ldots f_k$. We write $\mathbf{I}(V) = \langle f_1, \ldots, f_k \rangle$. Let ρ be the largest rank the $k \times m$ matrix $(\partial f_i / \partial x_j)$ attains at any point $x \in V$. A point $x \in V$ is* **nonsingular** *or* **simple** *if $(\partial f_i / \partial x_j)$ attains its maximal rank ρ at x and* **singular** *if*

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}\right) < \rho.$$

We remark that the Hilbert basis theorem guarantees that *any* ideal in $\mathbf{C}[x_1, \ldots, x_n]$ can be generated by a *finite* set of polynomials. Moreover, it is easy to check that the definition above is $\Sigma(V)$ denote the set of singular points of V. It is always a proper subset of V.

Theorem 2.6 (**Whitney**) *If V is a nonempty algebraic set, the set $V - \Sigma(V)$ is a smooth non-empty, complex manifold. It is complex analytic of dimension $n - \rho$ over $\mathbf{C}$.*

Our **main question** will be: what does V look like in a neighborhood of a singular point? To even begin to answer this question, we need to decide when two varieties look alike in some neighborhood. The definitions are easiest in the case when the variety can be defined by a single polynomial. Accordingly, in what follows, unless explicitly stated otherwise, we specialize to (complex) **hypersurfaces**.

Definition 2.7 *A variety $V \subset \mathbf{C}^n$ is called a* **hypersurface** *if V is the set of zeroes of a single polynomial.*

By a standard theorem in algebraic geometry, a variety in $\mathbf{C}^n$ is a hypersurface if and only if $\dim_{\mathbf{C}} V = n-1$. (This fails spectacularly over $\mathbf{R}$, where every algebraic set can be given by a single equation.)

Notice that if V is a hypersurface, $V = \{f = 0\}$, then

$$p \in V \quad \text{singular} \Longleftrightarrow \frac{\partial f}{\partial x_1}(p) = \ldots + \frac{\partial f}{\partial x_n}(p) = 0.$$

For hypersurfaces, we formalize the phrase "a singular point p of V **looks like** a singular point p' of V'" in one of the following ways.

Definition 2.8 *If $V, V' \subset \mathbf{C}^n$ are hypersurfaces with singular points p, p', respectively, then we say that the singular point p of V is* **topologically equivalent** *to p' of V' if there exists a neighborhood $B_\epsilon = \{x \in \mathbf{C}^n : |x-p| < \epsilon$ in $\mathbf{C}^n$ and a homeomorphism $h : B_\epsilon \longrightarrow h(B_\epsilon$ such that $h(p) = p'$ and $h(V \cap B_\epsilon) = V' \cap h(B_\epsilon)$. We say that p of V is* **analytically equivalent** *to p' of V' if there exist B_ϵ and H as above with h an analytic diffeomorphism.*

Other equivalences are possible: for example, we could insist that h be Lipschitz, or C^r for some r. Or we could drop the requirement that h be defined on the ambient space.

A usual tool in visualizing a singularity p is its link.

Definition 2.9 *Let $p \in V \subset \mathbf{C}^n$ and for each $\epsilon > 0$ let $B_\epsilon = \{x \in \mathbf{C}^n : |x - p| < \epsilon\}$ and $S_\epsilon = \partial B_\epsilon = \{x \in \mathbf{C}^n : |x - p| = \epsilon\}$. Then for ϵ sufficiently small, $V \cap S_\epsilon$ is called the* **link** *of the singularity.*

In order for this definition to make sense, it must be the case that for ϵ sufficiently small $V \cap S_\epsilon$ does not depend on ϵ in any important way. This is indeed the case as the following result will show. First, we recall that if K is a set in $\mathbf{C}^n$, then the **cone** over K based at the point p, denoted $\mathrm{Cone}_p(K)$, is the set $tp + (1-t)k : k \in K, t \in \mathbf{R}$. By the cone over a pair of sets (G, K), denoted $\mathrm{Cone}_p(G, K)$, we mean the pair of cones $(\mathrm{Cone}_p(G), \mathrm{Cone}_p(K))$. Note that the cone over a sphere based at the center of the sphere is just the ball centered on the same point.

Theorem 2.10 *Let V be a hypersurface and $p \in V$. Then for all ϵ sufficiently small, and S_ϵ as in definition 9, $p \in V$ is topologically equivalent to* $\mathrm{Cone}_p(V \cap S_\epsilon$. *In other words, if B_ϵ is the ball of radius ϵ centered on p, then $(B_\epsilon, B_\epsilon \cap V)$ and $Cone_p(S_\epsilon, S_\epsilon \cap V)$ are homeomorphic as pairs.*

This result tells us both that the link of a singularity is well-defined up to homeomorphisms, and that the link determines the singularity up to topological equivalence. A proof can be found in [21] in the case of isolated singularities (see below for the definition) and in [7] in general.

We say that a singularity $p \in V$ is **isolated** if there exists B_ϵ centered at p such that the only singularity of $V \cap B_\epsilon$ is p. In the case that p is isolated, then $V - \{p\}$ is a manifold near p and it is easy to see that S_ϵ intersects V transversely for small ϵ. It follows that the link of an isolated singularity is a smooth manifold. Here are some examples.

- If $V = \{x^2 - y^3 = 0\}$, then S_ϵ is the 3-sphere and $V \cap S_\epsilon$ is homeomorphic to a circle. It is, however, embedded in S_ϵ as a torus knot of type $(2, 3)$. That is, the link is homeomorphic to the type of knot one obtains in the three-sphere, by winding a curve twice around a torus in one direction and three times in the other. If one thinks of the torus as obtained by identifying opposite sides of a unit square, then a torus knot of type (3,2) is traced out by a straight line of slope 3/2. In particular, there is no homeomorphism of S_ϵ which carries the link onto an unknotted circle.

- If $V = \{x^k + y^l = 0\}$ with k, l relatively prime, then the link is a torus knot of type (k, l).

- If $V = \{x^k + y^{kl} = 0\}$ with k, l relatively prime, then the link is a torus link consisting of k unknotted circles, any two of which have linking number l.

- The above three items are easily proved by parameterizing V. In general, if V is a plane curve with one branch, then the link is an iterated torus knot (a torus knot on the boundary of a tubular neighborhood of a knot which is a torus knot on the boundary of a tubular neighborhood of ... of a torus knot). See [12]

- If $V = \{2xy - z^2 = 0\}$, then the link is homeomorphic to $\mathbf{RP}^3$ embedded in the 5-sphere S_ϵ. One sees this by parameterizing: $V = \{(s^2, t^2, \sqrt{2}st) | s, t \in \mathbf{C}\}$ where (s, t) and $(-s, -t)$ give the same point. On V,

$$|x|^2 + |y|^2 + |z|^2 = (|s|^2 + |t|^2)^2,$$

so that $V \cap S_\epsilon$ is homeomorphic to the unit sphere in (s, t) space modulo identification of antipodal points, and this is just real projective 3-space.

- If $V = \{x_1^3 + x_2 + \ldots + x_n^2 = 0\}$ and n is even, then the link is a topological sphere, which may fail to be diffeomorphic to the standard sphere. This fact was discovered by Hirzebruch: see [21].

One studies singularities by associating various invariants to them. These invariants can be geometric, algebraic or topological or (in the best cases) some combination of two of these. We are going to concentrate on two sorts of invariants: geometric invariants related to the tangent cone and algebraic (and topological) invariants related to "Milnor numbers". To compute the former, we will have to have some knowledge of computation in polynomial rings. To compute the latter, we will need to look at how one can do computations in local rings.

3 Tangent cones and limits of tangent spaces

Suppose that $V \subset \mathbf{C}^n$ is a variety (not necessarily a hypersurface). Then at any point p of V we can consider the tangent space $T(V,p)$, which is defined as follows.

Definition 3.1 *If $f \in \mathbf{C}[x_1,\ldots,x_n]$ is a polynomial, the* **linear part** *of f at p, denoted $d_p(f)$, is defined to be the polynomial*

$$d_p(f) = \frac{\partial f}{\partial x_1}(p)(x_1 - p_1) + \cdots + \frac{\partial f}{\partial x_n}(p)(x_n - p_n)\,.$$

Note that $d_p(f)$ has total degree ≤ 1. $V \subset \mathbf{C}^n$ and $p = (p_1,\ldots,p_n) \in V$ then the **tangent space** *to V at p, denoted $T(V,p)$ is the variety*

$$T(V,p) = \mathbf{V}(d_p(f) : f \in \mathbf{I}(V))\ .$$

If p is nonsingular, then V is a manifold near p and the definition of tangent space above coincides with the usual definition of the tangent space to a manifold at a point. However, if p is a singular point, then the dimension of the tangent space is "too big".

Example 3.2 *Suppose that $V = \{x^2 + y^3 = 0\}$. If $p = (1,1)$ then $T(V,p)$ is the line $2(x-1) + 3(y-1) = 0$ through (1,1) normal to (2,3). However, if $p = (0,0)$ is the singular point, then $T(v,p) = \{\mathbf{C}^2\}$ has dimension 2.*

To get something more useful we approximate V near p not by linear terms in the defining equations (which might well be zero), but by the lowest degree nonvanishing homogeneous pieces of the defining equations. More precisely, suppose that $p = (p_1, \ldots, p_n) \in \mathbf{C}^n$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n_{>0}$, let

$$(x - p)^\alpha = (x_1 - p_1)^{\alpha_1} \cdots (x_n - p_n)^{\alpha_n} ,$$

and note that $(x-p)^\alpha$ has total degree $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Now, given any polynomial $f \in \mathbf{C}[x_1, \ldots, x_n]$ of total degree d, we can write f as a polynomial in $x_i - p_i$, so that f is a $\mathbf{C}$-linear combination of $(x - p)^\alpha$ for $|\alpha| \leq d$. If we group terms according to total degree, we can write

$$f = f_{p,0} + f_{p,1} + \cdots + f_{p,d} ,$$

where $f_{p,j}$ is a $\mathbf{C}$-linear combination of $(x - p)^\alpha$ for $|\alpha| = j$. Note that $f_{p,0} = f(p)$ and $f_{p,1} = d_p(f)$ (as defined above). One can use Taylor's formula to express $f_{p,j}$ in terms of the partial derivatives of f at p. Notice also that if $p = 0$, then we are writing f as the sum of its homogeneous components. In most situations, it will be convenient to translate p to the origin. We now define the tangent cone.

Definition 3.3 *If $f \in \mathbf{C}[x_1, \ldots, x_n]$ is a nonzero polynomial, then $f_{p,min}$ is defined to be $f_{p,j}$, where j is the smallest integer such that $f_{p,j} \neq 0$ in the displayed formula above. The* **tangent cone** *of V at p, denoted $C(V, p)$, is the variety*

$$C(V, p) = \mathbf{V}(f_{p,min} : f \in \mathbf{I}(V)) .$$

The tangent cone of at a point on a hypersurface $V \subset \mathbf{C}^n$ is especially easy to compute. It is an easy exercise to show that if $\mathbf{I}(V) = \langle f \rangle$, then $C(V, p)$ is defined by the the single equation $f_{p,min} = 0$. However, when $\mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle$ has more generators, it need *not* follow that $C(V, p) = \mathbf{V}((f_1)_{p,min}, \ldots, (f_s)_{p,min})$. For example, suppose that V is defined by $xy = xz + z(y^2 - z^2) = 0$. One shows without difficulty that $\mathbf{I}(V) = \langle xy, xz + z(y^2 - z^2) \rangle$. To see that $C(V, 0) \neq \mathbf{V}(xy, xz)$, note that $f = yz(y^2 - z^2) = y(xz + z(y^2 -$

$z^2)) - z(xy) \in \mathbf{I}(V)$. Then $f_{0,min} = yz(y^2 - z^2)$ vanishes on $C(V, 0)$, but not on all of $\mathbf{V}(xy, xz)$.

The tangent cone $C(V,p)$ to a variety V at the point p is made up of lines through p. To describe which lines through p lie in $C(V,p)$ we recall first that a line L in $\mathbf{C}^n$ through p is called a **secant line** if it meets V in a point distinct from p. If we take secant lines determined by points of V getting closer and closer to p, then the "limit" of the secant lines lies on the tangent cone. To make this idea precise, suppose we have parametrized L as $p + tv$, where $v \in \mathbf{C}^n$ is a nonzero vector parallel to L and $t \in \mathbf{C}$. We say that a line $L \in \mathbf{C}^n$ through a point $p \in \mathbf{C}^n$ is a **limit of lines** $\{L_k\}_{k=1}^{\infty}$ through p if given a parameterization $p + tv$ of L, there exist parametrizations $p + tv_k$ of L_k such that $\lim_{k\to\infty} v_k = v$ in $\mathbf{C}^n$. A standard result in algebraic geometry then asserts that the tangent cone consists precisely of limits of secant lines (see, for example, [26].)

Theorem 3.4 *Let $V \subset \mathbf{C}^n$ be a variety. Then a line L through p in $\mathbf{C}^n$ lies in the tangent cone $C(V,p)$ if and only if there exists a sequence $\{q_k\}_{k=1}^{\infty}$ of points in $V - \{p\}$ converging to p with the property that the secant lines joining p and q_k converge to the given line L.*

Instead of introducing the tangent cone using its defining equations, we could have begun with the geometric characterization afforded by the result above. First note that the tangent cone can be viewed as the subspace of projective space whose points correspond to lines through a prescribed point. Limits of lines then translate into limits of points in projective space.

Now, let $V \subset \mathbf{C}^n$ be a variety and $p \in V$ a singular point. If ϕ maps $V - \{0\}$ to the set $\mathbf{P}^{n-1}$ of lines through p:

$$\phi : x \mapsto \overline{0x}, \quad x \in V - \{0\},$$

and if we consider the graph of ϕ,

$$\operatorname{gr}\phi \subset (V - \{0\}) \times \mathbf{P}^{n-1},$$

then the closure

$$\mathrm{cl}\,(\mathrm{gr}\,\phi) \subset V \times \mathbf{P}^{n-1}$$

projects onto V. Call this projection π. Then, one checks easily that

$$\pi^{-1}(p) = \{p\} \times [C(V,p)],$$

where $[C(V,p)]$ denotes the canonical image of $C(V,p)$ in $\mathbf{P}^{n-1}$.

The tangent cone provides an approximation to a surface near a singular point in the sense that, given any conical neighborhood of the tangent cone, a sufficiently small neighborhood of the surface lies entirely within the conical neighborhood (see [23] for a precise statement). It is also easy to see that the tangent cone transforms well under analytic diffeomorphism. Thus the tangent cone is an invariant of a singularity which is easy to compute. The difficulty with the tangent cone is that it is too coarse – it gives no indication of a number of features of a variety about which one would want to know, and entirely dissimilar varieties may have the same tangent cone. For example the surfaces $\{y^2+x^3-z^2x^2 = 0\}$ and $\{y-x^2-z^2 = 0\}$ have the same tangent cone, namely the plane $y = 0$.

A more subtle invariant is the space $K(V,p)$ of limiting tangent spaces to V at p. Since we shall restrict ourselves to hypersurfaces in what follows, we only provide a definition in the case that $V \subset \mathbf{C}^n$ is a hypersurface. Suppose that this is the case. Then, if $x \in V$ is a nonsingular point of V, the tangent space $T(V,x)$ to V at x is a hyperplane, which we label by its "conormal vector", $df(x)$, whose components are just the partials of f evaluated at x (its complex conjugate is normal to $T(V,x)$ with respect to the usual Hermitian metric on $\mathbf{C}^n$). Let Σ denote the singular set of V and let ψ map $V - \Sigma$ to the set $\mathbf{P}^{n-1}$ of planes in $\mathbf{C}^n$ through p by

$$\psi : x \mapsto [df(x)], \quad x \in V - \Sigma,$$

where $[df(x)]$ denotes the canonical image of the vector $df(x)$ in projective space (that is, the set of all nonzero scalar multiples of $df(x)$). Again, we consider the closure of the graph of ψ and let π be the map onto V induced by projection. We define $K(V,p)$ by demanding that its image $[K(V,p)]$ in $\mathbf{P}^{n-1}$ satisfy

$$\pi^{-1}(p) = \{p\} \times [K(V,p)].$$

We call an element of $K(V,p)$ a *limiting tangent space* to V at p.

An element of $K(V,p)$ consists of the set of vectors v in $\mathbf{C}^n$ based at the point p with the property that there exists a sequence of points $x_k \in V - \Sigma$, $x_k \to p$, for which the lines through x_k in the direction $df(x_k)$ converge to the line thru p in the direction v. Each such vector determines a hyperplane T in $\mathbf{C}^n$ through p (if $v = (v_1, \ldots, v_n)$ and $p = (p_1, \ldots, p_n)$, the equation of $T = \{u = (u_1, \ldots, u_n) \in \mathbf{C}^n : v_1(u_1 - p_1) + \ldots + v_n(u_n - p_n) = 0\}$, and each such hyperplane is the limit of some sequence of tangent hyperplanes $T(V, x_k)$ as x_k tends to p, the convergence of the tangent hyperplanes being understood as convergence in the "dual" projective space $\check{\mathbf{P}}^{n-1}$. In order to avoid trivialities, we shall suppose that $p \in \Sigma$ (otherwise $K(V,p)$ is the one point set $\{T(V,p)\}$).

As we shall see shortly, the invariant $K(V,p)$ contains a great deal of information about the surface. Since it is so easy to find the equation for $C(V,p)$ given the equation for V, one might ask if the same is true for $K(V,p)$: can we find its equation(s) given the equation for V? Sadly, things are not so straightforward as they are for tangent cones. In order to compute equations for $K(V,p)$, we will need to develop some computational techniques in polynomial rings. Before doing so, we develop a few other facts about the geometry of $K(V,p)$.

A moment's reflection will convince one that a limiting tangent space to the tangent cone of a variety "ought" to be a limiting tangent space to the variety itself: that is, $K(C(V,p)) \subset K(V,p)$. This inclusion was established by [27]. However, as Whitney points out, this inclusion can be, and typically is, strict. The question of characterizing the "extra" limiting tangent spaces is answered by a structure theorem due to Lê and Teissier [18] for arbitrary varieties.

Theorem 3.5 *There exists a finite (possibly empty) set of proper subcones*
$C_1, \ldots, C_r \subset C(V,p)$, *$C_i$ cones over p for all (called* **exceptional cones***) such that*

$$K(V,p) = K(C(V,p),p) \cup \ \cup_{i=1}^{r} K(C_i),$$

where $K(C_i)$ denotes the set of all hyperplanes in $\mathbf{C}^n$ tangent to C_i along some line in C_i through p.

Thus, one knows $K(V,p)$, given $C(V,p)$ and the exceptional cones. The data {tangent cone, exceptional cones} are called the *aureole* of V at p and signal important geometric features of the variety near p. For example, the z-axis is an exceptional cone for $\{y^2 + x^3 - z^2x^2 = 0\}$, and this signals the "vanishing fold". The fact that the z-axis is a line of singular points is incidental: one can easily concoct similar examples in which the surface has an isolated singularity at the origin.

Lê and Teissier also characterize the exceptional cones as the non-moving parts of tangent cones to polar varieties. We explain briefly in the case of surfaces. So suppose, temporarily, that $V \subset \mathbf{C}^3$ is a surface. Then the exceptional subcones of $C(V,p)$ will be lines through p. To find them, let $\pi : \mathbf{C}^3 \longrightarrow \mathbf{C}^2$ be a linear projection and define the *polar curve* P_π of V with respect to π to be the closure of the critical locus of the restriction of π to the smooth part of V:

$$P_\pi \equiv \mathrm{cl}\,(\mathrm{crit}\,(\pi|_{V-\Sigma})).$$

Generically, P_π is either empty or one-dimensional (over $\mathbf{C}$). Consider the tangent cone at 0 of P_π. If P_π is one-dimensional, the tangent cone will be a union of lines in $\mathbf{C}^3$ through p. In fact, it is easy to see that, as sets, $C(P_\pi, p) \subset C(V,p)$. As π varies, some lines in $C(P_\pi, p)$ will vary, while others will stay fixed.

Theorem 3.6 *The exceptional lines (of V at p) are precisely those lines in $C(P_\pi, p) \subset C(V,p)$ which do not vary as π varies.*

(The result, above, for surfaces was first proved by Lê and Henry for surfaces with isolated singularities and by Lé for arbitrary surfaces.) This result gives us another approach to computing $K(V,p)$, at least for surfaces. Here, again, however, we will need to develop some computational techniques for working with polynomials, an issue to which we now turn.

4 Computational Methods in Polynomial Rings

The modern algorithmic theory of computation in polynomial rings begins with the constructive solution, in 1965, of the ideal membership problem. This problem, first posed by Macaulay in [20] asks the following:

(Ideal Membership Problem). *Given an ideal* $I = \langle f_1, \ldots, f_r \rangle \subset \mathbf{C}[x_1, \ldots, x_n]$ *and a polynomial* $g \in \mathbf{C}[x_1, \ldots, x_n]$, *determine whether* $g \in I$.

The problem is easy if $n = 1$. For in this case every ideal can be generated by a single polynomial and to decide if a polynomial g belongs to the ideal, one merely divides g by the generating polynomial – if the remainder is zero, then g belongs to the ideal; otherwise it doesn't. The problem is likewise easy if $f_1, \ldots, f_r$ are linear and n is arbitrary. The problem in the general case is more difficult and Macaulay never did solve it. A fully algorithmic solution to the ideal membership problem only appeared in 1965, when Bruno Buchberger, a student of Groebner, created a working algorithm to test ideal membership [6]. (A year earlier in 1964, Hironaka had established a solution in principle as a part of his work on resolution of singularities [16].)

The solution of the ideal membership problem consisted of three pieces:

1. the introduction of a linear multiplicative order on all monomials

2. the extension of the division algorithm from polynomials of one variable to polynomials in an arbitrary number of variables, and

3. the notion of a Groebner basis,

each of which we describe in turn.

1. Multiplicative Orders on Monomials

If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n_{>0}$, we let x^α denote the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The first ingredient in the Hironaka-Buchberger solution was to linearly order all monomials in $\mathbf{C}[x_1, \ldots, x_n]$ in such a way that if $x^\alpha > x^\beta$, where $\alpha, \beta \in \mathbf{Z}^n_{>0}$, then $x^{\alpha+\gamma} > x^{\beta+\gamma}$ for all $\gamma \in \mathbf{Z}^n_{>0}$. Such an ordering is called a **multiplicative ordering**. Multiplicative orderings were already used to great effect by Macaulay.

The best known example of a multiplicative ordering is the **lexicographic** or **lex** ordering. Here we say that $x^\alpha >_{lex} x^\beta$ (and drop the subscript *lex* when the order is understood) if the first nonzero term of $\alpha - \beta$ is greater than 0. Thus, $x_1^2 x_2 x_3^5 x_4 > x_1^2 x_2 x_3^2 x_4^9$ because the first nonzero term (reading from the left) of $(2,1,5,1) - (2,1,2,9) = (0,0,3,-8)$ is 3 which is greater than 0. As another example, in the lex ordering we have

$$x^4 > x^3y^3 > x^2y^{10} > x^2 > xy^9 > x > y^{100} > y > 1.$$

A different multiplicative ordering is the **graded lexicographic** or **glex** ordering. This ordering sorts first by total degree and then uses the lexicographic order to sort monomials of the same degree. That is, we say that $x^\alpha >_{glex} x^\beta$ if $|\alpha| > |\beta|$ (here, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $|\beta|$ is defined similarly) or if $|\alpha| = |\beta|$ and the first nonzero term of $\alpha - \beta$ is nonzero. Again we drop the subscript *glex* when the ordering is clear from the context. With respect to the graded lex ordering we have

$$y^{100} > x^2y^{10} > xy^9 > x^3y^3 > x^4 > x^2 > x > y > 1.$$

In general, a multiplicative ordering is said to be **graded** if it first sorts by degree.

There are many other examples of monomial orderings, some better than others for different sorts of computations. In doing computations in polynomial rings, one usually restricts oneself to orderings that are **well-orderings**: that is, which are such that every non-empty subset of monomials has a least element. One can show that this is equivalent to the condition that $1 < x_i$ for all variables (see [8], §2.4). So, the lex and graded lex orders are well-orderings.

If $f = \sum_\rho a_\rho x^\rho \in \mathbf{C}[x_1, \ldots, x_n]$ is a polynomial, we say that a monomial $x^\rho = x_1^{\rho_1} \cdots x_n^{\rho_n}$ or a term $a_\rho x^\rho$ **belongs to** f if the corresponding coefficient a_ρ is nonzero. Once we have established an order $>$ on monomials, we define the **leading term** $\text{LT}_>(f)$ of f to be $a_\rho x^\rho$ where x^ρ is the largest monomial (with respect to the order $>$) which belongs to f. Thus, for example,

$$\text{LT}_{lex}(x^4 + xy^9 + x + 15y^{100} + y + 1) = x^4,$$

while

$$\text{LT}_{glex}(x^4 + xy^9 + x + 15y^{100} + y + 1) = 5y^{100}.$$

2. The Division Algorithm for Polynomials

The second ingredient of the solution to the ideal membership problem is the division algorithm for polynomials. This is a procedure for dividing a polynomial in any number of variables by an ordered collection of polynomials. This procedure works once a monomial order has been chosen. It is best explained by an example, which I have lifted from [24].

Consider the following question.

$$\text{Does } x^4y - xy^3 + y^4 + x^2y \in \langle xy + 1, x - y \rangle \ ?$$

With the case of one variable in our mind, the natural thing to do is to try to divide $x^4y - xy^3 + y^4 + x^2y$ by $xy + 1$ and $x - y$. Suppose we choose the graded lex order with $x > y$. Then the terms in the three polynomials are written in descending order and

$$\text{LT}(x^4y - xy^3 + y^4 + x^2y) = x^4y, \quad \text{LT}(xy+1) = xy, \text{ and } \text{LT}(x-y) = x.$$

The leading monomial xy of $xy + 1$ divides the leading monomial x^4y of $x^4y - xy^3 + y^4 + x^2y$ to give x^3. Multiplying x^3 by $xy + 1$ and subtracting gives $-xy^3 + y^4 - x^3 + x^2y$. The leading monomial xy of

$xy+1$ divides the leading term $-xy^3$ of $-xy^3+y^4-x^3+x^2y$ to give $-y^2$. Multiplying $-y^2$ by $xy+1$ and subtracting gives $y^4-x^3+x^2y+y^2$. Neither the leading monomial of $xy+1$ nor the leading monomial x of $x-y$ divide the leading monomial y^4 of $y^4-x^3+x^2y+y^2$. Thus, we put y^4 into the remainder and subtract it from $y^4-x^3+x^2y+y^2$ to get $-x^3+x^2y+y^2$. Now the leading monomial xy of $xy+1$ does not divide the leading term $-x^3$ of $-x^3+x^2y+y^2$, but the leading monomial x of $x-y$ does. Dividing the latter (that is, x) into $-x^3$ gives $-x^2$; multiplying $-x^2$ by $x-y$ and subtracting the result from $-x^3+x^2y+y^2$ gives y^2. Neither the leading monomial of $xy+1$ nor the leading monomial x of $x-y$ divide the leading monomial y^2 of y^2, so we put y^2 into the remainder leaving nothing left to divide. Thus, the process terminates and we have found that

$$x^4y-xy^3+y^4+x^2y=(x^3-y^2)(xy+1)-x^2(x-y)+y^4+y^2.$$

We leave it as an exercise to show that if we divide in a different order, putting $x-y$ before $xy+1$, then we get

$$x^4y-xy^3+y^4+x^2y=$$
$$(x^3y+x^2y^2+xy^3+y^4-y^3+xy+y^2)(x-y)+0(xy+1)+y^5+y^3.$$

Notice that the answers and, in particular, the remainders are sensitive to the order in which we divide! This is bad or, at least, highly undesirable. In particular, the results of our efforts might lead us to carelessly conclude that $x^4y-xy^3+y^4+x^2y$ does not belong to the ideal $\langle x-y, xy+1\rangle$, which is false because

$$x^4y-xy^3+y^4+x^2y=x^3(xy+1)+(x^2+y^3)(x-y).$$

In fact, the reader may enjoy finding an example of polynomials $g, f_1, f_2 \in \mathbf{C}[x,y]$ such that dividing g by f_1, f_2 gives nonzero remainder, but dividing g by f_2, f_1 gives zero remainder!

The last ingredient which allows the division algorithm to to be used to determine ideal membership is the notion of a Groebner basis, together with Buchberger's algorithm which allows one to compute such bases.

3. Groebner Bases

Fix a multiplicative ordering $>$ on the monomials in $\mathbf{C}[x_1, \ldots, x_n]$. If I is an ideal in $\mathbf{C}[x_1, \ldots, x_n]$, we let $\text{LT}(I)$ denote the ideal generated by all leading terms of elements of I:

$$\text{LT}(I) = \langle \text{LT}(f) : f \in I \rangle.$$

A **Groebner basis** of an ideal I (with respect to the ordering $>$) is a set of polynomials $f_1, \ldots, f_r \in I$ such that

$$\text{LT}(I) = \langle \text{LT}(f_1), \ldots, \text{LT}(f_r) \rangle.$$

We leave it as an easy exercise to show that if $f_1, \ldots, f_r$ is a Groebner basis of I, then $f_1, \ldots, f_r$ is actually a basis of I; that is, $I = \langle f_1, \ldots, f_r \rangle$.

Consider our example above and let $I = \langle xy + 1, x - y \rangle$. Note that

$$\text{LT}(\langle x - y, xy + 1 \rangle) \neq \langle \text{LT}(x - y), \text{LT}(xy + 1) \rangle = \langle x, xy \rangle = \langle x \rangle$$

because

$$-y(x - y) + (xy + 1) = y^2 + 1 \in I,$$

but

$$y^2 = \text{LT}(y^2 + 1) \notin \langle x \rangle.$$

An easy computation shows that $\text{LT}(I) = \langle x, y^2 \rangle$ so that $xy + 1, x - y, y^2 + 1$ is a Groebner basis of I. Dividing $x^4y - xy^3 + y^4 + x^2y$ by $xy + 1, x - y, y^2 + 1$ gives

$$x^4y - xy^3 + y^4 + x^2y = (x^3 - y^2)(xy + 1) + x^2(x - y) + y^2(y^2 + 1)$$

clearly showing that the former is in I. (In fact, the polynomial $xy + 1$ is redundant and $x - y, y^2 + 1$ is actually a Groebner basis of I.)

Bruno Buchberger introduced the notion of a Groebner basis in 1965 (naming in honor of his thesis supervisor) and, equally importantly, he discovered an algorithm for computing them. The idea behind the algorithm is simplicity itself. Given a set of generators $\{f_1, \ldots, f_r\}$ of an ideal, one successively considers the polynomials

$$S(f_i, f_j) = \text{LT}(f_i) f_j - \text{LT}(f_j) f_i.$$

If $\text{LT}\, S(f_i, f_j)$ is not divisible by some $\text{LT}\, f_k$ (that is, if $\text{LT}\, S(f_i, f_j) \notin \langle f_1, \ldots f_r \rangle$) then one adds $\text{LT}\, S(f_i, f_j)$ to the list of generators $\{f_1, \ldots, f_r\}$, otherwise one does nothing. The actual algorithm is a few lines of code and relies on a result which characterizes Groebner bases as those bases which give zero remainder when divided into all possible S-polynomials (eee [3], Chap 2.7). Simple arguments show that the algorithm terminates.

Groebner bases allow one to solve the ideal membership problem because the remainder obtained by dividing any $f \in \mathbf{C}[x_1, \ldots, x_n]$ by a Groebner basis is unique (that is, independent of the order in which one takes the elements of the basis) and $f \in I$ if and only if the remainder is 0. Algorithmically, one first uses Buchberger's algorithm to compute a Groebner basis, then divides the polynomial in question by this basis, and checks whether the remainder is zero.

Buchberger's algorithm and the division algorithm are implemented on many computer algebra systems. These two algorithms allow one to explicitly carry out operations on polynomial ideals. In particular, they allow us to carry out the operations we will need to compute limits of tangent spaces: namely, elimination and intersection.

Suppose that want to eliminate variables in a system of equations. In other words, suppose we are given an ideal $I \subset \mathbf{C}[x_1, \ldots, x_n]$, and we want to find the ideal $I \cap \mathbf{C}[x_{n-r+1}, \ldots, x_n]$ generated by the elements of I containing only the last r, say, variables. (This is the algebraic operation corresponding to computing the closure of the projection of the variety defined by I onto the coordinate subspace of $\mathbf{C}^n$ spanned by the last r coordinate vectors.) The following result shows that this can be done by ordering the variables lexicographically and computing a Groebner basis with respect to this ordering.

Theorem 4.1 *If $g_1, \ldots, g_s$ is a Groebner basis of $I \subset \mathbf{C}[x_1, \ldots, x_n]$ with respect to the lexicographic order in which*

$$x_1 > x_2 > \ldots > x_n,$$

then, for some $k \leq s$, the polynomials $g_{k+1}, \ldots, g_s$ will only involve the last r variables $x_{n-r+1}, \ldots, x_n$ (and any polynomial g_i with $i <$

k, will involve at least one of the variables $x_1, \ldots, x_{n-r}$ in a term with nonvanishing coefficient). The polynomials $g_{k+1}, \ldots, g_s$ are a standard basis for $I \cap \mathbf{C}[x_{n-r+1}, \ldots, x_n]$:

$$I \cap \mathbf{C}[x_{n-r+1}, \ldots, x_n] = \langle f_{n-r+1}, \ldots, f_n \rangle.$$

The proof consists in examining the division algorithm carefully (see [8], Chap. 2.8). To return to our example above, if $I = \langle xy+1, x-y \rangle$ and we want to compute $I \cap \mathbf{C}[y]$, then we compute a standard basis for I, finding that $I = \langle x - y, y^2 + 1 \rangle$ so that $I \cap \mathbf{C}[y] = \langle y^2 + 1 \rangle$.

Being able to eliminate variables immediately allows us to compute the equations of the image of an algebraic set. To be more precise, let $I \subset \mathbf{C}[x_1, \ldots, x_n]$ be an ideal and $V = \mathbf{V}(I)$ the corresponding algebraic set in $\mathbf{C}^n$ (here, and henceforth, we use the notation $\mathbf{V}(I)$ to denote the set of common zeroes of the polynomials in I). If $\Phi : \mathbf{C}^n \longrightarrow \mathbf{C}^m$ is a polynomial map in the sense that $\Phi = (\phi_1, \ldots, \phi_m)$ with each $\phi_i \in \mathbf{C}[x_1, \ldots, x_n]$, then the the image of $\Phi(V)$ of V under the map Φ need not be algebraic set. However, it is the case that the topological closure of $\Phi(V)$ is algebraic (and, in fact, the smallest algebraic set containing $\Phi(V)$ - see [23]) We can use theorem 4.1 to compute an ideal which defines the closure of $\phi(V)$. Namely, the following holds.

Proposition 4.2 *Let $I = \langle f_1, \ldots, f_r \rangle$ be an ideal in $\mathbf{C}[x_1, \ldots, x_n]$ and $\Phi = (\phi_1, \ldots, \phi_m)$ an algebraic map. Consider the ideal*

$$J \equiv \langle f_1, \ldots, f_r, y_1 - \phi_1, \ldots, y_m - \phi_m \rangle \cap \mathbf{C}[y_1, \ldots, y_m].$$

Then $\mathbf{V}(J)$ is the closure of $\Phi(V)$.

The proof is straightforward: see [8] Chap. 2.8.

This already allows us to compute the limits of tangent spaces at an isolated singularity because of the following lovely observation, due to Hénaut [15].

Proposition 4.3 *Let $V = \{x \in \mathbf{C}^n : f(x) =\}$, $f \in \mathbf{C}[x_1, \ldots, x_n]$ be a hypersurface with an isolated singularity at the origin. Let $\phi : \mathbf{C}^n, \mathbf{0} \longrightarrow \mathbf{C}^n, \mathbf{0}$ be the Jacobian map $\phi(x) = (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x))$. In a sufficiently small neighborhood U of the origin, $\phi(V \cap U)$ is a hypersurface in $\phi(U)$ (which we denote somewhat carelessly as $\phi(V)$). Then, as sets, the limit of tangent spaces (thought of as normals) $K(V, 0)$ to V at 0 is the tangent cone $C(\phi(V), 0)$ to $\phi(V)$ at 0.*

The proposition is almost a tautology when we represent tangent spaces by their "normals". We leave the easy proof to the reader (or see [24]). As an example of the proposition, consider the hypersurface $V = \{\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{3}z^3 = 0\}$. Here, $\phi(x, y, z) = (x, y, z^2)$ and we can parameterize V as $\{(s, t, (-\frac{3}{2}s^2 - \frac{3}{2}t^2)^{1/3}\}$. Thus, $\phi(V) = \{(s, t, (-\frac{3}{2}s^2 - \frac{3}{2}t^2)^{2/3}\}$. A local equation for $\phi(V)$ is the hypersurface $z^3 = (-\frac{3}{2}x^2 - \frac{3}{2}y^2)^2$ and the reduced tangent cone is $\{z = 0\}$. So $K(V, 0)$ consists of all normals lying in the xy-plane (or all planes containing the z-axis), which is certainly plausible from the (real) picture.

Combining the two previous propositions allows us to write out a procedure for computing the equation for $K(V, 0)$ when V has an isolated singularity.

Proposition 4.4 *Let $V = \{x \in \mathbf{C}^n : f(x) =\}$, $f \in \mathbf{C}[x_1, \ldots, x_n]$ be a hypersurface with an isolated singularity at the origin. Then $K(V, 0)$ is a hypersurface in $\mathbf{C}^n$ with a local equation given by the initial form of the generator of (the radical of) the ideal*

$$\langle f, y_1 - \frac{\partial f}{\partial x_1}, \ldots, y_n - \frac{\partial f}{\partial x_n} \rangle \cap \mathbf{C}[y_1, \ldots, y_n].$$

In practice, one does not bother computing radicals. We leave it as an easy computation to redo our example above to show directly that

$$\langle \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + \tfrac{1}{3}z^3, u - x, v - y, w - z^2 \rangle \cap \mathbf{C}[u, v, w] = \langle 4w^3 + 9u^4 + 18u^2v^2 + 9v^4 \rangle.$$

Being able to eliminate variables also allows us to compute intersections of ideals. This, in turn, allows us to compute equations for $K(V,0)$ in the case that 0 is not an isolated singularity. To pursue this, suppose that I and J are two ideals in $\mathbf{C}[x_1,\ldots,x_n]$ and let $t \cdot I$ denote the ideal in $\mathbf{C}[t,x_1,\ldots,x_n]$ generated by all multiples of elements of I by t and $(1-t)\cdot J$ the ideal generated by multiplying all elements of J by $1-t$). Then, the following holds.

Proposition 4.5 $I \cap J = t \cdot I + (1-t)\cdot J \cap \mathbf{C}[x_1,\ldots,x_n]$.

The proof is easy (see, for example, [8], §4). Moreover, because we can compute intersections, the following proposition is algorithmic.

Proposition 4.6 *Let* $V = \{x \in \mathbf{C}^n : f(x) =\}$, $f \in \mathbf{C}[x_1,\ldots,x_n]$ *be a hypersurface with an isolated singularity at the origin and let* $K(V,0) \subset \mathbf{C}^n$ *be the space of limiting tangent spaces. Let* A *denote the ideal in* $\mathbf{C}[x_1,\ldots,x_n,s,y_1,\ldots,y_n]$ *given by setting*

$$A = \langle f, y_1 - s\frac{\partial f}{\partial x_1}, \ldots, y_n - s\frac{\partial f}{\partial x_n}\rangle.$$

Then the ideal $\mathbf{I}(K(V,0))$ *of* $K(V,0)$ *is the radical of the ideal in* $\mathbf{C}[y_1,\ldots,y_n]$ *given by eliminating* s *and setting* $x_1,\ldots,x_n$ *equal to zero. That is,*

$$\mathbf{I}(K(V,0)) = \sqrt{A \cap \mathbf{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]/\langle x_1,\ldots,x_n\rangle}.$$

For a proof, see [24]. As an example, suppose again that $V = \{\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{3}z^3 = 0\}$. Then

$$A = \langle x^2 + y^2 + \tfrac{2}{3}z^3, u - tx, v - ty, w - tz^2\rangle.$$

Note that $zt^2(x^2 + y^2 + \frac{2}{3}z^3)$ (which belongs to A) is congruent to $zu^2 + zv^2 + \frac{2}{3}w^2$ modulo A and thus belongs to A. But the latter does not depend on t and belongs, therefore, to $A \cap \mathbf{C}[x,y,z,u,v,w]$. Thus, $w^2 \in (A \cap \mathbf{C}[x,y,z,u,v,w])/\langle x,y,z\rangle$ (we leave it as an exercise to show that the latter ideal is, in fact, equal to $\langle w^2\rangle$), in agreement with the calculation we did using Hénaut's observation.

One can also compute the exceptional cones, using their characterization as non-moving parts of tangent cones to polar varieties. However, the computational techniques needed for this would bring us beyond the scope of these lectures. For more information about this approach, see [24].

5 Milnor Numbers

Suppose henceforth that $V = \{x \in \mathbf{C}^n : f(x) = 0\}$, $f \in \mathbf{C}[x_1, \ldots, x_n]$ is a hypersurface with an isolated singularity at the origin. We have already seen several geometric invariants of V, namely $C(V,0)$ and $K(V,0)$. These are invariant under analytic equivalence of a singularity. Arguably, the most important invariant, however, of an isolated singularity of a hypersurface is its Milnor number.

Definition 5.1 *Let V and f be as above, then the* **Milnor number** *of V at 0 denoted $\mu(V,0)$ is the number*

$$\dim_{\mathbf{C}} \mathbf{C}[[x_1, \ldots, x_n]]/\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\rangle \mathbf{C}[[x_1, \ldots, x_n]]$$

where $\mathbf{C}[[x_1, \ldots, x_n]]$ *denotes the ring of formal power series in* $x_1, \ldots, x_n$.

An argument that we sketch later shows that the Milnor number is finite precisely when 0 is an isolated singularity.

This is a purely algebraic definition. However, the Milnor number can be characterized in a number of different ways. For $u = (u_1, \ldots, u_n)$ sufficiently close to the origin and generic, the Milnor number of f at the origin is precisely the number of critical points of $f_u = f(x) + u_1 x_1 + \ldots + u_n x_n$ near 0 (if the Milnor number is μ, what happens is that the critical point of f at 0 breaks up into a number of critical points near 0 which have smaller Milnor number. For u generic, these decomposed critical points all are nondegenerate (i.e. have Milnor number one), and there μ of them).

The Milnor number can also be characterized topologically. For ϵ sufficiently small, the ball $B_\epsilon = \{x \in \mathbf{C}^n : |x| \leq \epsilon\}$ contains no singularities of f other than the origin and is such that for all $\delta << \epsilon$ sufficiently small, $f^{-1}(z)$, $z \in \mathbf{C}$, $|z| < \delta$ meets S_ϵ transversally. For $z \neq 0, |z| < \delta$, the level hypersurfaces $f^{-1}(z) \cap B_\epsilon$ are all diffeomorphic, and nonsingular. Any one of them is called the **Milnor fiber** of f. They are $(2n-2)$-dimensional real manifolds with boundary, and no homology except in the middle dimension $n-1$. The rank of the homology group $H_{n-1}(F, \mathbf{Z})$, $F = f^{-1}(z) \cap B_\epsilon$ turns out to be the Milnor number. More is true: if μ is the Milnor number, then the Milnor fiber has the same homotopy type as a bouquet of μ real $(n-1)$-dimensional spheres. For proofs and more see [21].

Best of all, if two hypersurface singularities are topologically equivalent, then they have the same Milnor numbers [17]. Thus, the Milnor number is an invariant that has both topological and algebraic significance – hence, its importance.

There are a number of invariants of hypersurfaces that also appear as dimensions of algebras. Best known is the **Tjurina number** $\tau(V, 0)$ of a hypersurface, which is defined to be

$$\dim_{\mathbf{C}} \mathbf{C}[[x_1, \ldots, x_n]]/\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\rangle.$$

The Tjurina number is equal to the dimension of the base space of a so-called miniversal deformation of the singularity (see [19] for details), and is an analytic invariant.

Once the Milnor number of a hypersurface is defined, one can define a sequence of Milnor numbers as follows. If 0 is an isolated singularity of the hypersurface $V \subset \mathbf{C}^n$ and if $H \subset \mathbf{C}^n$ is a generic hyperplane passing through the origin, then $V \cap H$ is a hypersurface in $H \equiv \mathbf{C}^{n-1}$ with an isolated singularity at the origin. For almost all hyperplanes H, the Milnor number $\mu(V \cap H, 0)$ is constant (and a minimum). Call this number $\mu^{n-1}(V,0)$, Iterating, we get a sequence

$$\mu^*(V,0) = \{\mu \equiv \mu^n, \mu^{n-1}, \ldots, \mu^1\}$$

of numbers called the μ^***-invariant**. We remark that μ^1 is the Milnor number of the intersection of a generic line through the origin with V – an easy exercise shows that this is just one less than the multiplicity $m(V,0)$ of V at the origin. Here, the multiplicity is the usual multiplicity of a hypersurface; namely, the degree of the lowest degree term that belongs to f. Equivalently, it is the intersection multiplicity of a generic line through the origin with V.

Incidentally, the hyperplanes H for which $\mu(V \cap H, 0) = \mu^{n-1}$ are precisely those which do not correspond to elements in $K(V,0)$. Similarly, the lines L for which $\mu(V \cap L, 0) = \mu^1$ (that is, lines whose intersection number with V at 0 is the $m(V,0)$) are precisely those which do not belong to $C(V,0)$. Thus, being able to compute Milnor numbers would give us another way to determine membership in $K(V,0)$ and $C(V,0)$!

Teissier, who introduced the μ^*-invariants, showed that the μ^*-invariant is an analytic invariant (see [25] p.315). It is known that it is not a topological invariant (two singularities with are topologically equivalent can have different $\mu*$-invariants (see [5]). One of the most stubbornly resistant unsolved problems in singularity theory is whether multiplicity is invariant under topological equivalence of hypersurfaces (this question was asked by Zariski in 1971 (see [28]) and the assertion that the multiplicity is a topological invariant is often referred to as *Zariski's conjecture*, notwithstanding the fact that Zariski did not take any position on the issue, other than musing in public that it ought to be easy to settle.) It would be a major advance if one could determine whether all members of family of singularities in which the Milnor number is constant necessarily have the same multiplicity (except in dimension 3, it is known that all members of such families are topologically equivalent).

We would like to have a way to compute Milnor numbers and related invariants.

6 Computations in Local Rings

In order to compute Milnor numbers, it is useful to be able to make computations in local rings. Recall that a local ring is a ring with exactly one maximal ideal.

The ring $\mathbf{C}[[x_1, \ldots, x_n]]$ of formal power series

$$\mathbf{C}[[x_1, \ldots, x_n]] = \{\sum_{\alpha \geq 0} a_\alpha x^\alpha, \alpha \in \mathbf{Z}_+^n, a_\alpha \in \mathbf{C}\},$$

with the usual definitions of addition and multiplication, is a local ring with maximal ideal $\langle x_1, \ldots, x_n \rangle$. Any element not in $\langle x_1, \ldots, x_n \rangle$ is invertible as a formal series and is, therefore, a unit (in particular, it cannot belong to any proper ideal). The ring $\mathbf{C}\{x_1, \ldots, x_n\}$ consisting of power series in $x_1, \ldots, x_n$ which converge in some neighborhood of the origin is also a local ring. Another example of a local ring is the ring of complex-valued rational functions defined at the origin, denoted $\mathbf{C}[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle}$,

$$\mathbf{C}[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle} = \{\frac{f}{g} : f, g \in \mathbf{C}[x_1, \ldots, x_n], g(0) \neq 0\},$$

with the usual operations of addition and multiplication. We have natural inclusions

$$\mathbf{C}[x_1, \ldots, x_n] \subset \mathbf{C}[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle} \subset \mathbf{C}\{x_1, \ldots, x_n\} \subset \mathbf{C}[[x_1, \ldots, x_n]]$$

and ideals in $\mathbf{C}[x_1, \ldots, x_n]$ containing $< x >$ extend to ideals in these local rings.

In computing Milnor numbers, we are called to compute the dimension of the quotient ring by a zero-dimensional ideal. We remark, that it doesn't matter in which local ring we compute the dimension of the quotient.

Proposition 6.1 *Suppose that $f_1, \ldots, f_n \in \mathbf{C}[x_1, \ldots, x_n]$ have a common isolated zero which we take to be at the origin (isolated, means that there is a ball about the origin containing no common* *zeroes of $f_1, \ldots, f_n$ other than the origin). Then, the the quotient by the ideal generated by the f_i's is a finite dimensional algebra and*

$$\begin{aligned} & \dim_{\mathbf{C}} \mathbf{C}[x_1, \ldots, x_n]_{\langle x_1,\ldots,x_n\rangle}/\langle f_1, \ldots, f_n\rangle \mathbf{C}[x_1, \ldots, x_n]_{\langle x_1,\ldots,x_n\rangle} = \\ = \; & \dim_{\mathbf{C}} \mathbf{C}[[x_1, \ldots, x_n]]/\langle f_1, \ldots, f_n\rangle \mathbf{C}[[x_1, \ldots, x_n]] = \\ = \; & \dim_{\mathbf{C}} \mathbf{C}\{x_1, \ldots, x_n\}/\langle f_1, \ldots, f_n\rangle \mathbf{C}\{x_1, \ldots, x_n\}. \end{aligned}$$

To see why this is so, suppose that we want to compute the quotient by the ideal generated by x^2+x^3 and y^2. Then $\dim \mathbf{C}[x,y]_{\langle x,y\rangle}/\langle x^2+x^3, y^2\rangle \mathbf{C}[x,y]_{\langle x,y\rangle} = 4$ because $\langle x^2 + x^3, y^2\rangle \mathbf{C}[x,y]_{\langle x,y\rangle} = \langle x^2, y^2\rangle \mathbf{C}[x,y]_{\langle x,y\rangle}$ since $1/(1+x) \in \mathbf{C}[x,y]_{\langle x,y\rangle}$ and because the monomials $1, x, y, xy$ are a vector space basis of $\dim \mathbf{C}[x,y]_{\langle x,y\rangle}/\langle x^2, y^2\rangle$. We can represent $1/(1+x)$ as the formal power series $1 - x + x^2 - x^3 + x^4 - \ldots \in \mathbf{C}[[x,y]]$ and then

$$(x^2+x^3)(1 - x + x^2 - x^3 + x^4 - \ldots) = x^2$$

in $\mathbf{C}[[x,y]]$. This shows that $\langle x^2 + x^3, y^2\rangle \mathbf{C}[[x,y]] = \langle x^2, y^2\rangle \mathbf{C}[[x,y]]$ from which it follows that

$$\dim_{\mathbf{C}} \mathbf{C}[[x,y]]/\langle x^2, y^2\rangle \mathbf{C}[[x,y]] = 4$$

(as before, the monomials $1, x, y, xy$ are a vector space basis of $\mathbf{C}[[x,y]]/\langle x^2, y^2\rangle$). However, the power series $1-x+x^2-x^3+x^4-\ldots$ is, in fact, convergent, and precisely the same reasoning shows that $\langle x^2 + x^3, y^2\rangle \mathbf{C}\{x,y\} = \langle x^2, y^2\rangle \mathbf{C}\{x,y\}$ and, therefore,

$$\dim_{\mathbf{C}} \mathbf{C}\{x,y]\}\langle x^2, y^2\rangle \mathbf{C}\{x,y\} = 4.$$

One can generalize from this example to give an clumsy, but valid proof of the proposition assuming that one of the dimensions is finite. The dimension $\dim_{\mathbf{C}} \mathbf{C}\{x_1, \ldots, x_n\}/\langle f_1, \ldots, f_n\rangle \mathbf{C}\{x_1, \ldots, x_n\}$ is finite by the local analytic Nullstellensatz (the hypothesis that the origin is isolated and the local analytic Nullstellensatz say that $\sqrt{\langle f_1, \ldots, f_n\rangle} = \langle x_1, \ldots, x_n\rangle$ so that some power of the ideal $\langle f_1, \ldots, f_n\rangle$ must lie in the maximal ideal). The proposition is true, however, for quite general reasons. See [11].

In the case of the usual polynomial ring, the corresponding dimension bounds the number of common zeroes of the generators of the ideal. That is, if the polynomials $f_1, \ldots, f_m$ have a finite number of zeroes, then the ring $\mathbf{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_m\rangle$ is finite dimensional and the number of common zeroes is less or equal to $\dim_{\mathbf{C}} \mathbf{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_m\rangle$. More is true: if $m = n$ (so the number of polynomials is the same as the number of variables), then $\dim_{\mathbf{C}} \mathbf{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_n\rangle$ is exactly the number of common zeroes of $f_1, \ldots, f_n$ counted with multiplicity. (See [13] §2.9, Proposition 6.)

Happily, we know how to compute the dimension in the case of polynomial rings. For any ordering and any ideal I such that $\dim_{\mathbf{C}} \mathbf{C}[x_1, \ldots, x_n]/I < \infty$, we have

$$\dim_{\mathbf{C}} \mathbf{C}[x_1, \ldots, x_n]/I = \dim_{\mathbf{C}} \mathbf{C}[x_1, \ldots, x_n]/\langle \mathrm{LT} I\rangle,$$

and the latter is just the number of monomials

$$x^\alpha : x^\alpha \notin< \langle \mathrm{LT} I\rangle.$$

(See, for example, [8], Chapter 5.3.)

So, for example, if

$$I = \langle x^2 + x^3, y^2\rangle \subset \mathbf{C}[x, y],$$

and we are using lexicographic order, then

$$\dim \mathbf{C}[x, y]/I = \dim \mathbf{C}[x, y]/\langle \mathrm{LT} I\rangle = \dim \mathbf{C}[x, y]/\langle x^3, y^2\rangle = 6.$$

The rightmost equality follows because $1, x, x^2, y, xy, xy^2$ project to a vector space basis of $\mathbf{C}[x, y]/\langle x^3, y^2\rangle$. So there are at most 6 common zeroes of x^2+x^3 and y^2, in A_k^2. In fact, we see there are two solutions $(-1, 0)$ and $(0, 0)$.

So, in the polynomial case, the heart of the matter is to compute $\langle \mathrm{LT} I\rangle$ given generators $f_1, \ldots, f_s$ of I and this is done by computing a Groebner basis of I.

Our computation in the local case suggests that a similar strategy works for quotient of local rings. If $I = \langle x^2 + x^3, y^2\rangle$ and R were

$\mathbf{C}\{x,y\}_{\langle x,y\rangle}$ (or $\mathbf{C}[[x,y]]$ or $\mathbf{C}\{x,y\}$), we computed $\dim R/IR$ by replacing I by the monomial ideal $\tilde{I} = \langle x^2, y^2\rangle$. Note that $\tilde{I}$ consists of the *lowest degree terms* of I in contrast to the situation in which we compute $\dim \mathbf{C}[x,y]/I$ by replacing I by $\langle \text{LT} I\rangle = \langle x^2, y^3\rangle$, the highest degree terms in the generators.

It should come as no surprise to us that we need to examine lowest degree terms. When dealing with a power series (or a rational function), it is natural to define the initial form to be the lowest degree homogeneous piece.

There are two ways of proceeding: we could retain the term orderings we have been using with polynomials, but re-examine what we did in picking out terms which were maximal with respect to that ordering with a view to modifying the procedures to pick out terms minimal with respect to the ordering. There is a trick, often credited to Lazard, which allows one to do this easily. Or, we could admit orderings in which the maximal elements would have least degree, and try to carry over the theory of Groebner bases to that situation. We adopt the latter procedure, see[9] for more on the Lazard procedure.

So, we will we consider degree-anticompatible (or anti-graded) orderings:

$$|\alpha| < |\beta| \Longrightarrow x^\alpha > x^\beta.$$

We still insist that such orderings be total orderings and multiplicative.

Definition 6.2 *A* **degree-anticompatible ordering** *on* $\mathbf{C}[x_1,\ldots,x_n]$, $\mathbf{C}[x_1,\ldots,x_n]_{\langle x_1,\ldots,x_n\rangle}$, $\mathbf{C}\{x_1,\ldots,x_n\}$, *or* $\mathbf{C}[[x_1,\ldots,x_n]]$ *is a relation $>$ on $\mathbf{Z}^n_{>0}$, or equivalently, on the set of monomials $x^\alpha, \alpha \in \mathbf{Z}^n_{>0}$, satisfying:*

(i) *$>$ is a total ordering on $\mathbf{Z}^n_{>0}$*

(ii) *$>$ is a multiplicative ordering on $\mathbf{Z}^n_{>0}$*

(iii) *$>$ is degree-anticompatible on $\mathbf{Z}^n_{>0}$.*

Recall that the first property means that for any $\alpha, \beta \in \mathbf{Z}^n_{>0}$, exactly one of the following is true:

$$x^\alpha > x^\beta, \qquad , x^\alpha = x^\beta, \quad x^\alpha > x^\beta;$$

and the second means that

$$\text{for any } \gamma \in \mathbf{Z}^n_{>0}, \text{ if } x^\alpha > x^\beta, \text{ then } x^{\alpha+\gamma} > x^{\beta+\gamma}.$$

Notice that the third property implies that $1 > x_i$ for all $i, 1 \leq i \leq n$. Any ordering satisfying (i), (ii) and this latter property is called a **local ordering**. So a degree-anticompatible ordering is a local ordering (but not conversely).

Perhaps the simplest example is degree-anticompatible lexicographic order, abbreviated **alex**, which first sorts by degree, lower degree terms preceding higher degree terms, and then sorts monomials of the same degree lexicographically.

Definition 6.3 (Anti-graded Lex Order) *Let $\alpha, \beta \in \mathbf{Z}^n_{>0}$. We say $x^\alpha >_{alex} x^\beta$ if*

$$|\alpha| = \sum_{i=1}^n \alpha_i < |\beta| = \sum_{i=1}^n \beta_i, \quad or \quad |\alpha| = |\beta| \quad and \quad x^\alpha >_{lex} x^\beta.$$

Thus, for example, we have:

$$1 >_{alex} x >_{alex} y >_{alex} x^2 >_{alex} xy >_{alex} y^2 >_{alex} x^3 >_{alex} \cdots$$

Notice that a degree-anticompatible (and, more generally, any local) ordering is not a well-ordering and, hence, not a monomial ordering in the sense of most books on Groebner bases (which require that monomial orderings satisfy (i), (ii) and be well-orderings – see, for example, [8], Definition 2.2.1)). Since the property of being a well ordering is often used to assert that algorithms terminate, one needs to be especially careful in checking that procedures which use degree-anticompatible orderings terminate.

If $f = \sum_\alpha a_\alpha x^\alpha$ is a non-zero power series and $<$ a local order, we define the multidegree, the leading coefficient, the leading monomial, and the leading term of f exactly as for a well-ordering:

$$\text{multideg}(f) = \max_{<}\{\alpha \in \mathbf{Z}^n_{>0} : a_\alpha \neq 0\},$$

$$\textsc{lc}(f) = a_{\text{multideg}(f)}, \quad \textsc{lm}(f) = x^{\text{multideg}(f)}, \quad \text{and} \quad \textsc{lt}(f) = lc(f)\cdot\textsc{lm}(f).$$

If $f \in \mathbf{C}[x_1, \ldots, x_n]_{\langle x_1,\ldots,x_n\rangle}$, then we first re-write f (uniquely) in the form

$$f = \frac{g}{1+h} \quad \text{with} \quad g, h \in \mathbf{C}[x_1, \ldots, x_n], \quad h(0) = 0,$$

and set

$$\text{multideg}(f) = \text{multideg}(g), \quad \textsc{lc}(f) = \textsc{lc}(g),$$
$$\textsc{lm}(f) = \textsc{lm}(g), \quad \text{and} \quad \textsc{lt}(f) = \textsc{lt}(g).$$

Notice that the multidegree and leading term of $f \in \mathbf{C}[x_1, \ldots, x_n]_{\langle x_1,\ldots,x_n\rangle}$ agree with what one obtains upon viewing f as a power series.

Lemma 6.4 *Let f, g both be in $\mathbf{C}[x_1, \ldots, x_n]_{\langle x_1,\ldots,x_n\rangle}$, (or $\mathbf{C}\{x_1, \ldots, x_n\}$ or $\mathbf{C}[[x_1, \ldots, x_n]]$) and both be non-zero, Then*

i. multidegree*(fg)* = multidegree*(f)* + multidegree*(g)*.

ii. *If $f + g \neq 0$, then*

$$\text{multidegree}(f+g) \leq \max(\text{multidegree}(f), \text{multidegree}(g))$$

with equality if multidegree$(f) \neq$ multidegree(g).

Now, just as for well-orderings, given an ideal I in a local ring R, where $R = \mathbf{C}[x_1, \ldots, x_n]_{\langle x_1,\ldots,x_n\rangle}, \mathbf{C}[[x_1, \ldots, x_n]]$, or $\mathbf{C}\{x_1, \ldots, x_n\}$, and an ordering $<$, we define the *the set of leading terms of I*, denoted $\textsc{lt}(I)$, to be the set of leading terms of elements of I with respect to $<$, and define the ideal of leading terms of I, denoted

$\langle \text{LT} I \rangle$, to be the ideal generated by the set $\text{LT}(I)$. Just as for ideals in polynomial rings, it can happen that $I = \langle f_1, \ldots, f_s \rangle$, but $\text{LT}(I) \neq \langle \text{LT}(f_1), \ldots \text{LT}(f_s) \rangle$. In analogy to the notion of a Groebner basis, we define a **Groebner** or **standard basis** of an ideal I to be a set $f_1, \ldots, f_s \in I$ such that $\langle \text{LT} I \rangle = \langle \text{LT}(f_1), \ldots \text{LT}(f_s) \rangle$. It is easy to show that such a set necessarily generates the ideal I. In the literature, the phrase "standard basis" is more common than the expression "Groebner basis" when referring to local orderings, so that is what we use here.

It is also not too difficult to show that

$$\begin{aligned} & \dim_{\mathbf{C}} \mathbf{C}[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle} / I\mathbf{C}[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle} = \\ = \ & \dim_{\mathbf{C}} \mathbf{C}[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle} / \langle \text{LT} I \rangle \mathbf{C}[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle} \end{aligned}$$

where $\langle \text{LT} I \rangle$ is taken with respect to a graded anti-compatible order. The argument is similar to that in the polynomial case (see [9]).

Given generators of an ideal, how can we compute a standard basis for the ideal? Recall that the key elements in the polynomial case were the division algorithm and Buchberger's algorithm. Let us review each of them here and see what needs to be modified to allow us to use them with local orderings.

The key step in the division algorithm is the reduction of one polynomial f by another g. If $\text{LT}(f) = t\text{LT}(g)$, we define

$$\text{Red}\,(f, g) = f - \frac{\text{LC}(f)}{\text{LC}(g)} tg,$$

and say that we have *reduced f by g*. The polynomial $\text{Red}\,(f, g)$ is just what is left after the first step in dividing f by g – it is the first partial dividend. In the one variable case, the division algorithm in which one divides a polynomial f by another g is just the process of repeatedly reducing by g until either one gets 0 or a polynomial that cannot be reduced by g (because its leading term is not divisible $\text{LT}(g)$). In the case of several variables, the division algorithm, in which one divides a polynomial by a set of other polynomials, is just the process of repeatedly reducing the polynomial by members of the set, adding leading terms to the remainder, when no reductions are possible. This terminates in the case of polynomials because successive leading terms form a strictly decreasing sequence, and such sequences always terminate for well-orderings.

In the case of power series and local orders, one defines $\mathrm{Red}(f, g)$ exactly as above. However, a sequence of successive reductions need no longer terminate. For example, suppose $f = x$ and we decide to divide by $g = x - x^2$, so that we successively reduce by $x - x^2$. We get

$$\begin{aligned} f_0 &\equiv f = x, \\ f_1 &= \mathrm{Red}(f_0, g) = x^2, \\ f_2 &= \mathrm{Red}(f_1, g) = x^3, \\ f_3 &= \mathrm{Red}(f_2, g) = x^4, \\ &\vdots \\ f_n &= \mathrm{Red}(f_{n-1}, g) = x^{n+1}, \\ &\vdots \end{aligned}$$

which clearly does not terminate. The difficulty, of course, is that $x > x^2 > x^3 > \ldots$ is a strictly decreasing sequence of terms which does not terminate.

We can evade this difficulty, with a splendid idea of Mora's: when dividing f by a single polynomial or power series g say, we allow ourselves to reduce not just by g, but by the result of any previous reduction. More generally, when dividing a set of polynomials or power series, we allow ourselves to reduce by the original set together with the results of any previous reductions. So, in our example, where we are dividing $f = x$ by $g = x - x^2$, after the first reduction, we we allow ourselves to reduce with f (the result of the zeroth reduction) as well as g. The first reduction by g tells us that

$$f = 1 \cdot g + x^2.$$

Reducing the result x^2 with f instead of g gives $\mathrm{Red}(x^2, f) = 0$ so that we halt. Moreover, the fact that the reduction is zero gives

$x^2 = xf$ from which we obtain the relation

$$f = g + xf.$$

But this relation tells us that in $\mathbf{C}[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle}$

$$f = \frac{1}{1-x} g.$$

Looking at the above example, one might ask whether it would always suffice to first reduce by g, then subsequently reduce by f. Sadly, this is not the case: it is easy to construct examples where the sequence of reductions does not terminate. Suppose, for example, that we wish to divide $f \equiv x + x^2$ by $g \equiv x + x^3 + x^5$. Reducing f by g and then subsequently reducing the results by f gives the sequence

$$\begin{aligned} f_0 &\equiv f = x + x^2, \\ f_1 &= \mathrm{Red}\,(f_0, g) = x^2 - x^3 - x^5, \\ f_2 &= \mathrm{Red}\,(f_1, f) = -2x^3 - x^5, \\ f_3 &= \mathrm{Red}\,(f_2, f) = 2x^4 - x^5, \\ f_4 &= \mathrm{Red}\,(f_3, f) = -3x^5, \\ f_5 &= \mathrm{Red}\,(f_4, f) = 3x^6, \\ f_6 &= \mathrm{Red}\,(f_5, f) = -3x^7, \\ &\vdots \end{aligned}$$

which clearly does not terminate. We get something which terminates by reducing f_5 by f_4:

$$\begin{aligned} f_5 &= \mathrm{Red}\,(f_4, f) = -3x^6, \\ \tilde{f}_6 &= \mathrm{Red}\,(f_5, f_4) = 0. \end{aligned}$$

From this, we can easily write out an expression for f as a sum of a polynomial times g, plus a polynomial which vanishes at the origin times f plus a polynomial which vanishes at the origin times f_5:

$$\begin{aligned} f &= (\qquad) \cdot g + (\qquad) \cdot f + (\qquad) \cdot f_5 \\ x + x^2 &= 1 \cdot (x + x^3 + x^5) + (x - 2x^2 + 2x^3 - 3x^4) \cdot (x + x^2) \\ &\quad -x \cdot (-3x^5) \end{aligned}$$

Since, f_5 can be expressed as a sum of a polynomial times g plus a polynomial times f, we can actually express f as a sum of a polynomial times g plus a polynomial which vanishes at the origin times f by backsubstituting:

$$\begin{aligned}
x + x^2 &= 1 \cdot (x + x^3 + x^5) \\
&+ (x - 2x^2 + 2x^3 - 3x^4) \cdot (x + x^2) \\
&- x \cdot (-3x^5) \\
-3x^5 &= -1 \cdot (x + x^3 + x^5) + (1 - x + 2x^2 - 2x^3) \cdot (x + x^2) \\
\Longrightarrow & \\
x + x^2 &= (1 - x)(x + x^3 + x^5) + (-x^2 - x^4)(x + x^2) \\
\text{so} & \\
f &= (\text{polynomial}) \cdot g + (\text{polynomial vanishing at } 0) \cdot f + (\quad).
\end{aligned}$$

This, of course, is what we want because, upon transposing, we have

$$(\text{unit}) \cdot f = (\text{polynomial}) \cdot g.$$

In order to create an algorithm, we have to specify a way to choose a sequence of elements by which to reduce so that the reduction terminates, either by giving 0 or a polynomial whose leading coefficient is not divisible by the leading term of anything by which we can reduce.

Looking at the sequence $f_0, f_1, \ldots f_6$ in our division of $f = x + x^2$ by $g = x + x^3x^5$ above gives us some clue how to proceed. Notice that neither the degree of the dividends nor the order necessarily decreases. However, the difference between the degrees of the highest degree term and lowest degree terms in the partial dividends does decrease. Is there someday to guarantee that this will happen? Before addressing this, let us fix notation.

Definition 6.5 *Let $g \in \mathbf{C}[x_1, \ldots, x_n], g \neq 0$, and write g as a finite sum of homogeneous, non-zero polynomials of different degrees: $g = \sum_{i=1}^{k} g_i$, $g_i \neq$ homogeneous, $\deg(g_1) < \ldots < \deg(g_k)$. Define the* **order** *of g, denoted* $\operatorname{ord}(g)$ *to be the degree of g_1 and the* **écart** *of*

g, denoted $E(g)$, to be the difference of the degree of g and the order of g:

$$E(g) = \deg(g) - \operatorname{ord}(f).$$

By convention, we set $E(0) = -1$,

A simple argument which we leave as an exercise shows the following.

Lemma 6.6 *Let f and g be two nonzero polynomials such that $\text{LT}(g)$ divides $\text{LT}(f)$. Then*

$$E(\operatorname{Red}(f,g)) \leq \max\{E(f), E(g)\}.$$

If f, g are polynomials of a single variable, then the inequality is always strict. If, in the case of several variables, $E(g) \leq E(f)$ and $E(\operatorname{Red}(f,g)) = E(f)$, then $\operatorname{ord}(\operatorname{Red}(f,g)) = \operatorname{ord}(f)$.

In the one variable case, this gives us a strategy that guarantees termination. Namely, at each stage, among all the polynomials by which we can reduce, we reduce by the polynomial whose écart is least. This will ensure that the écarts of the sequence of partial dividends strictly decreases to zero, at which point we have a monomial which can be used to reduce any subsequent monomial to 0. In the multivariable, case we use the same strategy, but the écart may not strictly decrease – however, we can again assert that the sequence of reductions will terminate after a finite number of steps, essentially because whenever the écart stays fixed, the order must, too, and there are only finitely many monomials of a given order.

Making these remarks precise, gives us an algorithm for successively reducing one polynomial by others using an anti-graded local order, a proof that the algorithm is correct, and a proof that the algorithm terminates. see [14], [22], or [1].

With this in place, we now have a way of computing a standard basis of an ideal with respect to an anti-graded order, and hence the Milnor number of a singularity. Given an isolated singularity f, we compute the ideal of leading terms (with respect to an anti-graded order) of the ideal generated by the first partial derivatives of f. The Milnor number is just the number of monomials not contained in this (monomial) ideal. The same strategy allows is to compute Tjurina numbers and the μ^*-invariant.

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