The Classical Geometry of Vector Bundles

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1 Introduction

One of the expected products of this summer school is an answer to the following question:

What is ALGEBRAIC GEOMETRY?

(We write AG for short.) Actually this is a hard task, because everybody already has the fixed conviction that the objects of AG are algebraic varieties.

An irreducible algebraic variety X has a dimension $\dim X$, and this number is usually a rough indication of the level of completeness of the geometric theory describing it. Algebraic varieties of small dimension carry special names:

$\dim X = 0:$	set of points
$\dim X = 1:$	curves (Xavier Gomez-Mont's lectures)
$\dim X = 2:$	surfaces (Rick Miranda's lectures)
$\dim X = 3:$	3-folds (Miles Reid's lectures)
etc.	

To explain how my lectures fit into this list, I would like to remark that two algebraic varieties of different dimension can be *geometrically identical.* To see this, consider the following chain of examples:

$\dim = 0$:	a set of 6 distinct points on $\mathbb{P}^1$ up to $\mathrm{PGL}(2, \mathbb{C})$ action;
$\dim = 1$:	a curve of genus 2;
$\dim = 2$:	a cubic surface in $\mathbb{P}^3$ with one ordinary double point;
$\dim = 3$:	a nonsingular intersection of two quadrics in $\mathbb{P}^5$.

The identifications between the objects in $\dim < 3$ are absolutely obvious: the canonical map of a curve (see Rick Miranda's lectures [M] in this volume) of genus 2 is a double cover of $\mathbb{P}^1$ ramified in 6 points; considering $\mathbb{P}^1$ as a conic in $\mathbb{CP}^2$, blowing up 6 points on this conic and constructing the anticanonical map of the resulting surface, we get a cubic in $\mathbb{P}^3$ with an ordinary double point, see [M].

The threefold in our list carries the imposing full name of *Fano threefold of index* 2 *and degree* 4; its halfanticanonical map [M] displays it as the base locus of a pencil of quadrics in $\mathbb{P}^5$. The six singular quadrics of this pencil take us back to a set of 6 distinct points on $\mathbb{P}^1$.

This example of a chain of identifications is of course very classical and simple. A more recent example is Mukai's construction [Mu] of an identification of a plane quartic with a Fano variety V_{22}.

Slogan: *An algebraic geometer is skillful enough if he or she can* *recognize the geometric person under many guises of different dimensions.*

My first aim is to give you some experience in this direction. But my task is a little more complicated, because there is some new person in our game:

algebraic vector bundle.

In some sense this geometric object doesn't have any dimension (or, if you prefer, is infinite dimensional). But in any case, we can't avoid it. Even in our simplest chain of identifications, the intersection of two quadrics in $\mathbb{P}^5$ is a moduli space of stable vector bundles on the corresponding curve of genus 2.

So my second aim is to construct a simple but a new chain of geometric identifications including a vector bundle as a geometric object.

This new chain isn't quite as simple as the previous one, but it is perhaps the simplest illustration of new geometric observations showing that CLASSICAL AG is a slice of much more general GEOMETRY. Namely, some time ago Gromov observed that many results of ENUMERATIVE AG are true in SYMPLECTIC GEOMETRY. But a recent observation due to Donaldson is much more unexpected: *many constants of ENUMERATIVE AG are invariants of the underlying smooth structure of algebraic surfaces.*

Thus my third aim is to explain these relations between AG and differential geometry.

2 Clebsch and Darboux curves

Let $\mathbb{CP}^2$ be the complex projective plane:

$$\mathbb{CP}^2 = \mathbb{P}T, \quad \text{where } T = \mathbb{C}^3, \quad \text{so that } \text{Pic}\mathbb{CP}^2 = \mathbb{Z} \cdot l,$$

where l is a line. Then $|d \cdot l| = \mathbb{P}S^dT^*$ is the complete linear system of curves of degree d in $\mathbb{CP}^2$. So a homogeneous polynomial $\phi_C \in S^dT^*$ of degree d is the equation of a curve

$$C = \{\phi_C = 0\} \subset \mathbb{CP}^2, \quad \text{that is,} \quad C \in |d \cdot l|.$$

It is a classical enumerative problem in invariant theory to compute the degree $\deg V$ of some $\mathrm{PGL}(3, \mathbb{C})$ invariant subvariety $V \subset |d \cdot l|$.

Example 0: The discriminant hypersurface in $|d \cdot l|$:

$$V_{\text{sing}} = \{C \in |d \cdot l| \mid \text{Sing}C \neq \emptyset\}.$$

This is obviously a subvariety of $|d \cdot l|$ invariant under $\mathrm{PGL}(3, \mathbb{C})$, and an easy calculation shows that

$$\deg V_{\text{sing}} = 3 \cdot (d-1)^2.$$

Now a curve in $\mathbb{CP}^2$ which splits completely as a union of lines

$$\Delta_r = \bigcup_{i=1}^{r} l_i \tag{2.1}$$

is called a *polygon* or an *r-gon*. In §4 we will define an r-gon to be *regular* if all its sides l_i and all its vertices $l_i \cap l_j$ are distinct. Let $P_r \subset |d \cdot l|$ be the subvariety of all r-gons.

Useful exercise: What is $\deg P_r$?

Definition 2.1: *We say that a curve C circumscribes a regular r-gon Δ_r if for every pair (i, j) the vertex (= intersection of sides)*

$$l_i \cap l_j \in C.$$

Let

$$\mathrm{MP}_r^d = \{\Delta_r, C_d\} \subset P_r \times |d \cdot l| \tag{2.2}$$

be the closure of the incidence variety of pairs consisting of a regular r-gon Δ_r and a curve C_d of degree d circumscribing it. We have two projection maps:

$$\begin{array}{ccccc} & & \mathrm{MP}_r^d & & \\ & {}^{p_\Delta}\swarrow & & \searrow^{p_c} & \\ P_r & & & & |d \cdot l| \end{array} \tag{2.3}$$

Thus the subvariety

$$p_c(\mathrm{MP}_r^d) \subset |d \cdot l| \tag{2.4}$$

of curves of degree d circumscribing some r-gon is invariant under $\mathrm{PGL}(3, \mathbb{C})$.

Problem: What is $\deg p_c(\mathrm{MP}_r^d)$? More precisely, what is

$$s_r(d) = \deg p_c \cdot \deg p_c(\mathrm{MP}_r^d)? \tag{2.5}$$

In terms of the defining equations, it is easy to see that

$$(\Delta, C) \in \mathrm{MP}_r^{r-1} \iff \phi_C = \sum_{i=1}^{r} (\phi_\Delta / \phi_{l_i}) \tag{2.6}$$

where $\Delta = \bigcup_{i=1}^{r} l_i$.

Historically, the problem (2.5) is closely related to the following problem:

Definition 2.2: *We say that a polygon $\Delta = \bigcup_{i=1}^{r} l_i$ is apolar to a curve C if*

$$\phi_C = \sum_{i=1}^{r} (\phi_{l_i})^d. \tag{2.7}$$

Let

$$\mathrm{MPA}_r^d = \{\Delta_r, C_d\} \subset P_r \times |d \cdot l| \tag{2.8}$$

be the space of apolar pairs of polygons and curves. We again have two projection maps

$$\begin{array}{ccccc} & & \mathrm{MPA}^d_r & & \\ & p_\Delta \swarrow & & \searrow^{p_c} & \\ P_r & & & & |d \cdot l| \end{array} \tag{2.9}$$

Thus the subvariety

$$p_c(\mathrm{MPA}^d_r) \subset |d \cdot l| \tag{2.10}$$

is also invariant under $\mathrm{PGL}(3, \mathbb{C})$.

Problem: What is

$$c_r(d) = \deg p_c \cdot \deg p_c(\mathrm{MPA}^d_r)? \tag{2.11}$$

These problems were solved recently by Geir Ellingsrud and Stein Strømme. Using Bott's formula, they computed the constants $c_r(d)$ for $r < 9$ and $s_r(r-1)$ for $r = 6, 7, 8, 9, 10$.

For example, they find

$$\begin{aligned} 5! \cdot c_5(d) &= d^{10} - 100d^8 + 150d^7 + 3680d^6 - 10260d^5 \\ &\quad -52985d^4 + 224130d^3 + 127344d^2 \\ &\quad -1500480d + 1664640. \end{aligned} \tag{2.12}$$

The following particular cases of the general enumerative problem will be important for us:

Definition 3:

1. *A curve*
$$C \in p_c(\mathrm{MP}^d_{d+1})$$
is called a Darboux curve.

2. *A curve*
$$C \in p_c(\mathrm{MPA}^d_{d+1})$$
is called a Clebsch curve.

For special reasons, Darboux curves of degree 4 are called *Lüroth quartics*. These names have a historical explanation. Namely it is easy to see that the virtual (expected) dimension

$$\text{v.dimMP}^d_{d+1} = \text{v.dimMPA}^d_{d+1} = 3d + 2. \tag{2.13}$$

Remark: This dimension is one more than the dimension of the subvariety of rational curves of degree d.

But in 1865, Clebsch observed the following:

Clebsch's Theorem:

1. *The image $p_c(\text{MPA}^4_5)$ is a hypersurface in $|4 \cdot l| = \mathbb{P}^{14}$. (This is in spite of the fact that*

$$\text{v.dimMPA}^4_5 = \dim |4 \cdot l| = 14.)$$

2. *If C is nonsingular then*

$$C \in p_c(\text{MPA}^4_5) \iff p_c^{-1}(C) = \mathbb{P}^1.$$

3.

$$\deg p_c(\text{MPA}^4_5) = 6.$$

Exactly the same facts hold for MP^4_5, that is, for Lüroth quartics, as Lüroth observed in 1868. But the degree of the hypersurface of Lüroth quartics was only computed in 1918 by F. Morley [Mo]:

$$\deg p_c(\text{MP}^4_5) = 54. \tag{2.14}$$

This constant was reproduced in modern investigation (Tyurin, Le Potier, Ellingsrud and Strømme) under absolutely new motivations related to PDEs.

Remarks:

1. It follows from Clebsch's Theorem, that the polynomial (2.12) satisfies
$$c_5(4) = 0,$$
that is, 4 is a root of $s_5(d)$. Can you see this from the display of this polynomial (2.12)?

2. The fibres of the projection p_c of the diagram (2.9) were used by S. Mukai to describe special Fano varieties: let
$$p_c \, \mathrm{MP}_6^4 \to |4 \cdot l|$$
be the right side of the diagram (2.3). Then

(i) for general C, the inverse image $P_c^{-1}(C)$ is a Fano threefold;

(ii) if $C = 2q$ is a double nonsingular conic, then $P_c^{-1}(2q)$ is a compactification of $\mathbb{C}^3$, [Mu].

(iii) The exact formulas of Ellingsrud and Strømme also work when the degree of curve is not small with respect to the number of sides of polygons. More precisely, if $d \geq r-1$, S. Mukai proved that $p_c(\mathrm{MPA}_7^5) = |5 \cdot l|$, and that the map p_c is birational. But you can see that the constant $c_7(5)$ is negative.

3 Vector Bundles on an Algebraic Surface and Their Sections

Let me recall briefly the main constructions of sheaf theory on algebraic surfaces. The starting point is the structure sheaf $\mathcal{O}_S = \mathcal{O}$ of an algebraic surface S. For a first approach, it is enough to consider a nonsingular surface. Thus the stalk $\mathcal{O}_P$ of $\mathcal{O}$ at a point $p \in S$ is a 2-dimensional regular local ring. Every coherent sheaf F has a stalk F_P at each point $P \in S$, which is a module of finite type over $\mathcal{O}_P$; moreover, in a neighbourhood U of any point, there is a resolution
$$\mathcal{O}_U^p \to \mathcal{O}_U^q \to F_{|U} \to 0.$$

Thus each sheaf F on S defines a filtration

$$S_2 \subset S_1 \subset S$$

by the homological dimension of the stalk. If this filtration is trivial then F is called a *vector bundle* and we will note it as E.

For a sheaf F on S the canonical homomorphism

$$\mathrm{can} : F \to \mathrm{Hom}(\mathrm{Hom}(F, \mathcal{O}), \mathcal{O}) = F^{**} \tag{3.1}$$

can be completed to a 4-term exact sequence

$$0 \to T(F) \to F \xrightarrow{\mathrm{can}} F^{**} \to C(F) \to 0 \tag{3.2}$$

We say that F is a *torsion sheaf* if $F = T(F)$, a *torsion free sheaf* if $T(F) = 0$ and a *reflexive sheaf* if $F = F^{**}$.

It is easy to see that on a surface, *a reflexive sheaf is a vector bundle.* Moreover, for a torsion free sheaf F, we have $\dim \mathrm{Supp} C(F) = 0$; that is, in this case $C(F)$ is an *Artinian sheaf.*

A pair of sheaves F_1 and F_2 defines three vector spaces $\mathrm{Ext}^i(F_1, F_2)$, for $i = 0, 1, 2$ with the usual functorial properties.

In the short exact sequence of sheaves

$$0 \to F_2 \to F \to F_1 \to 0, \tag{*}$$

the sheaf F is called an *extension* of F_1 by F_2; such an extension is given by an element $e(F)$ in the vector space $\mathrm{Ext}^1(F_1, F_2)$, so the set of classes of such extensions has the structure of the vector space $\mathrm{Ext}^1(F_1, F_2)$. For the zero class $0 \in \mathrm{Ext}^1(F_1, F_2)$ we have $F = F_1 \oplus F_2$.

Exercises:

1. Prove that on an algebraic curve C, every coherent sheaf F is a direct sum
$$F = T(F) \oplus F^{**}.$$

2. Suppose that we have two extensions F and F' of F_1 by F_2 and F_1' by F_2, together with a homomorphism $\varphi\ F_1' \to F_1$. Then the identity map $F_2 = F_2$ and the given map φ extend to a homomorphism $F' \to F$ if and only if the homomorphism

$$\tilde{\varphi}\ \mathrm{Ext}^1(F_1, F_2) \to \mathrm{Ext}^1(F_1', F_2)$$

induced by φ satisfies

$$\tilde{\varphi}(e(F)) = e(F'). \tag{3.3}$$

Of course, we would prefer to work only with vector bundles, which is enough for working over algebraic curves. But over algebraic surfaces, it is absolutely necessary to use torsion free sheaves.

Any rank 1 torsion free sheaf J on an algebraic surface S admits an exact sequence of the form (3.2):

$$0 \to J \to \mathcal{O}_S(D) = J^{**} \to C(J) \to 0, \tag{3.4}$$

where D is some divisor on S, and we can untwist this sequence by tensoring with $\mathcal{O}_S(-D)$:

$$0 \to J(-D) \to \mathcal{O}_S \to \mathcal{O}_\xi \to 0. \tag{3.5}$$

The last sheaf is the structure sheaf of 0-dimensional subscheme (a *cluster*, or a "fine 0-cycle") ξ of S, and $J(-D) = \mathcal{I}_\xi \subset \mathcal{O}_S$ is the ideal sheaf of this subscheme. A cluster ξ defines a cycle of points

$$[\xi] = \sum \deg(\xi, p_i) \cdot p_i.$$

We say that ξ is *reduced* if we have

$$\deg(\xi, p_i) = 1 \quad \text{(or 0)} \quad \text{for every } i.$$

In this case $\xi = [\xi]$, and the cluster is a configuration of distinct points on S.

Thus a rank 1 torsion free sheaf admits two invariants: $c_1(J) = c_1(J^{**})$ and $c_2(J) = \deg \xi = h^0(\mathcal{O}_\xi)$.

Now let $s\ \mathcal{O}_S \to E$ be a section of a vector bundle E of rank 2. We say that a section is *regular* if its zero set is a 0-dimensional subscheme:

$$(s)_0 = \xi.$$

In this case, by definition, $\deg \xi = c_2(E)$.

Remark: If the zero set of a section contains an effective curve C, we can untwist it by $-C$ to obtain a regular section $s\,\mathcal{O}_S \to E(-C)$. In this case

$$\deg \xi = c_2(E) - C \cdot (c_1(E) - C).$$

For a rank 2 vector bundle E, the dual map to a regular section s can be extended to the Koszul resolution (as in David Eisenbud's lectures)

$$0 \to \bigwedge^2 E^* \xrightarrow{\wedge S^*} E^* \xrightarrow{s^*} \mathcal{O}_S \xrightarrow{\text{can}} \mathcal{O}_\xi \to 0 \tag{3.6}$$

of the zero set $(s)_0 = \xi$ of s.

The kernel of can is just the ideal sheaf of ξ from (3.5) and from the first part of the sequence (3.6) we get the exact sequence

$$0 \to \bigwedge^2 E^* \to E^* \to \mathcal{I}_\xi \to 0.$$

Tensoring this sequence by the invertible sheaf $\bigwedge^2 E = \det E$ we get finally the *short exact sequence of a regular section*:

$$0 \to \mathcal{O}_S \xrightarrow{s} E \to \mathcal{I}_\xi(c_1(E)) \to 0. \tag{3.7}$$

As we know, an extension of this type is given by an element $e \in \mathrm{Ext}^1(\mathcal{I}_\xi(c_1(E))$. For the last space, by Serre duality we have

$$\mathrm{Ext}^1(\mathcal{I}_\xi(D), \mathcal{O}_S) = \mathrm{Ext}^1(\mathcal{O}_S, \mathcal{I}_\xi(D + K_S))^* = H^1(\mathcal{I}_\xi(D + K_S))^* \tag{3.8}$$

where K_S is the canonical class of S.

Thus a pair (s, E) consisting of a vector bundle and a section is given by a cluster $(s)_0 = \xi$ and a hyperplane $p \subset H^1(\mathcal{I}_\xi(c_1(E) + K_S))$.

4 The First Interpretation – moduli spaces of stable pairs

Now we will consider the space MP^d_{d+1} of pairs (2.2) only. A polygon $\Delta = \sum l_i$ is called *regular* if $i \neq j \Rightarrow l_i \neq l_j$, and $l_i \cap l_j = l_k \cap l_n \Rightarrow (i,j) = (k,n)$. That is, all the sides of Δ are distinct, and all the vertices of Δ (= intersections of sides) are different too.

A pair (Δ, C) is called *regular* if Δ is regular and C is nonsingular.

Let P^0_r be the open subset of regular polygons. Then we have the open subset

$$M_0 P^d_{d+1} = p_\Delta(P^0_r)^{-1} \cap p_c^{-1}\Big(p_c(\mathrm{MP}^d_{d+1}) \setminus V_{\mathrm{sing}} \cap p_c(\mathrm{MP}^d_{d+1})\Big) \quad (4.1)$$

of regular pairs.

Every regular polygon $\Delta = \bigcup_{i=1}^r l_i$ defines a cycle of points

$$\Delta^* = l_1^* + \cdots + l_r^* \quad (4.2)$$

on the *dual plane* $\mathbb{P}^{2*}$. It is a fine cycle, and we want to consider it as a cluster (0-dimensional subscheme) of the dual plane.

On the other hand, Δ also determines the cycle of vertices

$$\mathrm{Ver}\Delta = \bigcup_{i,j} l_i \cap l_j \quad (4.3)$$

on the plane $\mathbb{CP}^2$ itself. The cluster Δ^* defines an ideal sheaf $\mathcal{I}_{\Delta^*}$, and the family of extensions

$$0 \to \mathcal{O}_{\mathbb{P}^{2*}} \to E \to \mathcal{I}_{\Delta^*}(2) \to 0 \quad (4.4)$$

parametrized by the space $\mathbb{P}H^1(\mathcal{I}_{\Delta^*}(-1))^*$ (see the end of the previous section).

From the exact sequence

$$0 \to \mathcal{I}_{\Delta^*}(-1) \to \mathcal{O}_{(\mathbb{CP}^2)^*}(-1) \to \mathcal{O}_{\Delta^*}(-1) \to 0$$

we get the isomorphism

$$H^1(\mathcal{I}_{\Delta^*}(-1) = H^0(\mathcal{O}_{\Delta^*}(-1)), \tag{4.4'}$$

so the extension (4.4) is given by a hyperplane in $H^0(\mathcal{O}_{\Delta^*}(-1))$.

On the other hand, the space of curves of degree d circumscribing Δ is the following:

$$|d \cdot l - \mathrm{Ver}\Delta| = \mathbb{P}H^0(\mathcal{I}_{\mathrm{Ver}_\Delta}(d)). \tag{4.5}$$

It's easy to see that ranks of the spaces (4.4') and (4.5) are equal. We would like to prove that

$$H^1(\mathcal{I}_{\Delta^*}(-1) = H^0(\mathcal{I}_{\mathrm{Ver}_\Delta}(d))^*. \tag{4.6}$$

Let me emphasize again that on the left-hand side we have a sheaf on $\mathbb{CP}^2$ but on the right-hand side we have a sheaf on the *dual* plane $(\mathbb{CP}^2)^*$.

Remark: Actually the proof of this equality is a very good exercise for David Eisenbud's lectures.

Here is the heart of our lectures: for a geometric object (Δ, C) on the plane $\mathbb{CP}^2$ we get a new interpretation as a pair (s, E) (see the end of the §3) on the dual plane $(\mathbb{CP}^2)^*$.

Let $\mathbb{CP}^2 = \mathbb{P}T$, where $T = \mathbb{C}^3$, and $(\mathbb{CP}^2)^* = \mathbb{P}T^*$. Let

$$H^1(\mathcal{I}_{\Delta^*}(k)) = V_k. \tag{4.7}$$

Then every line l on $(\mathbb{CP}^2)^*$ defines a homomorphism

$$H^1(\mathcal{I}_{\Delta^*}(-1)) = V_{-1} \to H^1(\mathcal{I}_{\Delta^*}) = V_0 \tag{4.8}$$

given by multiplication by ϕ_l.

When l sweeps out $\mathbb{P}T = ((\mathbb{CP}^2)^*)^*$, we get a homomorphism

$$T \otimes V_{-1} \to V_0, \tag{4.9}$$

which we can consider as a homomorphism of vector bundles on $\mathbb{CP}^2$!

$$V_{-1} \otimes \mathcal{O}_{\mathbb{CP}^2}(-1) \xrightarrow{\varphi} V_0 \otimes \mathcal{O}_{\mathbb{CP}^2}. \tag{4.10}$$

The homomorphism φ is nothing other than a $(d+1) \times d$ matrix of linear forms on $\mathbb{CP}^2$, which we can extend to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{CP}^2}(-d-1) \to V_{-1} \otimes \mathcal{O}_{\mathbb{CP}^2}(-1) \xrightarrow{\varphi} V_0 \otimes \mathcal{O}_{\mathbb{CP}^2} \to \mathrm{coker} \to 0 \tag{4.11}$$

It is easy to see that Suppcoker = VerΔ.

Now applying the functor $\mathrm{Hom}(*, \mathcal{O}_{\mathbb{CP}^2}(-1))$ to this exact sequence, we get *Eagon–Northcott resolution*

$$V_0^* \otimes \mathcal{O}_{\mathbb{CP}^2}(-1) \to V_{-1}^* \otimes \mathcal{O}_{\mathbb{CP}^2} \to \mathcal{I}_{\mathrm{Ver}\Delta}(d) \to 0 \tag{4.12}$$

of the ideal sheaf $\mathcal{I}_{\mathrm{Ver}\Delta}(d)$.

Remark: Our Eagon-Northcott resolution is a slight generalization of the Koszul complex (see Eisenbud's lectures).

Now the cohomology long exact sequence of (4.11) provides the required equality (4.6) and an embedding

$$M_0 P^d_{d+1} \hookrightarrow \mathcal{MP}(2, 2, d+1) \tag{4.13}$$

to the moduli space $\mathcal{MP}(2, 2, d+1)$ of *stable pairs* (s, E) where E is a vector bundle of rank 2 with $c_1 = 2$, $c_2 = d+1$. Here the zero set of s is a simple cluster in the dual plane, that is, a $(d+1)$-gon in $\mathbb{CP}^2$.

Now the left-hand side of (4.13) admits a projection map p_c to $|d \cdot l|$, and the right-hand side admits the projection on the second component – the vector bundle.

To compare these projections and to compute the fibres of p_c, we have to consider a new geometric object, the *noncommutative plane.*

5 Noncommutative Planes

For any pair $(\Delta, C) \in M_0 P^d_{d+1}$, the nonsingular curve C contains the effective divisor

$$\mathrm{Ver}\Delta = \bigcup_{i,j} (l_i \cap l_j) \tag{5.1}$$

(4.3) of degree $\frac{1}{2}d(d+1)$. Let $\mathcal{O}_C(h) = \mathcal{O}_{\mathbf{CP}^2}(1)_{|C}$.

Lemma 5.1: *The divisor class*

$$\mathrm{Ver}\Delta - 2h = \theta \tag{5.2}$$

is a regular theta characteristic of C. That is, $2\theta = K_C$ is the canonical class of C, and $h^0(\mathcal{O}_C(\theta)) = 0$, in other words, this theta characteristic is ineffective.

Proof: Consider the polygon as a curve of degree $d+1$. Then the support of the intersection

$$\mathrm{Supp}(\Delta \cdot C) = \mathrm{Ver}(\Delta),$$

because for every line l_i the intersection

$$C \cap l_i = l_i \cap (\bigcup_{j \neq i} l_j).$$

(Both sides have degree d and by definition the curve C contains the set in the right-hand side of this equality). Now by definition

$$\mathrm{Ver}\Delta = \mathrm{Sing}\Delta$$

and every singular point of Δ is quadratic. Hence as divisor classes on C, we have

$$2\mathrm{Ver}\Delta = C \cdot \Delta = (d+1)h,$$

and

$$2\mathrm{Ver}\Delta - 4h = (d-3)h = K_C$$

by the adjunction formula.

Now if $\mathrm{Ver}\Delta - 2h = \eta$ is effective then

$$\eta = (d-1)h - \mathrm{Ver}\Delta$$

and there exists a curve C' of degree $(d-1)$ which contains $\mathrm{Ver}\Delta$. But then C' and Δ have a common component, because

$$C' \cdot \Delta \geq 2 \deg \mathrm{Ver}\Delta = d(d+1) > \deg C' \cdot \deg \Delta = (d-1)\cdot(d+1).$$

Thus

$$C' = C_0 + \bigcup_{i=1}^{n} l_i,$$

where C_0 doesn't contain lines. Repeating this arguments for C_0 and Δ_{d+1-n}, we get a contradiction. Q.E.D.

Now the pair (C, θ) defines a net of quadrics. Namely, if θ is an ineffective theta characteristic on C then the complete linear system $|\theta + h|$ is base point free and $h^0(\mathcal{O}_C(\theta + h)) = d$. Consider $\mathcal{O}_C(\theta + h)$ as a $\mathcal{O}_{\mathbb{CP}^2}$-sheaf, and the canonical surjective map

$$H^0(\mathcal{O}_C(\theta + h)) \otimes \mathcal{O}_{\mathbb{CP}^2} \to \mathcal{O}_C(\theta + h) \to 0.$$

We have the exact sequence

$$0 \to \ker \xrightarrow{\alpha} H^0(\mathcal{O}_C(\theta + h)) \otimes \mathcal{O}_{\mathbb{CP}^2} \to \mathcal{O}_C(\theta + h) \to 0,$$

and it is easy to see that

$$\ker = H^0(\mathcal{O}_C(\theta + h))^* \otimes \mathcal{O}_{\mathbb{CP}^2}(-1)$$

and we have the net of correlations

$$\alpha\; H \otimes \mathcal{O}_{\mathbb{CP}^2}(-1) \to H^* \otimes \mathcal{O}_{\mathbb{CP}^2} \tag{5.3}$$

where $H = H^0(\mathcal{O}_C(\theta + h))^*$.

Under any identification $H = H^*$ and a choice of the homogeneous coordinates $(\lambda_0, \lambda_1, \lambda_2)$ of $\mathbb{CP}^2$, we can consider the homomorphism (5.3) as a linear combination of a triple of symmetric $d \times d$ matrices

$$\lambda_0 \cdot A_0 + \lambda_1 \cdot A_1 + \lambda_2 \cdot A_2, \tag{5.4}$$

where the equation of curve C is

$$\varphi_C = \det(\lambda_0 \cdot A_0 + \lambda_1 \cdot A_1 + \lambda_2 \cdot A_2) \tag{5.5}$$

and we can consider the line bundle $\mathcal{O}_C(\theta + h)$ as the family of cokernels of the net of correlations (5.3).

Now the group $\mathrm{GL}(d, \mathbb{C})$ acts on the set of triples in the usual way:

$$g(A_0, A_1, A_2) = (gA_0g^*, gA_1g^*, gA_2g^*) \tag{5.6}$$

Let $\{(A_0, A_1, A_2)\}^{\mathrm{SS}}$ be the set of semistable points with respect to this action. Then the variety

$$\{(A_0, A_1, A_2)\}^{\mathrm{SS}} \ / \ \mathrm{PGL}(d, \mathbb{C}) = \mathcal{P}_d^2 \tag{5.7}$$

is called the *noncommutative plane.*

C.T.C. Wall proved that

$$\begin{gathered}(A_0, A_1, A_2) \in \{(A_0, A_1, A_2)\}^{\mathrm{SS}} \Rightarrow \\ \phi_C = \det(\lambda_0 \cdot A_0 + \lambda_1 \cdot A_1 + \lambda_2 \cdot A_2) \neq 0.\end{gathered} \tag{5.8}$$

Thus we have a regular map

$$p_c\, \mathcal{P}_d^2 \rightarrow |d \cdot l| \tag{5.9}$$

sending a triple to ϕ_C (5.5). Thus

$$\deg p_c = 2^{g-1} \cdot (2^g + 1), \quad \text{where } g = \tfrac{1}{2}(d-1)(d-2)$$

is the number of even theta characteristics of a nonsingular plane curve of degree d.

Now assume that a triple (A_0, A_1, A_2) satisfies $\det A_0 \neq 0$, and consider the skew symmetric matrix

$$[A_0, A_1 \wedge A_2] = A_1 \cdot A_0^{-1} \cdot A_2 - A_2 \cdot A_0^{-1} \cdot A_1. \tag{5.10}$$

It easy to see that the rank of this matrix rankα is an invariant of a class of a net of quadrics. Thus we have a filtration

$$\mathcal{P}_d^2(0) \subset \mathcal{P}_d^2(2) \subset \cdots \subset \mathcal{P}_d^2, \tag{5.11}$$

where

$$\mathcal{P}_d^2(2r) = \{(A_0, A_1, A_2) \mid \operatorname{rank}[A_0, A_1 \wedge A_2] \leq 2r\}.$$

Now we have the pseudoclassical

Problem: What is

$$\deg p_c(\mathcal{P}_d^2(2r)) \cdot \deg p_c? \tag{5.12}$$

The relationship between our geometric objects is following

Proposition 5.1:

1. $p_c(\mathrm{MP}_{d+1}^d) \subset p_c(\mathcal{P}_d^2(2))$;
2. $d \leq 5 \Rightarrow p_c(\mathrm{MP}_{d+1}^d) = p_c(\mathcal{P}_d^2(2))$.

To prove these statements, we need a final interpretation of the geometric objects.

Consider the flag diagram

$$\begin{array}{ccc} & F = \{p \in l\} & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathbb{CP}^2 & & (\mathbb{CP}^2)^* \end{array} \tag{5.13}$$

Then we can apply the functor $q_* \circ p^*$ for any net of quadrics α (5.3). We get the exact sequence of sheaves on the dual plane $(\mathbb{CP}^2)^*$:

$$0 \to H \otimes \mathcal{O}_{(\mathbb{CP}^2)^*}(-1) \overset{q_* \circ p^*(-1)(\alpha(1))}{\longrightarrow} H^* \otimes \Omega(1) \to \operatorname{coker}\alpha \to 0 \tag{5.14}$$

and as second invariant of a net we have the number

$$\operatorname{rank}\operatorname{Hom}(\operatorname{coker}\alpha, \mathcal{O}_{(\mathbb{CP}^2)^*}).$$

But actually it isn't a new invariant, because of the following result:

Barth's theorem (see [B]):

$$\mathrm{rank}\mathrm{Hom}(\mathrm{coker}\alpha, \mathcal{O}_{(\mathbb{CP}^2)^*}) = d - \mathrm{rank}\alpha$$

where $2r$ *is the rank of the net of quadrics* (5.10)–(5.11).

Thus for every net α of rank 2 on $\mathbb{CP}^2$ we have the complex on its dual plane

$$0 \to H \otimes \mathcal{O}_{(\mathbb{CP}^2)^*}(-1) \overset{q_* \circ p^*(-1)(\alpha(1))}{\longrightarrow} H^* \otimes \Omega(1) \overset{\mathrm{can}}{\longrightarrow} \mathbb{C}^{d-2} \otimes \mathcal{O}_{(\mathbb{CP}^2)^*} \to 0 \qquad (5.15)$$

which is called a *monad*, and the middle cohomology

$$\ker \mathrm{can}/\mathrm{im} q_* \circ p^*(-1)(\alpha(1)) = E \qquad (5.16)$$

is a semistable torsion free sheaf of rank 2 with the Chern classes $c_1 = 0$, $c_2 = 2$.

So we have the map

$$\mathcal{P}_d^2(2)) \to \overline{M(2,0,d)}$$

to the Gieseker closure of the moduli space of stable vector bundles on $\mathbb{CP}^2$. The construction of the inverse map is as follows: a point $p \in \mathbb{CP}^2$ gives a line $l_p \in (\mathbb{CP}^2)^*$ in the dual plane. A line $l_p \in (\mathbb{CP}^2)^*$ is called a *jumping line* for E if

$$E_{|l_p} \neq \mathcal{O}_{l_p} \oplus \mathcal{O}_{l_p}, \quad \text{which happens iff} \quad h^0(E(-1)_{|l_p}) \neq 0. \qquad (5.17)$$

Thus in the dual plane $(\mathbb{CP}^2)^*$ we have the curve $C(E)$ of jumping lines of E.

Now it is easy to see that $H^1(E(-2)) = H = \mathbb{C}^d$ and the Serre dual space $H^1(E(-1)) = H^*$. Now multiplication by the equation of any line ϕ_{l_p} defines the correlation

$$H^1(E(-2)) = H \to H^1(E(-1)) = H^* \qquad (5.18)$$

as an element of the net of correlations (5.4) $C(E) = C(5.5)$.

Now consider a regular pair $(\Delta, C) \in M_0P^d_{d+1}$ (see (5.1)), where $\Delta = \bigcup l_i$. Then we have the chain of identifications:

$$(\Delta, C) = (\mathbb{C}^* \cdot s, E) \tag{5.19}$$

where s is a regular section of E;

$$E = (C, \theta) = (\alpha) \tag{5.19'}$$

So a pair (Δ, C) is geometrically equivalent to the exact sequence on $(\mathbb{CP}^2)^*$:

$$0 \to \mathcal{O}_{(\mathbb{CP}^2)^*}(-1) \to E \to \mathcal{I}_{\Delta^*}(1) \to 0 \tag{5.20}$$

where $\mathcal{I}_\Delta$ is the ideal sheaf of 0-dimensional cycle Δ^* on $(\mathbb{CP}^2)^*$.

Corollary: *The space of circumscribed $(d+1)$-gons to a Darboux curve C is a rational irreducible variety.*

Indeed, it is birationally equivalent to $\mathbb{P}H^0(E)$!

These geometric identifications were done for "regular" geometric objects. It is reasonable to construct some "natural" nonsingular compactification of the space of regular objects sending the computation of constants of type (5.12), (2.11), (2.5) to the regular procedure of computations of Chern classes of standard vector bundles on "moduli space" our geometric figures.

6 Compactifications

A regular $(d+1)$-gon (2.1) is of course a curve of degree $d+1$ on $\mathbb{CP}^2$, and so the space P_{d+1} of all polygons is a compact irreducible subvariety in the complete linear system $|(d+1) \cdot l|$. Geometrically,

$$P_{d+1} = S^{d+1}(\mathbb{CP}^2)^* \tag{6.1}$$

is the $(d+1)$-st symmetric power of $(\mathbb{CP}^2)^*$, a rather singular algebraic variety. Fortunately for algebraic surfaces, there is a canonical desingularization of it. Let us recall that a regular polygon Δ defines a zero dimensional subscheme Δ^* (4.2). Thus as a compactification of the space of regular polygons on $\mathbb{CP}^2$, we can consider the moduli space of zero dimensional subscheme of $(\mathbb{CP}^2)^*$ of degree d:

$$\mathrm{Hilb}^{d+1} = M(1,0,d+1) \qquad (6.2)$$

which is called the *Hilbert scheme* on $(\mathbb{CP}^2)^*$. The beautiful and very important theory of Hilbert schemes says that for a nonsingular algebraic surface, this scheme is again nonsingular (see [F]). The space of extensions of type (5.20) is given by the projectivization of the vector space $H^1(\mathcal{I}_\xi(-1))$ (see (3.8)), because of $c_1(E(1)) = 2h$, and because the canonical class of the plane is given by $K_{(\mathbb{CP}^2)^*} = -3h$. Thus it is natural to represent the space of all nontrivial extensions (5.20) as a projectivization of a vector bundle on Hilb^{d+1}. Of course this variety is nonsingular.

From now on, all of our geometric objects are defined on the dual projective plane $(\mathbb{CP}^2)^*$, and we omit the star.

First of all, on the Hilbert scheme we have the special divisor class H defined by clusters intersecting a fixed line. On the other hand, the Hilbert scheme defines the universal subscheme $Z \subset \mathbb{CP}^2 \times \mathrm{Hilb}$, and the two projection maps to the direct components define the diagram

$$\begin{array}{ccc} & Z_{d+1} & \\ {}^{p_{\mathbb{CP}^2}}\swarrow & & \searrow^{p_H} \\ \mathbb{CP}^2 & & \mathrm{Hilb}^{d+1} \end{array} \qquad (6.3)$$

For any divisor class $\mathcal{O}_{\mathbb{CP}^2}(k)$, consider the vector bundle

$$\mathcal{E}_k = R^0 p_H(p_S^* \mathcal{O}_{\mathbb{CP}^2}(k)). \qquad (6.4)$$

These sheaves are locally free, because the canonical homomorphism is surjective.

In particular, in our case $k = -1$, the fibre of this vector bundle over $\xi \in \mathrm{Hilb}$ is $H^0(\mathcal{O}_\xi(-1))$. Now the cohomology long exact sequence of

$$0 \to \mathcal{I}_\xi(-1) \to \mathcal{O}(-1) \to \mathcal{O}_\xi \to 0 \qquad (6.5)$$

gives the isomorphism

$$H^0(\mathcal{O}_\xi(-1)) = H^1(\mathcal{I}_\xi(-1)) \tag{6.6}$$

Thus the space of all extensions of type (5.20) is the projectivization

$$\mathbb{P}\mathcal{E}^*_{-1}. \tag{6.7}$$

The next thing we have to understand is that our constant $s_{d+1}(d)$ (2.5) is the top Segre class of the standard vector bundle on Hilb:

$$s_{d+1}(d) = s_{\text{top}}(\mathcal{E}_{-1}(H)) \tag{6.8}$$

To see this, we have to return to the isomorphism (4.6) and remark that for $s \in H^0(\mathcal{I}_{\text{Ver}_\Delta}(d)$ the condition "a curve $C = (s)_0$ passes through a point $l^* \in (\mathbb{CP}^2)^*$" defines a section of $\mathcal{E}_{-1}(H)$! Thus by the definition of the Segre class, $3d+2$ general points determine a general $(3d+2)$-dimensional subspace W of $H^0(\mathcal{E}_{-1}(H))$. Now the canonical homomorphism

$$W \otimes \mathcal{O}_{\text{Hilb}} \xrightarrow{\text{can}} \mathcal{E}_{-1}(H)$$

is general enough and

$$\deg \operatorname{coker} \text{can} =: s_{\text{top}}(\mathcal{E}_{-1}(H)) = s_{d+1}(d) \tag{6.9}$$

is the number of Darboux curves through $3d+2$ general points.

Using this beautiful interpretation, G. Ellingsrud and S. Strømme computed (2.12) (using Bott's formula for the $\mathbb{C}^*$ action on Hilb, see [E–S]):

$$\begin{aligned}
s_5(4) &= 54 \\
s_6(5) &= 2540 \\
s_7(6) &= 583020 \\
s_8(7) &= 99951390 \\
s_9(8) &= 16059395240 \\
s_{10}(9) &= 2598958192572.
\end{aligned}$$

This list can be extended if your computer is good enough and you have S. A. Strømme as a collaborator. But to understand the nature of these numbers (it's new shape of mathematical questions, isn't it?), you have to use new identifications, proposed below, and the collection of beautiful new results provided by differential topologists such as Fintushel and Stern, Kotschick and Lisca and many others. We will discuss this in the next section.

Thus this story isn't finished yet. The nonsingular compactification (6.7) of the moduli space of pairs (Δ, C) (on $(\mathbb{CP}^2)^*!$) is called the moduli space of stable pairs $(\mathbb{C}^* \cdot s, E)$ (see [T3], Lecture 6):

$$\mathbb{P}\mathcal{E}^*_{-1} = \mathcal{MP}(2,2,d+1). \tag{6.10}$$

Let us consider the general diagram of our identifications:

$$\begin{array}{ccccccc} & \mathbb{P}\mathcal{E}^*_{-1} & = & \mathcal{MP}(2,2,d+1) & & & \\ {}^{p_\Delta}\swarrow & & \searrow^{p_c} & & & \searrow^{p_E} & \\ \mathrm{Hilb}^{d+1} & & |d \cdot l| & & \overleftarrow{p_j} & & M(2,2,d+1) \end{array} \tag{6.11}$$

where $M(2,2,d+1)$ is the moduli space of semistable bundles, the map p_E sends a pair (E,s) to the vector bundle E, and p_j sends the vector bundle E to its curve of jumping lines (5.17). Now twisting sheaves by $\mathcal{O}_{\mathbb{CP}^2}(1)$ gives the isomorphism

$$M(2,0,d) = M(2,2,d+1)$$

and we can use the chain of identifications (5.19).

Remark: The extension of the map p_j to the compactification of moduli spaces is a nontrivial task.

The map p_E is only a rational map, as treated in detail in [T–T]: we described there what has to be blown up and what gets blown down. But in any case, we can describe the image of this rational map: the final moduli space contains the Brill–Noether locus

$$M_1 = \{E \in M(2,2,d+1) \mid h^0(E) \geq 1\}. \tag{6.12}$$

Thus we have

$$M_1 = p_E(\mathcal{MP}(2,2,d+1)) \tag{6.13}$$

This is just what we need. Now by the R–R theorem,

$$d \leq 5 \Rightarrow M_1 = M(2,2,d+1). \tag{6.14}$$

Moreover, if $d = 4$, that is, for the Lüroth quartics we have the following construction: let Q be a nonsingular plane conic and $|\eta|$ be a general linear pencil of divisors of degree 5 on Q. For an element of this pencil $p_1 + \cdots + p_5$, consider the pentagon $\Delta = \bigcup l_i$, where l_i is the tangent line to Q at p_i. When elements sweep out this pencil, the cycles Ver$l(\Delta)$ sweep out a quartic curve C.

So we have the divisor classes

$$\mathcal{O}_{\mathbf{P}\mathcal{E}^*_{-1}}(1) = p_c^*(\mathcal{O}_{|d\cdot l|}(1)) = \mathcal{O}_{\mathcal{MP}(2,2,d+1)}(D) \tag{6.15}$$
$$\mu(l) = p_j^*(\mathcal{O}_{|d\cdot l|}(1)) \tag{6.16}$$

on $\mathcal{MP}(2,2,d+1)$ and M_1, which are related by the birational map p_E. This situation is a beautiful exercise in practical birational geometry. As you know from Miles Reid's lectures, a birational map may well alter the degree of a divisor. But in our case (this beautiful observation is due to Dmitry Orlov) the existence of the regular maps p_c and p_j relating our divisors as in (6.15) and (6.16) gives the equality

$$s_{\text{top}}(\mathcal{E}_{-1}(H)) = c_{d+1}(d) = D^{3d+2} = (\mu(l))^{3d+2} \tag{6.17}$$

The last dearth of this beautiful chain of identifications of geometric objects is the following: the points of each of the moduli spaces $\mathcal{MP}(2,2,d+1)$, $M(2,2,d+1)$ and M_1 describe geometric objects on the same algebraic surface, the plane $\mathbb{CP}^2$. But the construction of the divisor class $\mu(l)$ sends us to geometric objects (jumping curves) on the dual plane $(\mathbb{CP}^2)^*$. From the *algebraic geometric* point of view this is reasonable, but we promised to extend these constructions to objects of *differential geometry*. As a differential topological object, the projective plane doesn't define the dual plane. Avoiding this obstacle, we would like to describe the constants $s_{d+1}(d)$ in terms of $\mathbb{CP}^2$. That is, we want to define μ-class $\mu(l)$ in terms of objects on $\mathbb{CP}^2$ only.

Now on the direct product $\mathbb{CP}^2 \times M(2,2,d+1)$, the universal sheaf $\mathcal{F}$ exists locally only (for technical details see [T2]), but the Pontrjagin class

$$p_1(\mathcal{F}) = 4c_2(\mathcal{F}) - c_1^2(\mathcal{F}) \in \mathrm{Pic}\mathbb{CP}^2 \otimes \mathrm{Pic}M(2,2,d+1) \quad (6.18)$$

is defined correctly in any case. The intersection number on $\mathrm{Pic}\mathbb{CP}^2$ gives the isomorphism $(\mathrm{Pic}\mathbb{CP}^2)^* = \mathrm{Pic}\mathbb{CP}^2$ so we can consider the Pontrjagin class (6.18) as the homomorphism

$$\mu = 1/4p_1(\mathcal{F}) : \mathrm{Pic}\mathbb{CP}^2 \to \mathrm{Pic}M(2,2,d+1). \quad (6.19)$$

Now it's easy to see that the divisor class $\mu(l)_{|M_1}$ is just (6.16).

Now to get our classical enumerative algebraic geometry constants (2.11) in the more general set-up, we have to use the equality

$$c_{d+1}(d) = (\mu(l))^{3d+2}, \quad (6.20)$$

and to extend the definitions of $M_1 \subset M(2,2,d+1)$ and $\mu(l)$ in differential geometric terms. Of course now $M_1 \subset M(2,2,d+1)$ will be compact spaces and $\mu(l) \in H^2(M(2,2,d+1)$ is a 2-cohomology class only but this is quite enough to define the constant (6.20)!

7 Differential Geometry

Algebraic geometry can be considered as a part of differential geometry, namely as Kähler geometry. Then we have to use new notions like connections, differential forms and so on; you can learn this approach from the standard monograph [G–H]. We also strongly recommend the monograph [D–K] as a unique source of new style to use these ideas. But it is very important to understand that classical algebraic geometry is a foundation of almost all the local constructions of Riemannian geometry. We will discuss the special case of the projective plane $\mathbb{CP}^2$, but you can determine the generality quantor yourself.

Let M be the underlying 4-manifold of the complex projective plane $\mathbb{CP}^2$. Any Riemannian metric g on M defines a decomposition of the complexified tangent bundle $TM_{\mathbb{C}}$ as a tensor product

$$TM_{\mathbb{C}} = (W^-)^* \otimes W^+$$

of two rank 2 Hermitian vector bundles $W^\pm$ with

$$c_1(W^\pm) = -3h,$$

where h is the generator of $H^2(M, \mathbb{Z})$.

Write $*$ for the Hodge star operator on $\Omega^2(M)$ determined by the metric g. Moreover, for any U(2)-bundle E on M of topological type $(c_1 = 2, c_2 = d+1)$ and any Hermitian connection $a \in \mathcal{A}_h$ on E, putting any Hermitian connection ∇_0 on $\bigwedge^2 W^\pm$ gives a coupled Dirac operator

$$D_a^{g,\nabla_0}\ \Gamma^\infty(E \otimes W^+) \to \Gamma^\infty(E \otimes W^-). \tag{7.1}$$

Now the orbit space of irreducible connections modulo the gauge group

$$\mathcal{B}(2,2,d+1) = \mathcal{A}_h^*(2,2,d+1)/\mathcal{G}$$

contains the subspace

$$\begin{aligned} \mathcal{M}^g(2,2,d+1) &= \{(a) \in \mathcal{M}^g(2,2,d+1) \mid *F_a = -F_a\} \\ &\subset \mathcal{B}(E) \end{aligned} \tag{7.2}$$

of antiselfdual connections with respect to the Riemannian metric g (here F_a is the curvature form of a connection a).

Now we can consider the subspace of jumping connections:

$$\begin{aligned} \mathcal{M}_1^g(d) &= \{(a) \in \mathcal{M}^g(2,2,d+1) \mid \operatorname{rank}\ker D_a^{g,\nabla_0} \geq 1\} \\ &\subseteq \mathcal{M}^g(2,2,d+1). \end{aligned} \tag{7.3}$$

The virtual codimension of $\mathcal{M}^g_{(d)}$ (that is, the expected codimension determined by the Atiyah–Singer index theorem) is given by

$$\text{v.codim}\mathcal{M}_1^g(d) = 2 - 2\chi = 2(5-d), \tag{7.4}$$

where χ is the index of the coupled Dirac operator (7.1), which depends only on the Chern classes of E.

So you can see that for $d \leq 5$

$$\mathcal{M}_1^g(d) = \mathcal{M}^g(2,2,d+1) \tag{7.5}$$

Please compare this fact with (6.14)!

For a generic metric g, the moduli spaces $\mathcal{M}^g(2,2,d+1)$ and $\mathcal{M}_1^g(d)$ (7.3) are smooth manifolds of the expected dimension with regular ends (see [D–K] and [P–T], Chap. 2, §3). Moreover, $\mathcal{M}^g(2,2,d+1)$ admits a natural orientation (see [D–K]) inducing an orientation on $\mathcal{M}_1^g$, because its normal bundle has a natural complex structure. This orientation is described in detail in [P–T], Chap. 1, §5.

Moreover there exists the so-called Uhlenbeck compactification of our moduli spaces

$$\overline{\mathcal{M}^g(2,2,d+1)} \supseteq \overline{\mathcal{M}_1^g(d)} \tag{7.6}$$

Now for any element of our filtration the first Pontrjagin class of the universal connection on the direct product $M \times \mathcal{M}^g(2,2,d+1)$ (by the slant product) defines cohomological correspondences

$$\begin{aligned} \mu_d : H_i(M,\mathbb{Z}) &\rightarrow H^{4-i}(\overline{\mathcal{M}^g(2,2,d+1)},\mathbb{Z}), \\ \mu_d^1 : H_i(M,\mathbb{Z}) &\rightarrow H^{4-i}(\overline{\mathcal{M}_1^g(d)},\mathbb{Z}), \end{aligned}$$

and two collections of numbers

$$D_g(d) = (\mu_d(h))^{4d-3}, \tag{7.7}$$

the so-called *Donaldson numbers* (Donaldson polynomials) of $\mathbb{CP}^2$, and

$$s\gamma_g^d = (\mu_d^1(h))^{3d+2}. \tag{7.8}$$

Now suppose as a special case that our metric g is the Fubini–Study metric g_{F-S}. In this case, by the Donaldson–Uhlenbeck identification theorem, we have

$$\mathcal{M}^{g_{F-S}}(2,2,d+1) = M(2,2,d+1),$$

where the right-hand side is the moduli space of holomorphic stable bundles on $\mathbb{CP}^2$ (6.11).

Making this identification $(a) = E$, we have identifications

$$\ker D_a^{g_{F-S}} = H^0(E) \oplus H^2(E) \text{ and } \operatorname{coker} D_a^{g_{F-S}} = H^1(E), \quad (7.9)$$

where $H^i(E)$ denote coherent cohomology groups (see [D–K]).

But by Serre duality $H^2(E) = H^0(E(-2))^* = 0$, by the stability of E. Thus the subspace $\mathcal{M}_1^{g_{F-S}}(d)$ is

$$\mathcal{M}_1^{g_{F-S}}(d) = \{E \in M(2,2,d+1) \mid h^0(E) \geq 1\}, \quad (7.10)$$

that is (see (6.12)),

$$\mathcal{M}_1^{g_{F-S}}(d) = M_1$$

and our constants (7. 8)

$$s\gamma_{F-S}^d = s_{d+1}(d) \quad (7.11)$$

are constants (2.5) and (6.8).

These integers don't depend on the metric g, because the space of all Riemannian metrics is contractible. (In fact, to be rigorous, we have to use much more sophisticated bordism arguments, similar to those in [D–K], where the same statement was proved for the Donaldson's numbers (7. 7)).

Let us remark that the initial terms of both collections (7.7) and (7.8) are coincidence by (7.5) and (6.14)

$$D_g(4) = s\gamma_g^4 = 54;$$
$$D_g(5) = s\gamma_g^5 = 2540;$$

and the collection of Donaldson's constants was extended to infinity by Ellingsrud and Göttsche (see [E–G]):

$$D_g(6) = 233208;$$
$$D_g(7) = 35825553;$$
$$D_g(8) = 8365418914;$$
$$D_g(9) = 2780195996868;$$
$$D_g(10) = 12535588470906000;$$

etc.

Let me remark that we have got the following striking fact:

Theorem: *The constants* (7.11) = (2.5) = (6.8) *are invariants of the underlying differentiable structure of* $\mathbb{CP}^2$.

It is well known that the complex structure on M is unique. Recently it was proved that the symplectic structure on M is unique. The next question is the following differentiable version of the Poincaré conjecture for $\mathbb{CP}^2$:

Conjecture (DPC for $\mathbb{CP}^2$): *The complex projective plane* $\mathbb{CP}^2$ *has a unique differentiable structure.*

In particular this statement would imply the following fact

Corollary: *The constants* (7.11) = (2.5) = (6.8) *are invariants of the topological structure of* $\mathbb{CP}^2$.

There is overwhelming direct evidence for this statement, and hence a partial confirmation of Conjecture DPC for $\mathbb{CP}^2$. Namely there are two possible methods to construct our constants using the topological structure of the plane only. The first approach is related to the following fundamental problem:

Hilbert scheme problem: *Can we give a purely topological construction of the Hilbert scheme* Hilb^d *and of the standard vector bundles* (6.3)*?*

Of course this problem is interesting in full generality for all 4-manifolds. If we could realize the scheme (6.3) topologically then, as proposed by G. Ellingsrud, we could use induction over d to prove

Proposition 7.1: *The constants* (6.8) *are topological invariants.*

The second approach to prove the statement of Corollary is related to proving of the same fact for Donaldson's constants (7.7) proposed by Kotschick and Lisca using the Kotschick–Morgan Conjecture from [K–M].

The idea of this program is the following: let us blow up one or two points on $\mathbb{CP}^2$ and for this new surface $\widetilde{\mathbb{CP}^2}$ let us consider the same numbers as (7.7). But now these numbers depend on the Riemannian metric in an essential way. However it can be shown that the dependence of these numbers on the metric can be controlled explicitly! Going from $\mathbb{CP}^2$ to $\widetilde{\mathbb{CP}^2}$, we have to consider the collection $\{\alpha\}_d \subset H^2(\widetilde{\mathbb{CP}^2}, \mathbb{Z})$ of classes such that

$$\alpha = 0 \bmod 2 \quad \text{and} \quad -4d \leq \alpha^2 < 0. \tag{7.12}$$

The intersection of the positive cone in $H^2(\widetilde{\mathbb{CP}^2}, \mathbb{R})$ with the hyperplane $\alpha^\perp$ is called a *d-wall* (or the d-wall defined by α). Let Δ_d be the set of open chambers into which the positive cone is divided by all d-walls.

Proposition 7.2:

1. *The constant $D_g(d)$ of (7.7) for $\widetilde{\mathbb{CP}^2}$ depends on the chamber $C \in \Delta_d$ which contains the g-self dual harmonic 2-form.*
2. *If C and C' are chambers then*

$$D_C(d) - D_{C'}(d) = \sum_\alpha \delta_d(\alpha) \tag{7.13}$$

where the sum is taken over all d-walls α such that

$$\alpha \cdot C' < 0 < \alpha \cdot C$$

that is over all walls dividing C and C'

Now the following result has been proved:

Proposition 7.3: *The constant $D_g(d)$ of (7.7) for $\widetilde{\mathbb{CP}^2}$ are determined by the difference terms δ_d only.*

The following fact is "almost" proved:

Kotschick–Morgan Conjecture (see [K–M]): *The difference terms δ_d are homotopy invariants.*

So finishing the proof of this conjecture implies the topological definition of the Donaldson constants (7.7).

Remark: Mixing the Hilbert scheme method and the difference terms method is a very fruitful technique. G. Ellingsrud and L. Göttsche [E–G] can use it to compute any Donaldson number exactly, but the real nature of these expressive numbers remains an open question.

Finally, to prove the topological nature of the constants $s\gamma^d_{F-S} = s_{d+1}(d)$ (7.11), we have to mimic these constructions for the moduli spaces of jumping instantons (7.3) or, in full generality, for the spin polynomials (see [T1]) in place of the Donaldson polynomials.

I would like to finish by drawing the following conclusion:

Moral: *Different interpretations of classical algebraic geometric figures provide very fruitful approaches to understanding the nature of results of ENUMERATIVE AG.*

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