

Arf characters of an algebroid curve

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Abstract

Two algebroid branches are said to be equivalent if they have the same multiplicity sequence. It is known that two algebroid branches R and T are equivalent if and only if their Arf closures, R' and T' have the same value semigroup, which is an Arf numerical semigroup and can be expressed in terms of a finite set of information, a set of characters of the branch.

We extend the above equivalence to algebroid curves with $d > 1$ branches. An equivalence class is described, in this more general context, by an Arf semigroup, that is not a numerical semigroup, but is a subsemigroup of \mathbb{N}^d . We express this semigroup in terms of a finite set of information, a set of characters of the curve, and apply this result to determine other curves equivalent to a given one.

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1 Introduction

By an algebroid branch we mean a one-dimensional domain of the form $R = k[[x_1, \dots, x_n]]/P$, where x_1, \dots, x_n are indeterminates over the field k (that we assume to be an algebraically closed field of characteristic zero) and P is a prime ideal in $k[[x_1, \dots, x_n]]$. The integral closure \bar{R} of R is isomorphic to the DVR $k[[t]]$. Thus every nonzero element in R has a value, considering R as a subring of \bar{R} . The set of values of nonzero elements in R constitute a numerical semigroup $v(R) = S$, i.e. an additive submonoid of \mathbb{N} , with finite complement in \mathbb{N} , and the multiplicity of the ring R , $e(R)$, is given by the smallest positive value in S . The blowup of $R = R_0$ is $R_1 = \bigcup_{n \geq 0} (m^n : m^n) = R[x^{-1}m]$, where m is the maximal ideal of R and x is an element of smallest value in m . The blowup R_1 is a local overring of R and, if R_{i+1} denotes the blowup of R_i , then $R_j = k[[t]]$, for $j \gg 0$. The multiplicity sequence of R is defined to be $e_0 = e(R_0), e_1 = e(R_1), \dots$

We define two algebroid branches as equivalent if they have the same multiplicity sequence. This equivalence extends the Zariski equivalence between

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plane branches (cf. [13]) to branches of any embedding dimension and has been studied by several authors (cf. e.g. [7, Definition 1.5.11]).

As has been proved by the work of Arf [1], Du Val [8], Lipman [9], two algebroid branches R and T are equivalent if and only if their Arf closures, R' and T' have the same value semigroup, that is an Arf numerical semigroup. For the convenience of the reader these results are set out in detail in Section 2. In Section 3 an explanation is given on how to express an Arf numerical semigroup (which always represents the equivalence class of a branch R) in terms of a minimal finite set of information, the characters of R . The content of Proposition 3.1 is more or less implicit but not explicitly proved in [8]. The main results of the paper are contained in Sections 4 and 5. In Section 4 we extend the above equivalence to algebroid curves with $d > 1$ branches. We have not found this natural definition in the literature.

As in the one branch case, an equivalence class is described, in this more general context, by an Arf semigroup, that is not a numerical semigroup, but is a subsemigroup of \mathbb{N}^d . On the other hand any such semigroup describes an equivalence class (cf. Theorem 4.1). Although the value semigroup $v(R)$ of an algebroid curve R with $d > 1$ branches is not finitely generated, the equivalence class of R , represented by an Arf semigroup contained in \mathbb{N}^d or by a multiplicity tree with d branches as well, can be expressed in terms of a finite (but not unique) set of information (cf. Proposition 5.3). Such a finite set, a set of characters of the curve R , is useful to determine subrings of R (for example projections of the curve R on spaces of lower dimension) that are equivalent to R (cf. Theorem 5.4). This is illustrated in the last two examples.

2 Equivalence between algebroid branches

We keep the hypotheses and notation of the Introduction: let R be an algebroid branch with value semigroup $v(R) = S$, let $R = R_0, R_1, \dots$ the sequence of successive blowups of R and let $e_0 = e(R_0), e_1 = e(R_1), \dots$ be its multiplicity sequence. Two algebroid branches are said to be *equivalent* if they have the same multiplicity sequence. If α is a nonnegative integer, let $R(\alpha) = \{r \in R; v(r) \geq \alpha\}$. Then $R(\alpha)$ is an ideal of R and an ideal I of R is integrally closed if and only if $I = R(\alpha)$, for some α . The ring R is said to be an *Arf ring* if $x^{-1}R(\alpha)$, where $v(x) = \alpha$, is a ring, for each $\alpha \in v(R)$. For each ring R there is a smallest Arf overring R' , called the *Arf closure* of R . Since the Arf closure maintains the multiplicity of the ring and commutes with blowingup (cf. [9, Theorem 3.5]), $(R_i)' = (R')_i$, the multiplicity sequence of R is the same as the multiplicity sequence of R' , i.e. R is equivalent to its Arf closure R' .

It is wellknown that the value semigroup of the Arf closure of R , $v(R')$ is an *Arf semigroup*, i.e. a numerical semigroup S such that $S(s) - s$ is a semigroup, for each $s \in S$, where $S(s) = \{n \in S; n \geq s\}$ (cf. [4, page 8]).

Given an Arf numerical semigroup $S = \{s_0 = 0 < s_1 < s_2 < \dots\}$, the *multiplicity sequence* of S is the sequence of differences $s_{i+1} - s_i$, for $i \geq 0$. The first element of this sequence $s_1 - s_0 = s_1$ is called the *multiplicity* of S . If

$S = v(R')$, the multiplicity sequence of the ring R is exactly the same as the multiplicity sequence of the Arf semigroup S (cf. [1] or [2, Proposition 5.10]).

We recall now how the multiplicity sequence of an Arf semigroup characterizes the semigroup completely.

It follows immediately from the definitions that the multiplicity sequence e_0, e_1, \dots of an Arf semigroup is such that, for each $i \geq 0$, $e_i = \sum_{h=1}^k e_{i+h}$, for some $k \geq 1$. Conversely any sequence e_0, e_1, \dots of natural numbers such that $e_n = 1$, for $n \gg 0$ and, for all i , $e_i = \sum_{h=1}^k e_{i+h}$, for some $k \geq 1$, is the multiplicity sequence of the Arf semigroup S given by the sums, $S = \{0, e_0, e_0 + e_1, \dots\}$. We call such a sequence of natural numbers a *multiplicity sequence*. For example, given the sequence $6, 3, 3, 3, 1, 1, \dots$, the Arf semigroup with such multiplicity sequence is $S = \{0, 6, 9, 12, 15, \rightarrow\}$ (with \rightarrow we mean that all consecutive integers are contained in the set). Thus to give an Arf numerical semigroup is equivalent to give its multiplicity sequence.

Moreover, if we consider $S_1 = (S \setminus \{0\}) - e_0 = \{0, e_1, e_1 + e_2, \dots\}$ and, for $i \geq 1$, $S_{i+1} = (S_i \setminus \{0\}) - e_i = \{0, e_{i+1}, e_{i+1} + e_{i+2}, \dots\}$, we get a chain of Arf semigroups $S \subset S_1 \subset S_2 \subset \dots \subset S_n = \mathbb{N}$ such that the multiplicity of S_i is e_i .

Recall also that, for each numerical semigroup S , there is a smallest Arf semigroup S' containing S , called the *Arf closure* of S . It is the intersection of all the Arf semigroups containing S . If s_1, \dots, s_n are natural numbers, we denote by $\text{Arf}(s_1, \dots, s_n)$ the smallest Arf semigroup that contains s_1, \dots, s_n . Given an Arf semigroup S , there is a uniquely determined smallest semigroup N such that the Arf closure of N is S . The minimal system of generators $\{n_1, \dots, n_h\}$ for such N is called in [11] the *Arf system of generators* for S . We also will use this terminology. The generators $\{n_1, \dots, n_h\}$ are called in [1] and [8] the *characters* of the ring R (if $v(R') = S$); so we refer to them as the *Arf characters* of R and in the next section we give an algorithm to find them.

In conclusion two algebroid branches R and T are equivalent if and only if their Arf closures R' and T' have the same value semigroup, that is an Arf semigroup. On the other hand each Arf semigroup S is the value semigroup of the algebroid branch $R = R' = k[[S]]$. It follows that the equivalence classes of algebroid branches are in one to one correspondence with Arf numerical semigroups.

In the particular case of two algebroid plane branches R and T it is well known that R and T are equivalent if and only if $v(R) = v(T)$ (cf. [14], [12] or [3, Theorem 2.5]). So in the plane case $v(R) = v(T)$ if and only if $v(R') = v(T')$. Notice however that $v(R')$ is an Arf semigroup that contains $v(R)$ and so the Arf closure of $v(R)$, but the inclusion $v(R)' \subseteq v(R')$ may be strict. A simple example is the following.

Example If $R = k[[t^4, t^6 + t^7]]$, then $v(R) = \langle 4, 6, 13 \rangle = \{0, 4, 6, 8, 10, 12, 13, 14, 16, \rightarrow\}$. Thus $v(R)' = \langle 4, 6, 13, 15 \rangle = \{0, 4, 6, 8, 10, 12, \rightarrow\}$. However $R' = k + kt^4 + k(t^6 + t^7) + t^8k[[t]]$, thus $v(R') = \langle 4, 6, 9, 11 \rangle = \{0, 4, 6, 8, \rightarrow\}$.

Notice also that, given an equivalence class of algebroid branches, or, equivalently, given an Arf numerical semigroup, not always there is a plane branch

in that class: there is a characterization of Arf semigroups representing a class that contains a plane branch (cf. e.g. [3, Theorem 3.2]).

3 Arf characters of an algebroid branch

As shown in the previous section the equivalence class of an algebroid branch R is determined by the value semigroup of its Arf closure $v(R') = S$ and hence is given in terms of finite information: the set of generators of S or, better, the Arf system of generators of S . The advantage with the Arf system of generators is that we can generalize it to curves with several branches and it is a finite set also in that case (as we will see in the last section), while the semigroup is no more finitely generated.

We will now give an algorithm to find the Arf system of generators of an Arf numerical semigroup.

Let S be an Arf numerical semigroup and let e_0, e_1, \dots be its multiplicity sequence. The *restriction number* $r(e_j)$ of e_j is defined to be the number of sums $e_i = \sum_{h=1}^k e_{i+h}$ where e_j appears as a summand. With this terminology we have the following result:

Proposition 3.1 *Let S be an Arf numerical semigroup and let e_0, e_1, \dots be its multiplicity sequence. Let $r(e_j)$ be the restriction number of e_j . Then the Arf system of generators for S is given by the elements $e_0 + e_1 + \dots + e_j$, where $r(e_j) < r(e_{j+1})$.*

To prove this result we need a lemma:

Lemma 3.2 *Suppose e_0, e_1, \dots is a multiplicity sequence and that $\{0, e_0, e_0 + e_1, \dots\}$ is its associated Arf semigroup. Then $r(e_j) < r(e_{j+1})$ if and only if there is no $k < j$ such that $e_k = e_{k+1} + \dots + e_j$. In this case $r(e_j) = r(e_{j+1}) - 1$.*

Proof. There is only one sum in which e_{j+1} but not e_j is a summand, namely $e_j = e_{j+1} + \dots$. All other sums that contain e_{j+1} as a summand also contain e_j as a summand. Thus $r(e_j) \geq r(e_{j+1})$ if and only if there is a $k < j$ such that $e_k = e_{k+1} + \dots + e_j$. Otherwise $r(e_j) = r(e_{j+1}) - 1$.

Proof of Proposition 2.1. We argue by induction on the number of elements $\neq 1$ in the multiplicity sequence of S . If this number is zero, i.e. the multiplicity sequence is $1, 1, \dots$, then $S = \mathbb{N} = \text{Arf}(1)$ and in fact only the first 1 of the multiplicity sequence has restriction number (equal to zero) strictly smaller than the next.

Let $S = \{0, e_0, e_0 + e_1, e_0 + e_1 + e_2, \dots\}$ be an Arf semigroup. Then $S_1 = \{0, e_1, e_1 + e_2, \dots\}$ is also an Arf semigroup and the number of elements $\neq 1$ in the multiplicity sequence of S_1 is one less than the number of elements $\neq 1$ in the multiplicity sequence of S . By the inductive hypothesis the minimal Arf system of generators for S_1 is given by $\{h'_j = e_1 + \dots + e_j; r'(e_j) < r'(e_{j+1})\}$ (where $r'(e_j)$ denotes the restriction number of e_j as an element of the multiplicity sequence of S_1). We will denote by J the set of the indices of the elements h'_j .

We now compare the restriction numbers $r(e_j)$ of the multiplicity sequence e_0, e_1, \dots of S with the restriction numbers $r'(e_j)$ of the multiplicity sequence e_1, e_2, \dots of S_1 . We have $r(e_0) = 0$ and, if $e_0 = e_1 + \dots + e_i$, $r(e_1) = r'(e_1) + 1, \dots, r(e_i) = r'(e_i) + 1, r(e_j) = r'(e_j)$, for $j > i$.

So, excluding the element e_0 (of course $r(e_0) = 0 < r(e_1)$), we have $r(e_j) < r(e_{j+1})$ if and only if $r'(e_j) < r'(e_{j+1})$, for every $j \neq i$. If $j = i$, then $r(e_i) = r'(e_i) + 1$ and $r(e_{i+1}) = r'(e_{i+1})$. Hence, even if $r'(e_i) = r'(e_{i+1}) - 1$, $r(e_i) \geq r(e_{i+1})$.

It follows that $\{h_j = e_0 + \dots + e_j; r(e_j) < r(e_{j+1})\} = \{e_0\} \cup \{e_0 + h'_j; j \in J, j \neq i\}$. Hence we need to prove that $S = \text{Arf}(e_0, e_0 + h'_j; j \in J; j \neq i)$ and that for every $j \neq i$, $e_0 + h'_j$ is necessary as Arf generator for S .

Since $e_0 = e_1 + \dots + e_i \in S_1$, then $S_1 = \text{Arf}(h'_j = e_1 + \dots + e_j; j \in J) = \text{Arf}(e_0, h'_j = e_1 + \dots + e_j, j \in J)$. By [11, Proposition 16], $S = \{0\} \cup e_0 + S_1 = \text{Arf}(e_0, 2e_0, e_0 + h'_j; j \in J)$.

Since $2e_0 = e_0 + e_1 + \dots + e_i$ is clearly superfluous, we get $S = \text{Arf}(e_0, e_0 + h'_j; j \in J; j \neq i)$.

Finally we notice that, for every $j \in J, j \neq i$, the elements $e_0 + h'_j$ are necessary as Arf generators for S . Otherwise we would have, for $H \subsetneq J \setminus \{i\}$, that $S = \text{Arf}(e_0, e_0 + h'_j; j \in H)$ and, again by [11, Proposition 16], $S_1 = \text{Arf}(e_0, h'_j; j \in H)$, a contradiction against the minimality of the set $\{h'_j; j \in J\}$ as Arf system of generators for S_1 .

Example If $S = \langle 6, 9, 16, 17 \rangle = \{0, 6, 9, 12, 15, 16, \rightarrow\}$, its multiplicity sequence is $6, 3, 3, 3, 1, 1, \dots$, then $r(e_0) = 0, r(e_1) = 1, r(e_2) = 2, r(e_3) = 1, r(e_4) = 1, r(e_5) = 2, r(e_6) = 2, r(e_j) = 1$, for $j \geq 7$. The Arf system of generators for S is $\{e_0 = 6, e_0 + e_1 = 9, e_0 + e_1 + e_2 + e_3 + e_4 = 16\}$.

Remark Given an Arf semigroup S , the cardinality of its minimal Arf system of generators is said to be the *Arfrank*(S) (cf.[11]).

Now let as above $S = \{0, e_0, e_0 + e_1, \dots\} \subset S_1 = \{0, e_1, e_1 + e_2, \dots\} \subset S_2 = \{0, e_2, e_2 + e_3, \dots\} \subset \dots \subset S_n = \mathbb{N}$ be the chain of Arf semigroups obtained by S .

In the proof of 3.1 it is shown that $\text{Arfrank}(S_i) = \text{Arfrank}(S_{i+1}) + 1$, if e_i is not in the Arf system of generators of S_{i+1} , and $\text{Arfrank}(S_i) = \text{Arfrank}(S_{i+1})$, if e_i is in the Arf system of generators of S_{i+1} . It follows that $\text{Arfrank}(S) \leq n + 1$.

It is not difficult to give examples of Arf semigroups S such that the equality holds. It is in fact enough to choose for each i the multiplicity e_i of S_i as an element of S_{i+1} , that is not in the Arf system of generators of S_{i+1} .

Example We give the construction of an Arf semigroup S with $n = 3$ and $\text{Arfrank}(S) = 4$.

Let $S_3 = \mathbb{N} = \text{Arf}(1)$ (hence its Arfrank is 1). Choose a multiplicity for S_2 in $S_3 \setminus \{1\}$, for example $e_2 = 3$. Then $S_2 = \{0, 3, 4, \rightarrow\} = \text{Arf}(3, 3 + 1) = \text{Arf}(3, 4)$ and $\text{Arfrank}(S_2) = 2$.

Choose a multiplicity for S_1 in $S_2 \setminus \{3, 4\}$, for example $e_1 = 5$. Then $S_1 = \{0, 5, 8, 9, \rightarrow\} = \text{Arf}(5, 5 + 3, 5 + 4) = \text{Arf}(5, 8, 9)$ and $\text{Arfrank}(S_1) = 3$.

Finally choose a multiplicity for S in $S_1 \setminus \{5, 8, 9\}$, for example $e_0 = 10$. Then $S_1 = \{0, 10, 15, 18, 19, \rightarrow\} = \text{Arf}(10, 10 + 5, 10 + 8, 10 + 9) = \text{Arf}(10, 15, 18, 19)$ and $\text{Arfrank}(S) = 4$.

The semigroup S has multiplicity sequence $10, 5, 3, 1, 1, \dots$ and the number of elements in this sequence different by 1 (or, equivalently, the length of the chain $S \subset S_1 \subset \dots \subset S_n = \mathbb{N}$) is $n = 3$.

Example A class of Arf semigroups $S(k)$ such that $\text{Arfrank}(S) = k + 1$ and $n = k$ is given by $S(k) = \langle 2^k, 2^k + 2^{k-1}, \dots, 2^k + 2^{k-1} + \dots + 2 + 1 \rangle$.

4 Equivalence between algebroid curves

Now we consider *algebroid curves*, i.e. one-dimensional reduced rings $R = k[[x_1, \dots, x_n]]/I$, where $I = P_1 \cap \dots \cap P_d$, and where P_1, \dots, P_d are prime ideals of $k[[x_1, \dots, x_n]]$. The rings $k[[x_1, \dots, x_n]]/P_j = R/P_j$ are called the branches of the curve. The integral closure \bar{R} of R is isomorphic to $k[[t_1]] \times \dots \times k[[t_d]]$.

In this case the blowup R_1 of $R = R_0$, i.e. $R_1 = \bigcup_{n \geq 0} (m^n : m^n)$, where m is the maximal ideal of R , is a semilocal ring. The sequence of overrings $R = R_0 \subseteq R_1 \subseteq \dots$ is defined blowing up at each step the Jacobson radical of the previous ring and $R_j = \bar{R} = k[[t_1]] \times \dots \times k[[t_d]]$, for $j \gg 0$.

Given a maximal ideal $n_j = k[[t_1]] \times \dots \times t_j k[[t_j]] \times \dots \times k[[t_d]]$ of \bar{R} the *branch sequence* of R along n_j is the sequence of local rings $(R_i)_{n_j \cap R_i}$.

We defined in [2] the *blowing up tree* of R (cf. also [10]): the nodes at level i are the local rings $(R_i)_{n_j \cap R_i}$, $1 \leq j \leq d$. For $i = 0$, $(R_0)_{n_j \cap R_0} = R_0 = R$, for each j . For $i \geq 1$, R_i has a certain number r_i of maximal ideals, $1 \leq r_i \leq d$, and $r_i \leq r_h$, if $i \leq h$. A node at level i is connected to a node at level $i + 1$ if and only if the corresponding local rings are in the same branch sequence along some maximal ideal n_j of \bar{R} , i.e. if and only if their maximal ideals are one over the other. Recall also that, for each $i \geq 0$, R_i is the product of its localizations at the maximal ideals, i.e. is the product of the nodes at level i in the blowing up tree (cf. [2, Proposition 3.1]).

Since $R \subseteq \bar{R} = k[[t_1]] \times \dots \times k[[t_d]]$, every nonzero divisor in R has a value in \mathbb{N}^d . The set of values of nonzero divisors in R constitute a subsemigroup of \mathbb{N}^d . This semigroup $v(R) = S$ satisfies the following conditions, cf. [2]:

(1) If $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$ are elements of S , then $\min(\alpha, \beta) = (\min\{\alpha_1, \beta_1\}, \dots, \min\{\alpha_d, \beta_d\}) \in S$.

(2) If $\alpha, \beta \in S$, $\alpha \neq \beta$ and $\alpha_i = \beta_i$ for some $i \in \{1, \dots, d\}$, then there exists $\epsilon \in S$ such that $\epsilon_i > \alpha_i = \beta_i$ and $\epsilon_j \geq \min\{\alpha_j, \beta_j\}$ for each $j \neq i$ (and if $\alpha_j \neq \beta_j$ the equality holds).

(3) There exists $\delta \in \mathbb{N}^d$ such that $S \supseteq \delta + \mathbb{N}^d$.

Any subsemigroup of \mathbb{N}^d , satisfying the three conditions above is called a *good semigroup*. A good semigroup is *local* if $\mathbf{0}$ is the only element of the semigroup which has some coordinate equal to 0. In fact the semigroup $v(R)$ is local if and only if the ring R is local. If $\alpha = (\alpha_1, \dots, \alpha_d)$ is the minimal positive value in a local semigroup $S = v(R)$, then the multiplicity of the algebroid curve R is given by $\alpha_1 + \dots + \alpha_d$ (cf. [10, Theorem 1]). We define the *fine multiplicity*

of R to be the vector $\alpha = (\alpha_1, \dots, \alpha_d)$. Since $v(R/\bar{P}_j)$, the value semigroup of the j -th branch of the curve, is the projection of $v(R)$ on the j -th component of \mathbb{N}^d , it turns out that α_j is the multiplicity of the branch R/\bar{P}_j .

Let U be the local ring corresponding to a node of the blowing up tree of the algebroid curve R . Suppose that U occurs in the j_1, \dots, j_r -th branch sequences of R , so that $\bar{U} = k[[t_{j_1}]] \times \dots \times [[t_{j_r}]]$. (Notice that U is the localization of R_i at its j -th maximal ideal $m_{i,j} = n_{j_1} \cap R_i = \dots = n_{j_r} \cap R_i$; hence j_1, \dots, j_r are consecutive indices.) Then U has r minimal primes q_{j_1}, \dots, q_{j_r} . We denote in this case the fine multiplicity $\mathbf{e}(U)$ of U by a vector with d (not r) components, setting $\mathbf{e}(U) = (e_1(U), \dots, e_d(U))$, where $e_{j_h}(U) = e(U/q_{j_h})$ and $e_j(U) = 0$, if $j \notin \{j_1, \dots, j_r\}$. If we replace the local rings in the tree with their fine multiplicities, we get the *multiplicity tree* of the algebroid curve R . If $j = j_h$, for some h , and if the j_1 -th, \dots , j_r -th branches are glued in a node at level i , we denote this node on the j -th branch of the multiplicity tree by

$$\mathbf{e}_{(i)}^j = (0, \dots, 0, e_i^{j_1}, \dots, e_i^{j_r}, 0, \dots, 0).$$

Example The following is a very simple example of an algebroid curve with two branches.

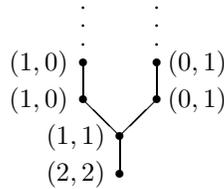
Let R be the subring $k[[t^2, u^2], (0, u^3), (t^3, 0)]$ of $k[[t]] \times k[[u]]$. This is the ring:

$$k[[x, y, z]]/(x^3 - z^2, y) \cap (x^3 - y^2, z)$$

Blowing up the maximal ideal of R we get $R_1 = k[[t^2, u^2], (0, u), (t, 0)]$, i.e. the ring

$$k[[x, y, z]]/(x - z^2, y) \cap (x - y^2, z)$$

that is still local. At next blowup the two branches split and we get the semilocal ring $R_2 = k[[t]] \times k[[u]]$. Thus the multiplicity tree of R is the following



Remark If we consider a branch $k[[x_1, \dots, x_n]]/P_j$ of an algebroid curve, the sequence of the local blowups of such a branch does not appear in the blowup tree of the curve, but the multiplicity sequence of the branch can be read off in the multiplicity tree of the curve, moving upwards along the j -th branch of the tree and taking the j -th component of each node. In fact the blowup R_1 of R has exactly one minimal prime, say Q_j contracting to P_j (it is the intersection of the corresponding minimal prime $k[[t_1]] \times \dots \times k[[t_{j-1}]] \times (0) \times k[[t_{j+1}]] \times \dots \times k[[t_d]]$ of \bar{R} with R_1) and $R/P_j \subset R_1/Q_j \subseteq k[[t_j]] = \bar{R}/\bar{P}_j$. Now the blow up of R/P_j

is exactly R_1/Q_j . Moreover, since $n_j \cap R_1$ is the unique maximal prime of R_1 containing Q_j , we have that $R_1/Q_j = (R_1)_{n_j \cap R_1}/Q_j$.

It follows by the previous remark that the multiplicity tree of an algebroid curve is determined by the multiplicity sequences of its branches and by the position of the branching nodes of the tree. In the multiplicity tree of the example above there is a branching node at level 1.

We define two algebroid curves to be *equivalent* if they have the same number of branches and the branches can be ordered in a way that gives the same multiplicity tree.

Of course this definition extends the equivalence between algebroid branches recalled in previous section. Let's see how it is also coherent with the classical equivalence definition for plane curves.

Consider two branches of an algebroid curve $R = k[[x_1, \dots, x_n]]/P_1 \cap \dots \cap P_d$, corresponding to the indices j and h , $1 \leq j, h \leq d$, $j \neq h$. Then $R_{jh} = k[[x_1, \dots, x_n]]/P_j \cap P_h = R/\bar{P}_j \cap \bar{P}_h$ is an algebroid curve with two branches and its multiplicity tree is given by the multiplicity tree of R , cancelling all the branches except the j -th and the h -th and all the coordinates of the vectors, except the j and h -th.

In case of a plane algebroid curve, the *intersection number* of two branches is defined as the intersection number of the ring R_{jh} , i.e. $\mu_{jh} = \mu(R/\bar{P}_j, R/\bar{P}_h) = l_{R_{jh}}(R_{jh}/\bar{P}_j + \bar{P}_h)$. By a theorem of Max Noether (cf. for example [6, Section 8.4, Theorem 13]), we have (with our notation):

$$\mu_{jh} = \sum_{0 \leq i \leq s} e_i^j e_i^h$$

where s is the level of the branching node of the tree. Thus the intersection number μ_{jh} says for plane curves at which level the two branches split.

Thus two plane algebroid curves $R = k[[x_1, x_2]]/P_1 \cap \dots \cap P_d$ and $T = k[[x_1, x_2]]/Q_1 \cap \dots \cap Q_d$ are equivalent (in our sense) if and only if, after renumbering the indices, for each j , $1 \leq j \leq d$, R/\bar{P}_j and T/\bar{Q}_j have the same multiplicity sequence and for each j and h , $1 \leq j, h \leq d$, $j \neq h$, $\mu(R/\bar{P}_j, R/\bar{P}_h) = \mu(T/\bar{Q}_j, T/\bar{Q}_h)$. This is exactly the classical definition of Zariski of equivalence between plane algebroid curves (cf. [13] and [12]).

Remark Notice that for non plane algebroid curves the numbers μ_{jh} does not indicate the level where the j -th and h -th branches split. In the previous example $l_R(R/P_1 + P_2) = 3$, but

$$\sum_{0 \leq i \leq 1} e_i^1 e_i^2 = 5.$$

Let $R = k[[x_1, \dots, x_n]]/P_1 \cap \dots \cap P_d$ be an algebroid curve with d branches and let $S = v(R)$ be its value semigroup. If $\alpha \in \mathbb{N}^d$, set $R(\alpha) = \{r \in R; v(r) \geq \alpha\}$. As in the one-branch case, R is an *Arf ring* if $x^{-1}R(\alpha)$, where $v(x) = \alpha$, is a ring for each $\alpha \in v(R)$ (equivalently R is Arf if each integrally closed ideal

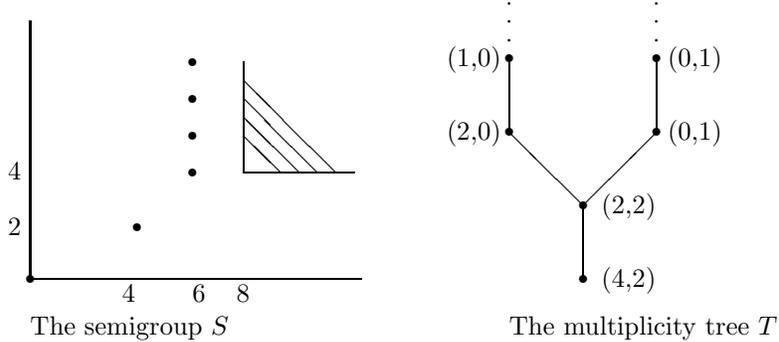
is stable, cf. for the equivalence [2, Lemmas 3.18 and 3.22]). There is also in this more general case a smallest Arf overring R' of R , called the Arf closure of R (cf. [9, Proposition-Definition 3.1]), that has the same multiplicity tree as R (cf. [2, Proposition 5.3]). Thus R is equivalent to its Arf closure R' .

A good semigroup $S \subseteq \mathbb{N}^d$ is called an *Arf semigroup* if $S(\alpha) - \alpha$ is a semigroup, for each $\alpha \in S$, where $S(\alpha) = \{\beta \in S; \beta \geq \alpha\}$ (equivalently if each semigroup ideal $S(\alpha)$ is stable, cf. [2, p.233]). To each (local) Arf semigroup $S \subseteq \mathbb{N}^d$ is associated a multiplicity tree (cf. [2, Section 5, p.247]) and, if $S = v(R')$, the multiplicity tree of the ring R is the same as the multiplicity tree of the semigroup S (cf. [2, Proposition 5.10]). Also in this case the multiplicity tree of a local Arf semigroup characterizes the semigroup completely and a tree $\mathbf{T} = \{\mathbf{e}_{(i)}^j\}$ of vectors of \mathbb{N}^d is the multiplicity tree of a local Arf semigroup if and only if it satisfies the following conditions:

- a) There exists $n \in \mathbb{N}$ such that, for $m \geq n$, $\mathbf{e}_{(m)}^j = (0, \dots, 0, 1, 0, \dots, 0)$ (the nonzero coordinate in the j -th position) for any $j = 1, \dots, d$.
- b) $e_i^j = 0$ if and only if $\mathbf{e}_{(i)}^j$ is not in the h -th branch of the tree (the h -th branch of the tree is the unique maximal path containing the h -th unit vectors).
- c) $\mathbf{e}_{(i)}^j = \sum_{\mathbf{e} \in \mathbf{T}' \setminus \mathbf{e}_{(i)}^j} \mathbf{e}$, for some finite subtree \mathbf{T}' of \mathbf{T} , rooted in $\mathbf{e}_{(i)}^j$ (cf. [2, Theorem 5.11]). We will call such a tree a *multiplicity tree* of \mathbb{N}^d .

As in the one branch case, we can get the Arf semigroup from the multiplicity tree taking $\mathbf{0}$ and sums of vectors lying on subtrees rooted in the root of our multiplicity tree. Conversely we can get the multiplicity tree from the Arf semigroup in the following way: let S_j be the projection of the Arf semigroup S on the j -th coordinate. S_j is a numerical Arf semigroup (cf. [2, Proposition 3.30]). Denote by $\{e_i^j\}_{i \geq 0}$ its multiplicity sequence. The multiplicity tree is determined by these multiplicity sequences and by the fact that the j -th and h -th branches are glued together as long as the projection of $S(\alpha) - \alpha$, with $\alpha = (e_0^j, e_0^h) + \dots + (e_k^j, e_k^h)$, on the " jh -plane" is local.

Example



The semigroup S contains for example the vectors $(4, 2)$, $(6, 5) = (4, 2) + (2, 2) + (0, 1)$, $(9, 5) = (4, 2) + (2, 2) + (0, 1) + (2, 0) + (1, 0)$, obtained summing vectors along subtrees of T .

By projecting S on the two coordinates, we get the multiplicity sequences $4, 2, 2, 1, \dots$ and $2, 2, 1, \dots$ respectively. $S(4, 2) - (4, 2)$ is local, but $S(6, 4) - (6, 4)$ is not local, so the tree splits here.

We identify two Arf semigroups S and \tilde{S} of \mathbb{N}^d (and the corresponding multiplicity trees) if there exists a permutation σ on $\{1, \dots, d\}$ such that $\alpha = (e_0^1 + \dots + e_{k_1}^1, \dots, e_0^d + \dots + e_{k_d}^d) \in S$ if and only if $\sigma(\alpha) = (e_0^{\sigma(1)} + \dots + e_{k_1}^{\sigma(1)}, \dots, e_0^{\sigma(d)} + \dots + e_{k_d}^{\sigma(d)}) \in \tilde{S}$.

In conclusion we have:

Theorem 4.1 *The equivalence classes of algebroid curves with d branches are in one-to-one correspondence with the Arf semigroups of \mathbb{N}^d .*

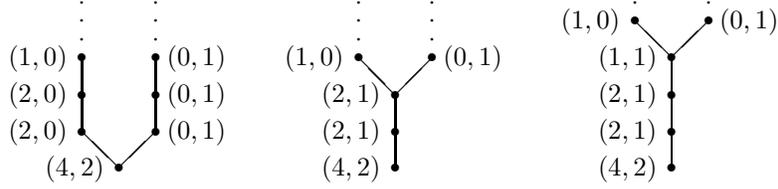
Proof. Two algebroid curves with d branches R and T are equivalent if and only if their Arf closures R' and T' are equivalent (cf. [2, Proposition 5.3]). This means that the branches of T' can be ordered in a way such that the multiplicity trees of R' and T' are the same, i.e. R' and T' have the same value semigroup, which is an Arf semigroups of \mathbb{N}^d . On the other hand each Arf semigroup $S \subset \mathbb{N}^d$ is the value semigroup of an algebroid curve with d branches (cf. [2, Corollary 5.8]).

As in the one branch case, given an equivalence class of algebroid curves (equivalently given an Arf semigroup of \mathbb{N}^d or a multiplicity tree of vectors of \mathbb{N}^d), not always there is a plane curve in that class: a characterization of multiplicity trees of \mathbb{N}^d representing a class that contains a plane curve is easily given, using [5, Theorem 5]. As a matter of fact, as we recalled in Section 2, for a plane branch R , knowing the semigroup $S = v(R)$ is equivalent to knowing the equivalence class of R , i.e. the multiplicity sequence of R . Moreover there is a characterization for numerical semigroups and for multiplicity sequences admissible for plane branches (cf. e.g. [3, Proposition 4.8 and Theorem 3.2]) and, by [5, Theorem 5], given two semigroups S and \tilde{S} admissible for plane branches, the possible intersection numbers of two plane branches with value semigroups S and \tilde{S} respectively are known.

Assume we have an ordered set of d multiplicity sequences admissible for plane branches and that the intersection number between every pair of consecutive branches is admissible according to [5, Theorem 5]. We claim that the multiplicity tree obtained in this way is the multiplicity tree of a plane curve. In fact it follows from the proof of [5, Theorem 5] that, given a plane branch R with $v(R) = S$ and a semigroup \tilde{S} admissible for plane branches, we can find a plane branch \tilde{R} such that $v(\tilde{R}) = \tilde{S}$ and the intersection number between R and \tilde{R} is μ , where μ is any of the intersection numbers determined in [5, Theorem 5].

Example Suppose we have a curve with two branches with semigroups $\langle 4, 6, 13 \rangle$ and $\langle 2, 3 \rangle$, respectively. By [5, Theorem 5] the possible intersection numbers are 8, 12, 13. Since the branches have multiplicity sequences $4, 2, 2, 1, \dots$ and $2, 1, \dots$, respectively (cf. e.g. [3, proof of Theorem 2.5]), it follows that they split on

level 0, 2, or 3. So we have the following three multiplicity trees. Below any tree is given a plane curve with that multiplicity tree.



$$k[[t^4, u^3], (t^6 + t^7, u^2)] \quad k[[t^4, 2u^2], (t^6 + t^7, u^3)] \quad k[[t^4, u^2], (t^6 + t^7, u^3)]$$

5 Arf characters of an algebroid curve

Our aim is to find, in analogy with the one-branch case, a finite minimal Arf system of generators for the local Arf semigroup $v(R')$, i.e. a set of characters of the algebroid curve R .

We recall some terminology. We call a *multiplicity tree* with d branches a tree satisfying conditions a), b), c) given in the previous section and a *multiplicity branch* a multiplicity tree with one branch, i.e. a multiplicity sequence of an Arf numerical semigroup.

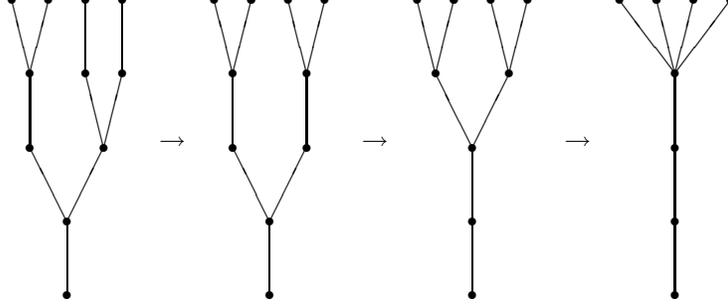
Suppose that E is a collection of d multiplicity branches $\{e_i^1\}_{i \geq 0}, \dots, \{e_i^d\}_{i \geq 0}$. Denote by $\tau(E)$ the set of all multiplicity trees having those d branches and by $\sigma(E)$ the set of the corresponding Arf semigroups.

Consider on $\tau(E)$ the order relation defined setting, for $T_1, T_2 \in \tau(E)$, $T_1 \leq T_2$ if and only if $S(T_1) \subseteq S(T_2)$, where $S(T_1)$ (resp. $S(T_2)$) is the Arf semigroup defined by T_1 (resp. T_2).

Before giving the next lemma we need some definitions and notation.

By *pinching* a (multiplicity) tree once, we mean modifying a tree identifying two nodes immediately over a branching node.

Example The following picture describes pinching of a tree on level 3, level 2, and level 3 consecutively.



Let T be a tree in $\tau(E)$ and let N be a level where all branches of T are distinct (there is such a level by definition of multiplicity tree). Then T is determined by $T^{(N)} = (n_1, \dots, n_{d-1})$, where $2n_j$ is the distance between the nodes on j -th and $j+1$ -th branch at level N (the distance between two nodes is the shortest walk between them in the tree).

Lemma 5.1 *Let $T_1, T_2 \in \tau(E)$. Then the following are equivalent:*

- (1) $T_1 \leq T_2$;
- (2) T_1 may be obtained from T_2 by a finite number of pinchings;
- (3) $T_1^{(N)}$ is coordinatewise less than or equal to $T_2^{(N)}$, where N is such that T_1 and T_2 have both d distinct branches at level N .

Proof. That (1) \Leftrightarrow (2) follows from the correspondence between Arf semigroups and multiplicity trees. That (2) \Leftrightarrow (3) follows from the observation that pinching a tree T once means subtracting 1 from one of the coordinates in $T^{(N)}$.

Lemma 5.2 *If $S_1, S_2 \in \sigma(E)$, then $S_1 \cap S_2 \in \sigma(E)$.*

Proof. We argue on the corresponding multiplicity trees $T_1, T_2 \in \tau(E)$, showing that, with respect to the order relation defined above, $\inf(T_1, T_2)$ exists in $\tau(E)$. Let $N \in \mathbb{N}$ such that T_1, T_2 both have d distinct branches at level N . If $T_1^{(N)} = (n_1^1, \dots, n_{d-1}^1)$ and $T_2^{(N)} = (n_1^2, \dots, n_{d-1}^2)$, we want to show that the tree T described by $T^{(N)} = (\inf(n_1^1, n_1^2), \dots, \inf(n_{d-1}^1, n_{d-1}^2))$ is a multiplicity tree of $\tau(E)$. Suppose that $T \notin \tau(E)$. So there is in T a node $\mathbf{e}_i^j \neq \sum_{\mathbf{e} \in T'} \mathbf{e}$, for each finite subtree T' of T , rooted in \mathbf{e}_i^j . This means that there are two nonzero coordinates $e_i^{j_1}$ and $e_i^{j_2}$ of \mathbf{e}_i^j such that

$$e_i^{j_1} = e_{i+1}^{j_1} + e_{i+2}^{j_1} + \dots + e_{i+k_1}^{j_1}$$

$$e_i^{j_2} = e_{i+1}^{j_2} + e_{i+2}^{j_2} + \dots + e_{i+k_2}^{j_2}$$

with $k_1 < k_2$ and that the j_1 -th and j_2 -branches split at a level strictly greater than $i + k_1$. It follows that in T_1 or in T_2 the j_1 -th and j_2 -th branches are glued at level i and split at a level strictly greater than $i + k_1$. This means that T_1 or $T_2 \notin \tau(E)$, a contradiction.

Suppose we have an Arf semigroup S in \mathbb{N}^d or, equivalently, a multiplicity tree with d branches. We will describe a method to find a finite set V of vectors in S which determines S .

For $1 \leq j \leq d$, let $\{e_i^j\}_{i \geq 0}$ be the multiplicity branches of the tree, let $S_j = \{e_0^j, e_0^j + e_1^j, \dots\}$ be the Arf numerical semigroup of the j -th branch and let $\{c_1^j, \dots, c_{n_j}^j\}$ be the Arf system of generators for S_j . To construct V , consider, for $1 \leq j \leq d$ and for $1 \leq i \leq n_j$, the vectors of S of the following form:

$$(\alpha_1, \dots, \alpha_{j-1}, c_i^j, \alpha_{j+1}, \dots, \alpha_d)$$

where, fixed c_i^j , the other coordinates are minimal (this choice is possible by property (2) of good semigroups (cf. Section 4)). Each of these vectors corresponds, in the multiplicity tree, to a subtree with only one branch, i.e. to a node. For any couple of consecutive branches (the j -th and the $j+1$ -th branch) there is a node in the tree where they split. Let's call it the $(j, j+1)$ -branching node of the tree. For each j , $1 \leq j \leq d-1$, check if in the set V there is a vector corresponding to a node of the tree over the $(j, j+1)$ -branching node. If not, add to the set V such a vector.

Proposition 5.3 *The smallest Arf semigroup containing V is S .*

Proof. By our choice, all the numbers $c_1^j, \dots, c_{n_j}^j$ appear as j -th coordinate of some of the vectors in V . So the numerical semigroups $S_j = \text{Arf}(c_1^j, \dots, c_{n_j}^j)$ (and the collection E of their multiplicity sequences) are determined. It follows that any Arf semigroup containing V is in $\sigma(E)$ and there exists at least one such (the Arf semigroup corresponding to a tree with a unique branching node at level 0).

Notice that the presence in an Arf semigroup of $\sigma(E)$ of a vector $\mathbf{v} = (\alpha_1, \dots, \alpha_d)$, with $\alpha_j = e_0^j + \dots + e_{k_j}^j$ for $1 \leq j \leq d$, and with $k_j > k_{j+1}$ for some j , forces the j -th and $j+1$ -th branches to split at most at level k_{j+1} . If moreover the vector \mathbf{v} above corresponds to a node on the j -th branch over the $(j, j+1)$ branching node (hence $k_j > k_{j+1}$), then it forces the j -th and $j+1$ -th branches to split exactly at level k_{j+1} .

By the choice of the vectors in V , any Arf semigroup containing V corresponds to a multiplicity tree of $\tau(E)$ with d distinct branches at level N , where N is the greatest k_j which appears in the expressions $\mathbf{v} = (\alpha_1, \dots, \alpha_d)$, with $\alpha_j = e_0^j + \dots + e_{k_j}^j$ for $1 \leq j \leq d$, \mathbf{v} in V . It follows that the number of Arf semigroups of $\sigma(E)$ containing V is finite. By Lemma 5.2, the intersection H of all these is still an Arf semigroup of $\sigma(E)$ and so it is the smallest Arf semigroup containing V . Moreover $H = S$, because the corresponding trees are the same: indeed, if $\mathbf{v} = (\alpha_1, \dots, \alpha_d)$, with $\alpha_j = e_0^j + \dots + e_{k_j}^j$ for $1 \leq j \leq d$, and with

$k_j > k_{j+1}$ for some j is a vector of V and if $2n_j$ is the shortest walk between the j -th and $j + 1$ -th branch at level N , then n_j is equal to $N - k_{j+1}$ in the tree corresponding to H , for its minimality, and it is also equal to $N - k_{j+1}$ in the tree corresponding to S , because the vector \mathbf{v} has been chosen on the j -th branch over the $(j, j + 1)$ -branching node.

Remark The set V is not uniquely determined, as is clear by its construction. We can e.g. get a smaller set by eliminating vectors in V with the following rule: if $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3 = \min(\mathbf{v}_1, \mathbf{v}_2) \neq \mathbf{v}_i$ (for $i = 1, 2$) are in V , then eliminate \mathbf{v}_3 .

But even for minimal subsets of V determining S the cardinality is not unique. A very simple counterexample is the following. Let

$$S = \{(0, 0, 0), (1, 1, 1)\} \cup ((2, 2, 2) + \mathbb{N}^3)$$

Then $V_1 = \{(1, 1, 1), (2, 3, 2)\}$ and $V_2 = \{(1, 1, 1), (3, 2, 2), (2, 2, 3)\}$ both are minimal subsets determining S .

In any case it seems proper to call a minimal set V determining S a set of *characters* of R , where $v(R') = S$.

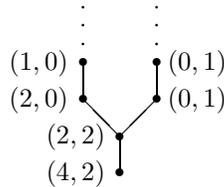
From the discussion above the following result follows:

Theorem 5.4 *Let R be an algebroid curve which is an Arf ring, with $v(R) = S$, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_N\}$ be a set of characters of R . Then any subring $U = k[[f_1, \dots, f_N]]$ of R with $v(f_i) = \mathbf{v}_i$, $i = 1, \dots, N$, has Arf closure R and so U is an algebroid curve equivalent to R .*

In some cases, for a particular Arf ring R , a proper subset of $\{f_1, \dots, f_N\}$ is enough to generate a ring with Arf closure R . In that case the curve $k[[f_1, \dots, f_N]]$ can be projected into a space of lower dimension without changing its multiplicity tree, i.e. without changing its equivalence class.

We conclude with two examples.

Example We consider the following multiplicity tree



which corresponds to the Arf semigroup

$$\{(0, 0), (4, 2), (6, n); n \geq 4\} \cup ((8, 4) + \mathbb{N}^2).$$

The Arf rings over k with that semigroup (that are all equivalent by Theorem 4.1) are the rings

$$k + (t^4 + \alpha_5 t^5 + \alpha_7 t^7, u^2 + \beta_3 u^3)k + (t^6 + \gamma_7 t^7, 0)k + (t^8, u^4)k[[t, u]],$$

where $\alpha_5, \alpha_7, \beta_3, \gamma_7 \in k$. The characters for the first branch are 4, 6, 9, and for the second 2, 5. The vectors containing these characters are $(4, 2)$, $(6, 4) = (4, 2) + (2, 2)$, $(9, 4) = (4, 2) + (2, 2) + (2, 0) + (1, 0)$ and $(6, 5) = (4, 2) + (2, 2) + (0, 1)$. We can delete $(6, 4) = \min(\{(9, 4), (6, 5)\})$ and get a minimal set of Arf generators $\{(4, 2), (9, 4), (6, 5)\}$. For the choice of parameters $\alpha_5 = \alpha_7 = \beta_3 = \gamma_7 = 0$, we get the ring

$$R = k + (t^4, u^2)k + (t^6, 0)k + (t^8, u^4)k[[t, u]]$$

Any subring of R generated by elements of values $(4, 2)$, $(9, 4)$, and $(6, 5)$, such as

$$U = k[[t^4, u^2], (t^9, u^4), (t^6, u^5)] = k[[x, y, z]]/(x^3 - z^2, x^2z - y^2) \cap (y - x^2, xy^2 - z^2)$$

has Arf closure R , i.e. is equivalent to R . With another choice of parameters, $\alpha_5 = \alpha_7 = \beta_3 = 0, \gamma_7 = 1$, we get the ring

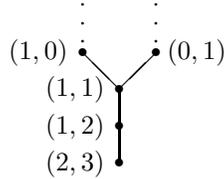
$$\tilde{R} = k + (t^4, u^2)k + (t^6 + t^7, 0)k + (t^8, u^4)k[[t, u]]$$

Already the ring

$$\tilde{U} = k[[t^4, u^2], (t^6 + t^7, u^5)] = k[[x, y]]/(y^4 - x^7 + x^6 - 4x^5y - 2x^3y^2) \cap (x^5 - y^2)$$

has Arf closure \tilde{R} , so any ring $\tilde{U}_r = k[[t^4, u^2], (t^6 + t^7, u^5), r]$ with $v(r) = (9, 4)$ is equivalent to \tilde{R} and has a projection to a plane curve equivalent to \tilde{R} .

Example We consider the following multiplicity tree



which corresponds to the Arf semigroup

$$\{(0, 0), (2, 3), (3, 5)\} \cup ((4, 6) + \mathbb{N}^2)$$

The Arf rings over k with that semigroup are the rings

$$k + (t^2, u^3 + \beta_4u^4 + \beta_5u^5)k + (t^3, \gamma u^5)k + (t^4, u^6)k[[t, u]], \gamma \neq 0,$$

where $\beta_4, \beta_5, \gamma \in k$. The characters for the first branch are 2, 3, and for the second 3, 5. The vectors containing these characters are $(2, 3)$ and $(3, 5) = (2, 3) + (1, 2)$, so we have to add one more generator corresponding to a node above the branching point, say $(4, 7) = (2, 3) + (1, 2) + (1, 1) + (0, 1)$. If we choose the parameters $\beta_4 = \beta_5 = 0, \gamma = 1$, we can take the ring

$$k[[t^2, u^3], (t^3, u^5), (t^4, u^7)] = k[[x, y, z]]/(z - x^2, xz - y^2) \cap (yz - x^4, xz - y^2, x^3y - z^2)$$

as an example of a ring with

$$k + (t^2, u^3)k + (t^3, u^5)k + (t^4, u^6)k[[t, u]]$$

as Arf closure. The choice of a node above the branching point is arbitrary. If we choose $(6, 6) = (2, 3) + (1, 2) + (1, 1) + (1, 0) + (1, 0)$ we can, with the same choice of parameters, take

$$k[[t^2, u^3], (t^3, u^5), (t^6, u^6)]] = k[[x, y, z]] / (x^3 - y^2, z - y^2) \cap (z - x^3, y^3 - xz^2)$$

as an example of a ring with

$$k + (t^2, u^3)k + (t^3, u^5)k + (t^4, u^6)k[[t, u]]$$

as Arf closure.

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