NOTE ON CAHIT ARF'S "UNE INTERPRÉTATION ALGÉBRIQUE DE LA SUITE DES ORDRES DE MULTIPLICITÉ D'UNE BRANCHE ALGÉBRIQUE"*

By PATRICK DU VAL

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Cahit Arf's results being severely algebraic in form, their geometrical meaning may not, at first sight, be evident to all geometers; it is felt therefore that a word of explanation may not be out of place.

A branch being parametrized as in § 8 of C.A., the orders of the elements of the ring

$$H = k[Y_1(t), \ldots, Y_n(t)]$$

are the possible intersection numbers of all hypersurfaces with the branch; in fact the intersection number of the hypersurface

$$\Sigma a_{i_1 \ldots i_n} Y_1^{i_1} \ldots Y_n^{i_n} = 0$$

is clearly the order of the element

$$\Sigma a_{i_1 \ldots i_n} (Y_1(t))^{i_1} \ldots (Y_n(t))^{i_n}$$

of the ring. If the ring H is canonical, theorem 7 shows that these intersection numbers are all the multiplicity sums of the branch. This accordingly is the characteristic property of a canonical branch. It does not, as one might suppose, follow from this that there exist hypersurfaces passing simply through every number of consecutive points of the branch; for the ring

$$H = k + k[t]\,t^2$$

corresponding to an ordinary cusp

$$Y_1 = t^2, \quad Y_2 = t^3$$

is clearly canonical, whereas there are no curves passing simply through more than the first two points of the branch.

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* *Supra*, 256–287. Referred to as C.A.

Thus the uniqueness of the canonical closure of a ring means that every branch is a projection of sequence, unique save for a transformation regular at the origin.

The number of base characters of the branch clearly gives the minimum space in which there can be a branch canonically equivalent to the given one.

To fix our ideas, let us consider the ring generated by

$$X_1 = t^4, \quad X_2 = t^{10}(1+t^5)$$

considered by C.A. The classical theory of Enriques shows that the points $P_1, P_2, \ldots$ consecutive along this plane branch are of multiplicities

$$4,\ 4,\ 2,\ 2,\ 2,\ 2,\ 1,\ 1,\ \ldots,$$

P_4 and P_8 being satellite and the rest free. Since typical elements of orders

$$0,\ 4,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18,\ 20,\ 22,\ 24,\ 25,\ 26,\ 28,\ 29,\ 30,\ 32,\ 33,\ 34,\ 35$$

are

$$1,\ X_1,\ X_1^2,\ X_2,\ X_1^3,\ X_1X_2,\ X_1^4,\ X_1^2X_2,\ X_1^5,\ X_1^3X_2,\ X_1^6,\ X_2^2-X_1^5,\ X_1^4X_2,$$
$$X_1^7,\ X_1X_2^2-X_1^6,\ X_1^5X_2,\ X_1^8,\ X_1^2X_2^2-X_1^7,\ X_1^6X_2,\ X_2^3-X_1^5X_2,\ \ldots$$

we can write

$$\begin{aligned} H &= k[X_1X_2] \\ &= k+kt^4+kt^8+kt^{10}(1+t^5)+kt^{12}+kt^{14}(1+t^5)+kt^{16}+kt^{18}(1+t^5) \\ &\quad +kt^{20}+kt^{22}(1+t^5)+kt^{24}+kt^{25}(2+t^5)+kt^{26}(1+t^5)+kt^{28}+kt^{29}(2+t^5) \\ &\quad +kt^{30}(1+t^5)+k[t]\,t^{32}, \end{aligned}$$

which is the same thing as

$$\begin{aligned} H &= k+kt^4+kt^8+kt^{10}(1+t^5)+kt^{12}+kt^{14}(1+t^5)+kt^{16}+kt^{18}(1+t^5) \\ &\quad +kt^{20}+kt^{22}(1+t^5)+kt^{24}+kt^{25}+kt^{26}(1+t^5)+kt^{28}+kt^{29}+kt^{30}+k[t]\,t^{32}; \end{aligned}$$

we have thus

$$\begin{aligned} \frac{I_4}{X_1} &= k+kt^4+kt^6(1+t^5)+kt^8+kt^{10}(1+t^5)+kt^{12}+kt^{14}(1+t^5) \\ &\quad +kt^{16}+kt^{18}(1+t^5)+kt^{20}+kt^{21}+kt^{22}(1+t^5)+kt^{24}+kt^{25}+kt^{26}+k[t]\,t^{28}, \end{aligned}$$

and, as the ring generated by this contains the elements

$$\left(\frac{X_2}{X_1}\right)^2-X_1^3, \quad \left(\frac{X_2}{X_1}\right)^2X_2-X_1^3X_2$$

of order 17, 23 respectively, we have

$$\begin{aligned} H_1 = [I_4] &= k+kt^4+kt^6(1+t^5)+kt^8+kt^{10}(1+t^5) \\ &\quad +kt^{12}+kt^{14}(1+t^5)+kt^{16}+kt^{17}+kt^{18}+k[t]\,t^{20}. \end{aligned}$$

By exactly the same method we see that

$$H_2 = k + kt^2(1+t^5) + kt^4 + kt^6 + k[t]\,t^8,$$

and since this is clearly canonical we find that the canonical closure of H_1 is

$$^*H_1 = k + t^4 H_2 = k + kt^4 + kt^6(1+t^5) + kt^8 + kt^{10} + k[t]\,t^{12},$$

and that of H is

$$^*H = k + t^{4*}H_1 = k + kt^4 + kt^8 + kt^{10}(1+t^5) + kt^{12} + kt^{14} + k[t]\,t^{16},$$

of which, as we expect, the characters are, 4, 10, 17, the first, third, and seventh multiplicity sums of the branch. Since

$$^*H = k[X_1, X_2, X_3, X_4],$$

where $X_3 = t^{17}$, $X_4 = t^{19}$, we see that the canonical branch of which the given branch is a projection is in four dimensions. Any projection of this into two dimensions is represented by a ring of the form $k[Y_1, Y_2]$, where Y_1, Y_2 belong to H, and can clearly be chosen to be of orders 4, 10 respectively, i.e. we may take

$$Y_1 = t^4 + a_1 t^{17} + a_2 t^{18} + \ldots, \quad Y_2 = t^{10} + t^{15} + b_1 t^{17} + b_2 t^{18} + \ldots;$$

for different values of the coefficients $a_1, a_2, \ldots, b_1, b_2, \ldots$, these branches are not regularly but only canonically equivalent.

An interesting feature is the apparent unimportance of the term t^{15} in the canonical ring, whereas of course this is of fundamental significance in determining the characters of the plane branch. In fact the canonical ring

$$^*H = k + kt^4 + kt^8 + kt^{10}(1+t^5) + kt^{12} + kt^{14} + k[t]\,t^{16}$$

clearly has the same characters as

$$^*H' = k + kt^4 + kt^8 + kt^{10} + kt^{12} + kt^{14} + k[t]\,t^{16},$$

which is also canonical. This latter, however, cannot be generated by two elements; in fact a base for it must be in some such form as

$$X_1' = t^4, \quad X_2' = t^{10}, \quad X_3' = t^{17};$$

thus of the two canonical branches

$$X_1 = t^4, \quad X_2 = t^{10}(1+t^5), \quad X_3 = t^{17}, \quad X_4 = t^{19}$$

and $$X_1' = t^4, \quad X_2' = t^{10}, \quad X_3' = t^{17}, \quad X_4' = t^{19},$$

both of which have two fourfold followed by four twofold and a succession of simple points, the former can and the latter cannot be projected into a plane branch with the same multiplicity sequence. In fact a general plane projection of the latter is of the form

$$Y_1' = \frac{t^4+a_1t^{10}+a_2t^{17}}{1+c_1t^4+c_2t^{10}+c_3t^{17}} = t^4+\alpha_1t^8+\alpha_2t^{10}+\alpha_3t^{12}+\alpha_4t^{14}+\alpha_5t^{16}+\alpha_6t^{17}+\ldots,$$

$$Y_2' = \frac{t^{10}+bt^{17}}{1+c_1t^4+c_2t^{10}+c_3t^{17}} = t^{10}+\beta_1t^{12}+\beta_2t^{14}+\beta_3t^{16}+\beta_4t^{17}+\ldots,$$

or in terms of

$$\tau = (Y_1')^{\frac{1}{4}} = t(1+p_1t^4+p_2t^6+p_3t^8+p_4t^{10}+p_5t^{12}+p_6t^{13}+\ldots),$$
$$t = \tau(1+q_1\tau^4+q_2\tau^6+q_3\tau^8+q_4\tau^{10}+q_5\tau^{12}+q_6\tau^{13}+\ldots)$$

and $\quad Y_1' = \tau^4, \quad Y_2' = \tau^{10}+B_1\tau^{12}+B_2\tau^{14}+B_3\tau^{16}+B_4\tau^{17}+\ldots;$

which by Enriques's theory clearly represents a branch with two fourfold followed by not four but five twofold points; the canonical closure of $k[Y_1', Y_2']$ is in fact not $^*H'$ but its canonical subring

$$k+kt^4+kt^8+kt^{10}(1+at^7)+kt^{12}+kt^{14}+kt^{16}+k[t]\,t^{18},$$

where a is a fixed constant depending on those in the expansions of Y_1', Y_2'. $^*H'$ is, however, the canonical closure of

$$k[X_1', X_2', X_3'],$$

or of the ring representing a general projection of the branch corresponding to $^*H'$ into three dimensions. This canonical branch can accordingly be projected into three but not into two dimensions without changing its characters; in short, whereas the two canonical branches considered both have the characters 4, 10, 17, the base characters of the former are 4, 10, and those of the latter 4, 10, 17.

Projection of any branch from a general point clearly gives one represented by a subring of the ring representing the given branch; if the projection alters the characters (i.e. the multiplicity sequence) of the branch, this means that the canonical closure of the subring is not that of the given ring, but is a subring of the latter; i.e. the multiplicity sums

$$\nu_1', \quad \nu_1'+\nu_2', \quad \nu_1'+\nu_2'+\nu_3', \quad \ldots$$

for the projected branch are a certain selection, not the whole, of the multiplicity sums

$$\nu_1, \quad \nu_1+\nu_2, \quad \nu_1+\nu_2+\nu_3, \quad \ldots$$

of the given branch. Thus we must have for some $i_1, i_2, \ldots$

$$\nu_1' = \sum_{j=1}^{i_1}\nu_j, \quad \nu_2' = \sum_{j=i_1+1}^{i_2}\nu_j, \quad \nu_3' = \sum_{j=i_2+1}^{i_3}\nu_j, \quad \ldots.$$

It seems natural to regard the first point of the projected branch as corresponding to the first i_1 points of the given branch, the next point of the former to the next i_2 points of the latter, and so on. Thus in the projection of the branch

$$k+kt^4+kt^8+kt^{10}+kt^{12}+kt^{14}+k[t]\,t^{16}$$

into a plane we may hold that the seventh and eighth points of the canonical branch (which are its first two simple points) are projected into the same point of the plane, which is accordingly a fifth double point of the plane branch. This is seen to agree with the genesis of a cusp by projection from a point on the tangent, where the double point on the projected branch certainly arises from two consecutive simple points on the original, the second point on the projected branch from the third simple point on the original, and so on.

Thus where a branch, such as

$$X_1' = t^4, \quad X_2' = t^{10}, \quad X_3' = t^{17},$$

cannot be projected into a lower space without altering its characters, this means that every cone (of whatever vertex) which passes through certain of its points inevitably passes through certain others; in the present case every cone which passes through the seventh point passes also (and with the same multiplicity) through the eighth; whereas the branch

$$X_1 = t^4, \quad X_2 = t^{10}(1+t^5), \quad X_3 = t^{17}$$

has not this property.

The canonical rings

$$H_1 = [I_{i_1}], \quad H_2, \quad H_3, \quad \ldots,$$

obtained from a canonical ring H represent the branches obtained by resolving the points of the branch in succession, as is seen in the course of the proof of C.A., theorem 7. Thus the base characters of these rings show the dimensions of the least spaces into which these resolved branches can be projected without altering their characters. For instance, of the branches

$$\left.\begin{aligned} {}^*H &= k+kt^{4\nu}(1+t)+kt^{6\nu}(1+t)+kt^{7\nu}(1+t)+k[t]\,t^{8\nu}, \\ {}^*H' &= k+kt^{4\nu} \qquad\quad +kt^{6\nu}(1+t)+kt^{7\nu}(1+t)+k[t]\,t^{8\nu}, \end{aligned}\right\} \quad (\nu>1)$$

considered in C.A. §7, both are capable of being projected into three dimensions without altering their characters, and each has a 4ν-ple, a 2ν-ple, and two ν-ple followed by simple points. The branches

$${}^*H_1 = k+kt^{2\nu}+kt^{3\nu}+k[t]\,t^{4\nu},$$

$${}^*H_2 = k+kt^{2\nu}(1+t)+kt^{3\nu}(1+t)+k[t]\,t^{4\nu},$$

obtained by resolving the first points of each, have of course the same multiplicity sequence, namely a 2ν-ple, and two ν-ple, followed by simple points; but whereas the former of these cannot be projected into a plane without altering its characters, the latter can. In fact the general plane projection of the former is represented by a ring of the form $k[Y_1, Y_2]$ where

$$Y_1 = t^{2\nu} + at^{4\nu+1} + \dots, \quad Y_2 = t^{3\nu} + bt^{4\nu+1} + \dots.$$

By the same method as in the first example the canonical closure of this is found to be

$$k + kt^{2\nu}(1 + at^{2\nu+1} + \dots) + kt^{3\nu}(1 + bt^{\nu+1} + \dots) + kt^{4\nu} + k[t]\, t^{5\nu}$$

the characters of which are 2ν, 3ν, $5\nu + 1$, so that the projected branch has a 2ν-fold followed by not two but three ν-ple and a succession of simple points, i.e. the first ν simple points of the branch $^*H_1$ are projected into a single ν-ple point; the same result is obtained by expressing the ring in terms of $\tau = Y_1^{1/2\nu}$, when it takes the form

$$Y_1 = \tau^{2\nu}, \quad Y_2 = \tau^{3\nu} + b\tau^{4\nu+1} + \dots.$$

By similar methods it can be seen that the projection of either of the original branches $^*H$, $^*H'$ into a plane has one 4ν-ple and two 2ν-ple followed by simple points, i.e. the two ν-ple points are projected into a single 2ν-ple point.

In the same way it can be seen that the branches

(i) $X_1 = t^{702}, \quad X_2 = t^{1620}, \quad X_3 = t^{2340}, \quad X_4 = t^{2383},$

(ii) $X_1 = t^{702}(1 + t^{72})^3, \quad X_2 = t^{1620}(1 + t^{72})^7, \quad X_3 = t^{2383}(1 + t^{72})^9,$

(iii) $X_1 = t^{702}(1 + t^{115})^3, \quad X_2 = t^{1620}(1 + t^{115})^7, \quad X_3 = t^{2340}(1 + t^{115})^9,$

(iv) $X_1 = t^{702}(1 + t^7)^{13}, \quad X_2 = t^{1620}(1 + t^7)^{30}, \quad X_3 = t^{2340}(1 + t^7)^{44},$

(v) $X_1 = t^{702}(1 + t^7)^{13}(1 + t^{79})^3, \quad X_2 = t^{1620}(1 + t^7)^3(1 + t^{79})^7,$

considered in C.A. § 7, all of which have the characters

702, 1620, 2340, 2383

and the multiplicity sequence

702 (twice), 216 (three times), 54 (five times),

18 (three times), 7 (twice), 4, 3,

followed by simple points, differ in the minimum space into which their canonical equivalents can be projected without altering their characters. (ii), (iii), (iv), indeed all exist in three dimensions and cannot be projected into a plane; but whereas the branch obtained by resolving the first five points of (iv) can be projected into a plane without altering its characters, those obtained from (ii) and (iii) in the same way cannot. The difference between these latter is not quite of the same kind, since the number of base characters is the same at every stage; only their values differ; thus whereas in (iii) the ring $k[X_1, X_2, X_3]$ contains an element of order 2340, in (ii) it does not; this means that the branch (iii), or any other in three dimensions canonically equivalent to it, can be cut by a surface (in the given form the plane $X_3 = 0$) so that the intersection number is 2340, i.e. a surface can be drawn passing simply through the first eleven points of the branch (as far as the first 18-ple point), while in the case of (ii) this is not possible—every surface passing through the eleventh point either passes through some further points after it, or has higher multiplicity at some of the earlier points.

The branch (v) is a plane branch, and can therefore by the classical theory be expressed in the form

$$X_1 = \tau^{702}, \quad X_2 = \tau^{1620} + a\tau^{1674} + b\tau^{1692} + c\tau^{1699} + \dots \quad (b \neq 0,\ c \neq 0).$$

The transformation from the variable t to $\tau = X_1^{1/702}$ is lengthy but straightforward.

It is hoped that enough has been said to make clear the geometrical significance of the canonical branch of which a given branch is a projection, and of the number and values of its base characters.

The University
Istanbul