Theorem [Sertöz ☺]: *Let $a_1, \dots, a_n$ be non-negative integers and $m_1, \dots, m_n$ be positive integers, where $n > 1$. Then we have:*

$$\text{If } \sum_{i=1}^{n} \frac{a_i}{2m_i} > 1, \text{ then } \lim_{(x_1,\dots,x_n)\to(0,\dots,0)} \frac{\prod_{i=1}^n x_i^{a_i}}{\sum_{i=1}^n x_i^{2m_i}} = 0.$$

Proof (Mefharet Kocatepe) .

First let $\sum_{i=1}^{n} x_i^{2m_i} = R$, where $R > 0$. Then clearly

$$(x_1, \dots, x_n) \to (0, \dots, 0) \text{ if and only if } R \to 0.$$

Now for any $i = 1, \dots, n$, we have $x_i^{2m_i} \leq R$, and hence $|x_i| \leq R^{1/2m_i}$. It then follows that

$$0 \leq \left| \frac{\prod_{i=1}^n x_i^{a_i}}{\sum_{i=1}^n x_i^{2m_i}} \right| = \frac{\prod_{i=1}^n |x_i|^{a_i}}{\sum_{i=1}^n x_i^{2m_i}} \leq \frac{\prod_{i=1}^n R^{\frac{a_i}{2m_i}}}{R} = R^{\sum_{i=1}^n \frac{a_i}{2m_i} - 1}.$$

But $\lim_{R\to 0} R^{\sum_{i=1}^n \frac{a_i}{2m_i} - 1} = 0$, since $\sum_{i=1}^{n} \frac{a_i}{2m_i} - 1 > 0$.

Hence by the squeeze theorem

$$\lim_{(x_1,\dots,x_n)\to(0,\dots,0)} \frac{\prod_{i=1}^n x_i^{a_i}}{\sum_{i=1}^n x_i^{2m_i}} = 0. \qquad \square$$

See also this link for other proofs and some discussion.

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