ON ARF RINGS

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ABSTRACT. In this note we summarize Arf's work on Arf rings and discuss its relevance in geometry. We also briefly mention current literature on the subject and some related research problems.

Study of curves is full of lessons in modesty. As soon as you pass the level of basic definitions you run into simple questions whose answers lie in deeper waters than one dares to dive. The simple question of understanding space curves with singularities, after the plane curves were relatively understood, proved to be mightier than the mortals who asked it. Among those who attacked this problem we can quote **Semple-1938** who analyzed the geometry of successive blow-ups on a singular curve branch in 3-space. Later **Du Val-1942**, while he was in İstanbul University, showed that if the multiplicity sums up to certain geometrically significant steps in the successive blow-up process were known then the whole multiplicity sequence could be obtained from these sums through a modified version of the Jacobian algorithm. While Du Val was explaining his findings, a young mathematician in the audience, none other than Cahit Arf, observed and claimed that there was an efficient algebra behind these geometrical arguments. Du Val challenged him into making his claims explicit, which he did the next week! In his article **Arf-1949** shows that the completion of the local ring at the singularity of the branch carries all the information necessary to obtain the multiplicity sequence.

The process of finding the Arf characters, as described in Arf's article goes as follows: If R is the completion of the local ring in question then

i) We first construct ${}^*R$, the Arf closure of R.

ii) Then we construct $S({}^*R)$, the sub-semigroup of the natural numbers consisting of the orders of the elements of ${}^*R$.

iii) We then construct the smallest semigroup g whose Arf closure is $S({}^*R)$, i.e. ${}^*g = S({}^*R)$.

iv) Finally, the generators of g over the natural numbers are the Arf characters of R.

The Arf characters are precisely the critical multiplicity sums of Du Val which yield the multiplicity sequence of the branch as mentioned earlier.

This process totally solves the problem of understanding the multiplicity sequence and hence the resolution process of cusp type singularities of algebraic space curves. Normally when a mathematician writes such a hard core mathematical article, his students and colleagues 'jump' on it and examine the new land into which this article is a door, which did not happen in this case for reasons that are none of my business at this point! After **Du Val-1949** explained the geometric significance of Arf's results there was a long silence before **Lipman-1971** generalized

Appendix in **The Collected Papers of Cahit Arf**, a publication of **Turkish Mathematical Society**, 1990, pp 416-419.

the nature of rings to which Arf's results can be applied. Lipman's article end with a comparison of Arf rings to saturated rings. Unfortunately the comparison gives the impression, probably not intended, that Arf rings do not accomplish much in the way of explaining the geometry of curves. With this silent 'verdict' Arf rings fell out of fashion, which they never enjoyed anyway. Moreover algebraic geometers turned their attention to higher dimensional geometry and curve theorists mainly focused on Riemann surfaces. It was also discovered that cusp singularities are not the generic singularities for curves but in a sense lie on the boundary of the moduli space of curves.

However despite the fact that researchers did not focus their attention on Arf rings at any time Arf rings surfaces in the study of several researchers from time to time. Among these we can quote a few; **Kubota-1982** gives an explicit description of Arf closure for certain classes of rings. **Campillo-1983** while studying singularities of plane curves over any characteristic introduces the concept of presaturation and observes that his presaturated rings are Arf rings when the residue field is infinite. **Tamone-1984** also runs into Arf rings while studying the coordinate rings obtained during the blow-up process. **Castellanos-1986** studies certain matrices obtained in the resolution process of singular curves and observes that these matrices can be used to give an explicit construction of the Arf closure associated to the curve.

A second look at the process used in producing Arf characters of the curve shows that the value-semigroup $S({}^*R)$ mentioned above is of interest in itself. **Kunz-1970** examines one such case. Moreover looking for the largest integer which is not included in this semigroup, in case its generators are relatively prime, has been an intellectual challenge to number theorists and to the rest of us as well for a long time and is known as the Frobenius' problem. This problem has a vast literature and we content by mentioning only **Selmer & Beyer-1978** and **Rodseth-1978** and the references given there.

The reason why one might study a problem may differ:

i) Everybody else is interested in this problem and hence it will be easy to publish whatever comes out of it (fashion).

ii) The solution seems not so far away and it is time to publish something (publish or perish).

iii) The problem is interesting (altruism).

It is difficult to find people of the third category these days (nostalgia). Even if you begin your career in the third category circumstances may force you to change your problem, and hence your category; now you are in the second category. After a while you will discover that what pleases your dean is only the number of articles that are being produced and not even the number of total pages, let alone the content. Now you are in the first category and at nights in your dreams you see Gauss waving his finger at you! If you wake up one morning with a determination to graduate to the third category again you might try to understand the relation between Arf characters and the **Hironaka-1964** characters used in checking how much a ring is resolved after a blow-up. You might also try to see the geometric significance of Frobenius' problem through the link established by the value-semigroups. You may examine the full power of Arf rings as far as the classification of curve singularities are concerned, which is partly done by **Arf-1949** using only Arf characters.

Of course trying to see a higher dimensional equivalent of Arf phenomena will also safely carry you into the third category, but none of this may cheer your dean!

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