ON BOTT'S VANISHING THEOREM AND APPLICATIONS TO SINGULAR FOLIATIONS

SİNAN SERTÖZ

ABSTRACT. Let M be a complex manifold with tangent bundle T which can be decomposed as $T = A \oplus B$ and let E be a subbundle of A. If E and B are integrable, then the graded chern ring $Chern^*(A/E)$ vanishes beyond the corank of E in A. This slightly extends Bott's vanishing theorem which is the $B = 0$ case. Bott has also observed that not only the above mentioned characteristic classes vanish but a connection can be chosen such that the differential forms giving the classes themselves vanish. This is then applied to a singular foliation F with singular set S. On $M - S$ we can choose a connection such that the differential form representing any $\alpha \in Chern^i(T/F|M - S)$ with $i >$ rank $T -$ rank F is zero. This connection can be extended to M which then gives for α a differential form with compact support. The Poincare dual of this form can be mapped into S to define a homology residue class first observed by Baum and Bott. We summarize two methods including Baum and Bott's original method for the calculation of this residue.

VANISHING THEOREM

A complex manifold M of complex dimension n is a topological manifold whose transition functions are holomorphic. A holomorphic foliation L of rank k on a complex manifold M of dimension n is a decomposition of M into disjoint connected sets $L = \{L_\alpha\}_{\alpha \in \Lambda}$ with Λ some indexing set, satisfying the following condition: for every point $p \in M$ there exists an open neighbourhood U of p with holomorphic coordinate map

$$x = (x_1, ..., x_n) : U \longrightarrow \mathbb{C}^n$$

such that for every $\alpha \in \Lambda$, either $L \cap U = \emptyset$ or

$$L \cap U = \{q \in U \;\big|\; x_i(q) = t_i^\alpha,\; k + 1 \leq i \leq n$$

where $(t_{k+1}^\alpha, ..., t_n^\alpha) \in \mathbb{C}^{n-k}$ depends on α and U. Each L_α is called a leaf of the foliation. Any foliation of rank k is locally isomorphic to the trivial foliation which is formed as the fibres of the map

$$pr : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-k}$$

where $\mathbb{C}^n$ is considered as $\mathbb{C}^k \oplus \mathbb{C}^{n-k}$ and pr is the projection on the second component. This local isomorphism is established through the local coordinate system (U, x) which is described above in the definition of the foliation. Such coordinate systems are called distinguished.

There is also a vector bundle approach to foliations. Let T be the holomorphic tangent bundle of M, and let E be a subbundle of T with rank k. In the classical terminology smooth subbundles of T are called smooth distributions. E is called

Appeared in **Doğa Math. Journal (Turkish Journal of Math.)**, 11 (1987), 62-67.

integrable if at each point p in M there exists a complex submanifold whose holomorphic tangent space at p is E_p. Each such submanifold is part of a leaf of a foliation on M. It is easy to see that in this case E is closed under the bracket operation. Integrable and involutive bundles are related to each other by the classical theorem f Frobenius.

Frobenius Theorem: *A subbundle of the tangent bundle is integrable iff it is involutive.*

It is not true that every subbundle is of the tangent bundle is integrable. Nor is it true that any subbundle can be deformed into one which is integrable. A necessary condition is given by the vanishing of certain obstruction classes.

Theorem: *Let $T = A \oplus B$ and E be a subbundle of A. If E and B are integrable, then the graded chern ring $chern^*(A/E)$ vanishes beyond the corank of E in A, i.e.*

$$chern^i(A/E) = 0 \text{ if } i > \text{rank } A - \text{rank } E.$$

For a proof of this which exploits the existence of distinguished coordinates for foliations see [1]. When $B = 0$ this is Bott's vanishing theorem, [2]. In fact the proof of the theorem says more than the statement. Let us first develop some notation.

For any $n \times n$ matrix A define $c_i(A)$ by

$$\det(I + tA) = 1 + tc_1(A) + \cdots + t^n c_n(A)$$

where I is the $n \times n$ identity matrix. When A is the diagonal matrix $d_t = \text{diag}(t_1, ..., t_n)$ then $c_i(d_t)$ is called the i-th elementary symmetric polynomial on $t_1, ..., t_n$ For any homogeneous symmetric polynomial $P \in \mathbb{C}[t_1, ..., t_n]$ there exists a polynomial P_s such that

$$P(t_1, ..., t_n) = P_s\big(c_1(d_t), ..., c_n(d_t)\big).$$

Then for any matrix A define $P(A) = P_s\big(c_1(A), ..., c_n(A)\big)$. If B is an $r \times r$ matrix with $r < n$ then $P(B) = P_s\big(c_1(B), ..., c_r(B), 0, ..., 0\big)$. Now we return to the vanishing theorem.

The proof of the theorem reveals that if E in T is integrable then using distinguished coordinates one can construct a connection D, called the basic connection, on T/E such that the associated curvature matrix K_D satisfies the following condition: for any symmetric, homogeneous polynomial $P \in \mathbb{C}[t_1, ..., t_n]$, $P(K_D) = 0$ if $\deg P > \text{rank } T - \text{rank } E$. For details on basic connections see [3].

Applications to Singular Holomorphic Foliations

A singular holomorphic foliation is defined as a coherent subsheaf of the tangent sheaf which is closed under the bracket operation. The singular set S of the foliation is the proper subvariety of M where the coherent subsheaf is not locally free. Then on $M - S$ there is a foliation whose leaves may intersect on S or may avoid S. Assume that S is compact. For example when M is compact S is always compact. Vanishing theorem applies here to define a residue on S. The fundamental observation which allows such an application is explained in the following set up: Let E be a subbundle of the holomorphic tangent bundle T such that for some proper subvariety S of M, $E|M-S$ is integrable. Then on $M-S$ we have a foliation and the vanishing theorem gives us some differential forms which vanish on $M-S$. To be able to say something for all of M we localize our attention to a neighbourhood of S. To be precise let Z be a connected component of S and let U be an open neighbourhood of Z which deformation retracts to it. Let $r : U \to Z$ be this retraction. By the vanishing theorem we can find a basic connection D_{U-Z} for $T/E|U-Z$. Let Σ be a compact set which contains Z in its interior and $\Sigma \subset U$. Define any connection D_Σ for $T/E|\Sigma$. Then glue these two connections together, i.e. let $f : U \to \mathbb{R}$ be a smooth function such that $f|M-\Sigma = 0$ and $F|Z = 1$ Now define a connection D for $T/E|U$ as

$$D = (1-f)D_{M-Z} + fD_\Sigma.$$

Then the curvature matrix K_D of D satisfies

$$P(K_D)|U-\Sigma = 0 \ \text{ if } \ \deg P > \text{rank } T - \text{rank } E,$$

since E is integrable on $U-\Sigma$ and $D|U-\Sigma = D_{U-\Sigma}|U-\Sigma$ is a basic connection. We then have a closed differential form $P(K_D)$ on U with compact support which uniquely defines a cohomology element $[P(K_D)]$ in $H^*_c(U;\mathbb{C})$ cohomology with compact supports. The Poincare duality isomorphism $PD : H^*_c(U;\mathbb{C}) \to H_*(U;\mathbb{C})$ defines a homology cycle $PD[P(K_D)]$ on U and $r_*PD[P(K_D)]$ gives us a homology cycle on Z, where $r_* : H_*(U;\mathbb{C}) \to H_*(Z;\mathbb{C})$ is the isomorphism induced by the deformation retraction r. We call $r_*PD[P(K_D)]$ the residue $\text{Res}_p(E;Z)$ at Z of the singular foliation E.

In the general case where we have an integrable coherent subsheaf $\mathcal{F}$ of the tangent sheaf T, $T/\mathcal{F}$ is not locally free and hence does not give rise to a vector bundle. Then we must maneuver around to bring the situation to a set up similar to the one above so that we can construct our global closed differential form with compact support. The process of getting a residue out of it is then a mere technicality. We mention two methods to build this global form.

A. Resolve $T/\mathcal{F}$ by locally free analytic sheaves. Notice that a global resolution by locally free analytic sheaves is possible in the real category but not in the complex case. Each locally free analytic sheaf of this resolution defines a connection and on $U-Z$ these connections can be used to define a connection on $T/\mathcal{F}|U-Z$, which is locally free there. We require that this connection be basic. Then we define our global differential form using the curvature matrices of these connections of the resolution. For details on this method see [3].

B. At every point $x \in U-Z$, the stalk $\mathcal{F}_x$ defines a k dimensional vector subspace of T_x, hence gives a point in the Grassmann bundle $\pi : G(k,T) \to U$ of k planes in T. The closure U' of the image of $U-Z$ in $G(k,T)$ is an analytic space [1]. On U' there is a natural vector bundle W which agrees with the pull back of $T/\mathcal{F}$ outside $\pi^{-1}(Z)$. Hence the basic connection of $T/\mathcal{F}|U-Z$ carries onto $W|U'-\pi^{-1}(Z)$ and when U' is smooth, can be extended to a connection on $W'|U'$ to give a differential form with compact support.For details on this construction see [1]. However this form defines a different residue than the one described above in **(A)**. The difference between these two residues can be computed using a Grassmann graph construction [1]. For details on the Grassmann graph technique in the algebraic category see [4].

Meromorphic Vector Field Theorem

When the rank k of the singular foliation is 1 we have a relatively well understood residue. The calculations of the residue in this case uses the Jacobian matrix rather than the curvature matrix and is due to the Meromorphic Vector Field theorem [3,5]. We illustrate this theorem on a vector field of [6]. Let X_1, X_2, X_3 be the Euclidean coordinates on a local chart of $\mathbb{P}^3$, and define a vector field

$$V = (X_2 - X_1^2)\frac{\partial}{\partial X_1} + (X_3 - X_1X_2)\frac{\partial}{\partial X_2} - (X_1X_3)\frac{\partial}{\partial X_3}.$$

Origin is the only zero of this vector field which corresponds to the flow

$$\exp(t \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix})$$

which in turn gives a singular holomorphic foliation of rank one with the origin the only singularity. We define the Jacobian matrix A as

$$A = (\frac{\partial f_i}{\partial X_j} \quad \text{where} \quad f_1 = X_2 - X_1^2, f_2 = X_3 - X_1X_2, \quad \text{and} \quad f_3 = X_1X_2.$$

By the Nullstellensatz there exist holomorphic functions b_{ij} and positive integers n_i, $1 \leq i, j \leq 3$, such that

$$X_i^{n_i} = \sum_{j=1}^{3} b_{ij} f_j.$$

We explicitly calculate these functions and find them as

$$\begin{aligned} X_1^4 &= X_1^2 f_1 - X_2 f_2 - f_3 \\ X_2^2 &= -X_1^2 f_1 - X_2 f_2 - f_3 \\ X_3^3 &= 0 f_1 + X_3 f_2 - x_2 f_3 \end{aligned}$$

Let

$$\begin{aligned} \sigma_1(t_1, t_2, t_3) &= t_1 + t_2 + t_3 \\ \sigma_2(t_1, t_2, t_3) &= t_1t_2 + t_1t_3 + t_2t_3 \\ \sigma_3(t_1, t_2, t_3) &= t_1t_2t_3. \end{aligned}$$

We will calculate the residue of V at the origin with respect to the polynomials $\sigma_3, \sigma_1\sigma_2, \sigma_1^3$; note that the degrees of these polynomials are greater than the rank of $T\mathbb{P}^3$ minus the rank of the foliation.

a. $\sigma_3(A) = \det(A) = 2X_1^3 - X_3 - X_1X_2$. The residue is the coefficient of $(X_1X_2X_3)^{-1}$ in the Laurent expansion of $\big(\sigma_3(A)\det(B)\big)/(X_1^4X_2^2X_3^2)$ where $B = (b_{ij})$. The residue can be found to be 4.

b. $\sigma_1(A)\sigma_2(A) = -20X_1^3 - 4X_1X_2$. And the residue, again the coefficient of $(X_1X_2X_3)^{-1}$ in the Laurent expansion of $\big(\sigma_1(A)\sigma_2(A)\det(B)\big)/(X_1^4X_2^2X_3^2)$, is 24.

c. $\sigma_1(A)^3 = -64X_1^3$. The coefficient of $(X_1X_2X_3)^{-1}$ in $\big(\sigma_1(A)^3\det(B)\big)/(X_1^4X_2^2X_3^2)$ is 64.

The relevance of these numbers to $\mathbb{P}^3$ is as follows: Let $\omega \in H^*(\mathbb{P}^3;\mathbb{C})$ be the fundamental class of a hyperplane in $\mathbb{P}^3$. Then the total chern class of $\mathbb{P}^3$ is given by $c(\mathbb{P}^3) = (1+\omega)^4$, i.e. $c_1(\mathbb{P}^3) = 4\omega$, $c_2(\mathbb{P}^3) = 6\omega^2$, $c_3(\mathbb{P}^3) = 4\omega^3$. Letting $\cap$

denote the cap product and $\cup$ the cup product we calculate the chern numbers of $\mathbb{P}^3$:

$$c_3(\mathbb{P}^3) \cap [\mathbb{P}^3] = 4$$
$$\big(c_1(\mathbb{P}^3) \cup c_2(\mathbb{P}^3)\big) \cap [\mathbb{P}^3] = 24$$
$$\big(c_1(\mathbb{P}^3)\big)^3 \cap [\mathbb{P}^3] = 64$$

which are the residues calculated above in (**a**), (**b**) and (**c**) respectively. For a proof of the Meromorphic Vector Field theorem see [5].

References

[1] Sertöz, S., Singular Holomorphic Foliations, UBC thesis, 1984.

[2] Bott, R., *On a Topological Obstruction to Integrability,* Proceedings of Sympos. in Pure Math., AMS, Vol. 16, 127-131, 1970.

[3] Baum, P., Bott, R., *Singularities of Holomorphic Foliations*, Journal of Differential Geometry, 7, 279-342, 1972.

[4] Fulton, W., *Intersection Theory,* A Series of Modern Surveys in Mathematics, Springer-Verlag, 1984.

[5] Baum, P., Bott, R., *On the Zeros of Meromorphic Vector Fields*, Memoires dedies a Georges de Rham, Springer-Verlag, 29-47, 1970.

[6] Carrell, J. B., Lieberman, D., *Vector Fields and Chern Numbers*, Math. Ann., 225, 263-273, 1977.

Mathematics Department, Bilkent University, 06533 Ankara, Turkey

E-mail address: `sertoz@@fen.bilkent.edu.tr`