ON A DIOPHANTINE PROBLEM OF FROBENIUS

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Communicated by C. Arf

ABSTRACT. Frobenius problem is to find the largest integer g which cannot be expressed as a linear combination of some given natural numbers $1 < a_1 < \cdots < a_n$ with nonnegative integer coefficients, where $a_1, \ldots, a_n$ are relatively prime. We consider this problem as the investigation of the lattice points of the region $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1, \ldots, x_n > 0\}$. It can then be shown that g is the largest element of a finite set S, (lemma 1). Even though S is finite it is constructed using an infinite set H. Our main result (theorem 4) is to show that a finite subset E of H suffices to determine g. These methods also yield an upper bound for g, (lemmas 2 and 3).

1. Introduction

The Diophantine problem of Frobenius consists of finding the largest integer N which cannot be expressed as a linear combination of some given relatively prime integers with nonnegative coefficients. To be more precise let $1 < a_1 < \cdots < a_n$ be integers whose greatest common divisor is 1. For a given $N \in \mathbb{N}$ we look at the existence of solutions of the equation $N = a_1y_1 + \cdots + a_ny_n$ with $(y_1, \ldots, y_n) \in \mathbb{N}^n$. It is known that for all N sufficiently large a solution vector $(y_1, \ldots, y_n) \in \mathbb{N}^n$ exists. Following [4] and [3] we denote by $g(a_1, \ldots, a_n)$ or simply by g the largest integer N for which no such solution vector exists. In this paper we describe a procedure for general n to find $g+1$ which consists of finding the maximal element of a certain restricted set. Such maximality procedures are common in the literature, see [2], [8] and [6]. The method we use is to find g from a finite set which is formed by using an infinite set H, see lemma 1. Our main observation is theorem 4 where we show that a finite subset E of H can be used for the same purpose. Our method also gives an upper bound for g which is in some cases comparable to the bounds given in [3]. For an extensive bibliography of the literature see Selmer's paper [7].

2. Cut-off point for the General Case

2.1. Let W be the semigroup generated by $a_1, \ldots, a_n$ with $(a_1, \ldots, a_n) = \nu$. We order the elements of W in increasing order

$$W = \{i_0 = 0, i_1, i_2, \ldots\}.$$

It is a folk theorem that all sufficiently large multiples of ν are elements of W.[1] For completeness we record two interesting proofs here.

Appeared in the Bulletin of the Technical University of İstanbul, vol: 39 (1986), 41-51.

In this retyping all the known typos are corrected. All the footnotes are added during this retyping.

[1]We thank Richard A. Smith of Honolulu for sending in a correction in 2004.

THEOREM: *For r sufficiently large we have*

$$i_{r+l} = i_r + l\nu, \quad l \in \mathbb{N}.$$

PROOF I: (Arf [A])
Let $\nu_l = (i_1, \dots, i_l)$, $l = 1, 2, \dots,$ Since ν_l divides ν_{l-1} for all $l > 1$, there exists q such that

$$\nu_q = \nu_{q+1} = \cdots = \nu.$$

Write

$$\nu = m_1 i_1 + \cdots + m_q i_q \quad \text{with} \quad m_1, \dots, m_q \in \mathbb{Z}$$

and let

$$m = \max_{1 \le h \le q.} \mid m_h(\frac{i_1}{\nu} - 1) \mid$$

Now it is claimed that all the multiples of ν that are greater than

$$i = m i_1 + \cdots + m i_q$$

are in W.

First observe that i itself is a multiple of ν. Multiples for ν larger than i are of the form $i + l\nu$ for $l \in \mathbb{N}$. We look at three cases.

I) $l = 0, 1, \dots, \dfrac{i_1}{\nu} - 1.$
$i + l\nu = (m + lm_1)i_1 + \cdots + (m + lm_q)i_q = n_1 i_1 + \cdots + n_q i_q$
where $n_h \in \mathbb{N}$ since $m \ge |m_h l|$. Hence $i + l\nu \in W$.

II) $l = \dfrac{i_1}{\nu}.$
$i + l\nu = i + i_1$ which is clearly in W.

III) $l > \dfrac{i_1}{\nu}$. Then $l = s\dfrac{i_1}{\nu} + t$, $0 \le t < \frac{i_1}{\nu}$,
$i + \nu = i + s i_1 + t\nu$, and this is also in W due to the considerations in **I** and **II** above.

PROOF II: Write

$$\nu = m_1 a_1 + \cdots + m_n a_n \qquad (*)$$

and let

$$-N = \text{sum of negative terms in } (*)$$
$$P = \text{sum of positive terms in } (*)$$

Note $N \geq 0$, $P > 0$. We thus have $\nu = -N + P$, where each of N and P are non-negative linear combinations of $a_1, a_2, \ldots, a_n$.

$$\begin{aligned} \text{Let} \quad M &= (a_1 - 1)N. \\ \text{Then} \quad M + \nu &= (a_1 - 2)N + P \\ M + 2\nu &= (a_1 - 3)N + 2P \\ &\cdots \\ M + (a_1 - 1)\nu &= 0 + (a_1 - 1)P. \end{aligned}$$

Thus each of $M, M + \nu, M + 2\nu, \ldots, M + (a_1 - 1)\nu$ are non-negative linear combinations of $a_1, a_2, \ldots, a_n$ and therefore so is $M + k\nu$, for all integers $k \geq 0$.

2.2. **Upper Bound.** We assume that the integers $a_1, \ldots, a_n$ satisfy the conditions

$$(a_1, \ldots, a_n) = 1 \quad \text{and} \quad 1 < a_1 < \cdots < a_n.$$

Define the following sets:

$$\begin{aligned} H &= \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \;\big|\; a_1x_1 + \cdots + a_nx_n = -1\} \\ A &= \{(y_1, \ldots, y_n) \in \mathbb{N}^n \;\big|\; \forall (x_1, \ldots, x_n) \in H, \exists i \text{ s.t. } y_i + x_i < 0\} \end{aligned}$$

and

$$S = \{n \in \mathbb{N} \;\big|\; \exists (y_1, \ldots, y_n) \in A \text{ s.t. } n = a_1y_1 + \cdots + a_ny_n\}$$

where $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is the set of non-negative integers.

Lemma 1:

$$g = \big(\max_{n \in S} n\big) - 1.$$

Before giving a proof let us develop some notation which will be used throughout;

π_N is the plane defined by $a_1x_1 + \cdots a_nx_n = N$, i.e.

$\pi_N = \big\{(X_1, \ldots, X_n) \in \mathbb{R}^n \;\big|\; a_1X_1 + \cdots + a_nX_n = N\big\}$.

L_N is the lattice on π_N, $L_N = \pi_N \cap \mathbb{Z}^n$, and

L_N^+ is the part that lies in the first quadrant, $L_N^+ = L_N \cap \mathbb{N}^n$.

Proof of Lemma: It is clear that $A \subset \bigcup_{N \geq 0} L_N^+$. In fact $A \cap L_N^+ \neq \emptyset$ iff $L_N^+ \neq \emptyset$ and $L_{N-1}^+ = \emptyset$; If $L_{N-1}^+ \neq \emptyset$ then let $q \in L_{N-1}^+$ and $p \in A \cap L_N^+$. Then $q - p \in H$ and $p + (q - p) \in \mathbb{N}^n$ contradicting the fact that $p \in A$. Now $g + 1 \in S$. If $N > g + 1$ is also in S then $L_{N-1}^+ = \emptyset$ contradicting the definition of g. Hence the lemma.

Remark: An alternate description of H can be given for any fixed point $p \in L_N$ for any N,

$$H = \{q - p \mid q \in L_{N-1}\}.$$

This alternate description disposes of the unpleasant possibility of having two different points $p, p' \in L_N^+$ such that $p \in A$ but $p' \notin A$; If $p' \notin A$ then $\exists q' \in L_{N-1}^+$,[2] but $x = q' - p' + (p' - p) \in H$ and $p + x \in L_{N-1}^+$ contrary to $p \in A$. In order to describe a procedure of employing the lemma in calculating g, we give a decomposition of H which will facilitate our work; Let

$$H = H_1 \cup \cdots \cup H_n \cup H'$$

where $H_i = \{(x_1, \ldots, x_n) \in H \mid x_i < 0 \ \text{ and } \ x_j \geq 0 \ \text{ for } \ j \neq i\ \}$ for $i = 1, \ldots, n$ and H' consists of those vectors of H which have at least two negative entries.

For $i = 1, \ldots, n$, choose a vector $(x_{i1}, \ldots, x_{in}) \in H_i$ such that[3]

$$-x_{ii} = \min\{\ |x_i| \ \ \big| \ \ (x_1, \ldots, x_n) \in H_i\ \}.$$

Let $(y_1, \ldots, y_n) \in A$ be such that

$$g + 1 = a_1 y_1 + \cdots + a_n y_n.$$

Then in particular we have

$$\begin{gathered} y_1 + x_{11} < 0 \\ \vdots \\ y_n + x_{nn} < 0 \end{gathered}$$

This proves the following

Lemma 2: *With $(x_{11}, \ldots, x_{1n}), \ldots, (x_{n1}, \ldots, x_{nn})$ chosen as above, we have*

$$g \leq a_1(-x_{11} - 1) + \cdots + a_n(-x_{nn} - 1) - 1.$$

Example A: $a_1 = 137$, $a_2 = 251$, $a_3 = 256$.

$$\begin{aligned} (x_{11}, x_{12}, x_{13}) &= (-11, 6, 0) \\ (x_{21}, x_{22}, x_{23}) &= (20, -15, 4) \\ (x_{31}, x_{32}, x_{33}) &= (6, 11, -14) \end{aligned}$$

$$g + 1 \leq 137(10) + 251(14) + 256(13) = 8212.$$

Erdös and Graham give the following bounds, see [3];

$$\begin{aligned} g &\leq (a_1 - 1)(a_n - 1) - 1 = 34679 \\ g &\leq a_3 a_1 + a_3 \frac{(a_1, a_2, a_3)}{(a_1, a_2)} = 35328 \\ g &\leq 2a_{n-1}\left[\frac{a_n}{n}\right] - a_n = 42414. \end{aligned}$$

[2] i.e. $L_{N-1}^+ \neq \emptyset$ so pick any q' there.

[3] Note that only the i-th entry is negative in H_i and x_{ii} is the largest of the negative entries, hence $|x_{ii}| = -x_{ii}$ is minimum.

2.3. **Sufficiency.** We now aim towards our main result that a certain subset of H suffices for the above computations. To be more specific we need the following subset of H;

$$E = \{(x_1, \dots, x_n) \in H \;\big|\; |x_i| < a_1, \;\; i = 2, \dots, n \;\}.$$

Similar to the decomposition of H we decompose E

$$E = E_1 \cup \cdots E_n \cup E'$$

where $E' = E \cap H'$, $E_i = E \cap H_i$, $i = 1, \dots, n$.

The following lemma establishes the fact that the x_{ii}, defined through H_i, can be defined using only E_i, $i = 1, \dots, n$.

Lemma 3:[4]

$$-x_{ii} = \min\{\; |x_i| \;\big|\; (x_1, \dots, x_n) \in E_i \;\}, \quad i = 1, \dots, n.$$

Proof: Assume that $(x_{i1}, \dots, x_{in}) \notin E_i$.[5] We distinguish two cases according to $i = 1$ or not.

Case 1: $i = 1$, $(x_{11}, \dots, x_{1n}) \notin E_1$, then $x_{1j} \geq a_1$ for some $j > 1$. Notice that $x_{11} + a_j < 0$ since

$$\begin{aligned} -1 &= (a_1, \dots, a_n)(x_{11}, \dots, x_{1n}) \\ &= (a_1, \dots, a_n)(x_{11} + a_j, x_{12}, \dots, x_{1j} - a_1, \dots, x_{1n}) \end{aligned}$$

and the LHS being negative, all the entries on the right hand side cannot be non-negative. hence $x_{11} + a_j < 0$. But then $(x_{11} + a_j, x_{12}, \dots, x_{1j} - a_1, \dots, x_{1n}) \in H_1$ and the minimality of x_{11} is contradicted. Therefore $(x_{11}, \dots, x_{1n}) \in E_1$.
Case 2: [6]
Subcase 2.1: $x_{ii} \leq -a_1$.[7]
Let $(y_1, \dots, y_n) \in E_i$ with $-y_i = \min\{|x_i| \;\big|\; (x_1, \dots, x_n) \in E_i \;\}$. Then clearly $-a_1 < y_i \leq x_{ii} < 0$. Hence this subcase is not possible.
Subcase 2.2: $x_{ij} \geq a_1$ for some $i \neq j$, $j = 2, \dots, n$.
Let $x_{ir} = K_r a_1 + u_r$ where $K_r \geq 0$, $0 \leq u_r < a_i$, $i \neq r$, $r = 2, \dots, n$. If we denote

$$u_1 = x_{i1} + K_2 a_2 + \cdots + K_{i-1} a_{i-1} + K_{i+1} a_{i+1} + \cdots + K_n a_n$$

then

$$(u_1, u_2, \dots, u_{i-1}, x_{ii}, u_{i+r}, \dots, u_n) \in E_i.$$

This shows that the vector $(x_{11}, \dots, x_{nn})$ which is defined using $H_1, \dots, H_n$ can instead be defined using only $E_1, \dots, E_n$.[8]

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[4]The claim is that $-x_{ii}$, which was chosen by the same minimality condition in H_i, can also be found by restricting our attention to the smaller set E_i.

[5]We assume that $(x_{i1}, \dots, x_{in})$ is in H_i but not in E_i.

[6]Now we set i to a number larger than 1.

[7]i.e. $|x_{ii}| \geq a_1$, which is one reason why the definition of a vector belonging to E_i may be violated.

[8]We have actually shown that even if the vector $(x_{i1}, \dots, x_{ii}, \dots, x_{in})$ is not in E_i, another vector such as $(u_1, \dots, u_{i-1}, x_{ii}, u_{i+1}, \dots, u_n)$ which is constructed above can be shown to be in E_i. Hence the number x_{ii} can be obtained from the set E_i.

We have shown the sufficiency of $E_1, \dots, E_n$ in the above computations. To show the sufficiency of E in computing g we define the following sets

$$A_E = \{(y_1, \dots, y_n) \in \mathbb{N}^n \ \mid \ \forall (x_1, \dots, x_n) \in E, \ \exists i \text{ s.t } y_i + x_i < 0 \ \}$$

and

$$S_E = \{n \in \mathbb{N} \ \mid \ n = y_1 a_1 + \cdots + y_n a_n \ \text{ for some } \ (y_1, \dots, y_n) \in A_E \ \}.$$

Our main observation is the following theorem.

THEOREM 4:

$$g = \big(\max_{n \in S_E} \ n\big) - 1.$$

PROOF: It suffices to show that $A = A_E$. That $A \subseteq A_E$ is trivial. To show the converse let $(y_1, \dots, y_n) \in A_E$. Assume that $(y_1, \dots, y_n) \notin A$, then there exists $(x_1, \dots, x_n) \in H$ such that $x_i + y_i \geq 0$, $i = 1, \dots, n$. Note that in the light of lemma 3 if $(y_1, \dots, y_n) \in A_E$ then $y_i < -x_{ii}$, $i = 1, \dots, n$ and in particular for $i = 2, \dots, n$, we have $y_i < a_1$. Hence if $x_r < 0$ for $r = 2, \dots, n$ then $x_r + y_r \geq 0$ implies that $|x_r| < a_1$. Define a vector $(u_1, \dots, u_n) \in E$ as follows. If $x_i < 0$ then let $u_i = x_i$, $i \neq 1$. If $x_i \geq 0$ and $i \neq 1$ define u_i as x_i mod a_1, i.e.

$$x_i = k_i a_1 + u_i, \ \ k_i \geq 0, \ \ 0 \leq u_i < a_1.$$

And for $i = 1$, $u_1 = x_1 + k_{i_1} a_{i_1} + \cdots + k_{i_r} a_{i_r}$ where $x_{i_1}, \dots, x_{i_r}$ are those x_i's , $i \neq 1$, which are nonnegative. Then $(u_1, \dots, u_n) \in E$ and clearly $u_i + y_i \geq 0$ for $i = 2, \dots, n$. For $i = 1$, observe that $x_1 \leq u_1$, and if $y_1 + x_1 \geq 0$, then $y_1 + u_1 \geq 0$. But this contradicts the fact that $(y_1, \dots, y_n) \in A_E$. Hence $(y_1, \dots, y_n) \in A$.

EXAMPLE B: $a_1 = 137, \quad a_2 = 251, \quad a_3 = 256.$
Start with the vector $(10, 14, 13) = (-x_{11} - 1, -x_{22} - 1, x_{33} - 1)$, see example A. The only vectors in E' that gives a nonnegative entry when added to $(10, 14, 13)$ are $(13, -2, -5)$, $(-4, -7, 9)$ and $(37, -10, -10)$. Thus we modify $(10, 14, 13)$ so that when added to $(13, -2, -5)$ it gives a negative entry and its dot product with $(137, 251, 256)$ is maximal with this property. This gives us two candidates

$$U = (10, 1, 13), \quad V = (10, 14, 4).$$

We modify these vectors with respect to $(-4, -7, 9)$; U already gives negative entries when added to $(-4, -7, 9)$ so does not change whereas V gives rise to two other candidates.

$$Z = (3, 14, 4), \quad \text{and} \quad W = (10, 6, 4).$$

These three vectors, U, Z, W now satisfy the required condition, i.e. they are in A_E, and

$$g + 1 = \max_{n \in S_E} \ n \ = \max_{n \in S_{\{U,Z,W\}}} \ n \ = \max_{n \in \{3900, 4949\}} \ n = 4949.$$

Examples A and B are worked with the aid of a computer.[9]

REMARK 1: A few words are in order to explain what we mean by "modification"

[9] A friend of us did these calculations for us on the computer but we did not trust the machine and went over these calculations by hand. Then was the age of innocence!

of a vector (c_1, c_2, c_3) by (b_1, b_2, b_3) where $(b_1, b_2, b_3) \in H$ and $c_1, c_2, c_3 \in \mathbb{N}$. This particular process consists of carrying out the following algorithm and choosing all vectors satisfying the stated conditions;

1) If there is an i such that $c_i + b_i < 0$, then (c_1, c_2, c_3) remains unaltered.
2) If all $c_i + b_i \geq 0$, then
   a) choose $(-b_1 - 1, c_2, c_3)$ if $b_1 < 0$ and $c_1 + b_1 \geq 0$.
   b) choose $(c_1, -b_2 - 1, c_3)$ if $b_2 < 0$ and $c_2 + b_2 \geq 0$.
   c) choose $(c_1, c_2, -b_3 - 1)$ if $b_3 < 0$ and $c_3 + b_3 \geq 0$.

Remark 2: In order to obtain E it suffices to find a basis for the lattice

$$L = \{(y_1, \ldots, y_n) \in \mathbb{Z}^n \mid a_1 y_1 + \cdots + a_n y_n = 0\}.$$

For this purpose the following method seems appropriate; Let

$$1 = \sum_{i=1}^{n} m_{0i} a_i, \qquad d_1 = \sum_{i=2}^{n} m_{1i} a_i,$$
$$\cdots \qquad \cdots$$
$$d_{n-2} = \sum_{i=n-1}^{n} m_{n-2\ i} a_i, \qquad d_{n-1} = a_n$$

be the greatest common divisors of the sets

$$\{a_1, \ldots, a_n\}, \{a_2, \ldots, a_n\}, \ldots, \{a_{n-1}, a_n\}, \{a_n\}$$

respectively, where $m_{ij} \in \mathbb{Z}$. Using this data, define the vectors

$$\mathbf{A}_1 = (d_1, -a_1 m_{12}, -a_1 m_{13}, \ldots, -a_1 m_{1n-1}, -a_1 m_{1n})$$
$$\mathbf{A}_2 = (0, \frac{d_2}{d_1}, -\frac{a_2}{d_1} m_{23}, \ldots, -\frac{a_2}{d_1} m_{2n-1}, -\frac{a_2}{d_1} m_{2n})$$
$$\cdots\cdots$$
$$\mathbf{A}_{n-1} = (0, \ldots, 0, \frac{d_{n-1}}{d_{n-2}}, -\frac{a_{n-1}}{d_{n-2}}).$$

These vectors constitute a basis for L. This construction does not produce unique $\mathbf{A}_i$'s, however a canonical procedure for choosing m_{ij}'s can be given.

It is comforting to see that our method agrees with the classical result in case $n = 2$.

Corollary 5: *When $n = 2$, $g + 1 = a_1 a_2 - a_1 - a_2 + 1$.*

Proof: In this case

$$E_1 = \{(x_{11}, x_{12})\}, \quad E_2 = \{(x_{21}, x_{22})\}, \quad E' = \emptyset.$$

And according to our method $(-x_{11} - 1, -x_{22} - 1) \in A_E$ will give the maximal value, i.e.

$$\begin{aligned} g + 1 &= a_1(-x_{11} - 1) + a_2(-x_{22} - 1) \\ &= -a_1 x_{11} - a_1 - a_2 x_{22} - a_2 \\ &= -a_1 x_{11} - a_1 - a_2(x_{12} - a_1) - a_2 \\ &= -(a_1 x_{11} + a_2 x_{12}) - a_1 + a_1 a_2 - a_2 \\ &= 1 - a_1 + a_1 a_2 - a_2 \quad \text{as required.} \end{aligned}$$

Our method can also be used to find g in the special case of $(m, m+1, \ldots, m+n-1)$, see [5].

COROLLARY 6: *Let $a_i = m+i-1$, $i = 1, \ldots, n$; $n < m$. Then*

$$g+1 = m\{\frac{m-1}{n-1}\}$$

where for $r \in \mathbb{R}$, $\{r\} = k \in \mathbb{Z}$ such that $k-1 < r \leq k$ (Note that when $r \notin \mathbb{Z}$ then $\{r\} = [r]+1$, and when $r \in \mathbb{Z}$ then $\{r\} = r$).

Proof: We have

$$(1, -1, 0, \ldots, 0) \in E_2, \ldots, (0, \ldots, 0, 1, -1) \in E_n,$$

therefore $x_{22} = \cdots = x_{nn} = -1$. If $(y_1, \ldots, y_n) \in A_E$ then $0 \leq y_i < -x_{ii}$, $i = 1, \ldots, n$. Hence

$$y_2 = \cdots = y_n = 0.$$

We now know that $g+1$ is a multiple of a_1. The semigroup generated by $a_2, \ldots, a_n$ consists of the intervals $\bigcup [k(m+1), k(m+n-1)]$. To find x_{11} we find the smallest multiple of m that comes close to the end of one of these intervals within a unit, i.e. find the smallest k such that

$$k(m+n-1) - (k+1)m \geq -1.$$

This gives $k \geq \dfrac{m-1}{n-1}$.

Then $x_{11} = -\{\dfrac{m-1}{n-1}+1\}$ and $y_1 = \{\dfrac{m-1}{n-1}\}$ as required.

ACKNOWLEDGMENTS We thank professors C. Arf and T.C. Brown for valuable discussions.

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Received February 2, 1986

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