An Overview of the Search for Minimal Models of Algebraic Threefolds

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1 Introduction

The classification theory of algebraic threefolds had an exciting decade in the eighties and culminated in the major contributor Shigefumi Mori winning a Fields medal with [18] and [21]. The origins of the major ideas go back to the works of Hironaka [5] in the sixties. The road to the final accomplishment of the classification program is paved with the most prominent names of algebraic geometry. A list of the contributors suffices to awe a newcomer and the same applies to the beauty and delicacy of the theory.

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My aim in this manuscript is to expose the grand classification scheme of threefolds with as little machinery as possible. In particular I make no attempt to motivate the subject by testing new ideas on surfaces, which the reader should do systematically. Also I omit a tremendous amount of information on several aspects of the theory in order to reach the destination; the description of the Minimal Model Program. In particular I made no mention of the log version of the program but I hope that after this paper the reader will be well equipped to read these from other sources.

Miles Reid wrote [25] "...you learn to get around your own village before you set about memorising the entire motorway network". Expounding on this allegory this exposition can be considered as an address, a description of a route from your own village to the capital where the MMP resides. Many beautiful scenic spots are only briefly observed and numerous chances for exciting excursions along the highway are carefully avoided in order to arrive on schedule. Having once surveyed the main route of the highway the reader, the venturer, can go back and take the first exit of his or her choice. That at least is what I intend to do ...

The plan of this survey is as follows: In the following Section 2 I give a long and tedious proof of the existence of rational curves contracted by a proper birational map following [13]. This is perhaps the most elementary part of the theory and I want a beginning graduate student to build up confidence by observing that a fundamental observation can be proved totally within his or her existing vocabulary. This is the only place where a formal proof is given. In Section 3 the reader will find a collection of new concepts which play a part in the theory. The list of definitions is short enough to make the manuscript readable but long enough to equip the reader with tools to appreciate what is to come. In Section 4 the reader is brought into contact with the necessity of admitting singularities to the program. Section 5 gives a brief description of these singularities *après* Reid. Again we exercise great will power in avoiding a deeper understanding of the singularity machinery of higher dimensional geometry. Section 6 describes the existence and finiteness of the flip map which is a way of avoiding singularities worse than those we are ready to handle. And finally in Section 7 a full statement of the classification theorem is given with its implications on the structure of the cone of curves.

Any article written on this topic carries a long list of references because the achievements in higher dimensional geometry result from a truly international collaboration. After each section I give some references to lead the graduate student into the literature. This has the effect of mostly choosing those articles which are either expository in nature or include abundant motivation and explanation to ease the student's way into the topic. It is inevitable that I have failed to mention numerous researchers whose work merits mention by any standards. This I hope will be forgiven given my explicit aim of paving a graduate student's way into the subject.

The final version of this manuscript owes its merit and accuracy to the careful reading and correcting of Miles Reid. All the inaccuracies owe their survival to my unenviable dexterity in concealing them. I am also grateful to Reid for coming to Ankara and lecturing on these topics. In preparing this note, among other sources, I occasionally made use of his Bilkent lecture notes but I hereby acquit him of any responsibility thereof ...

2 Contracting Curves

In this section a variety means a smooth projective variety over $\mathbb{C}$ of dimension two or more. We agree to say that a variety Y is 'simpler' than another variety X when there is a regular birational map $f\colon X \longrightarrow Y$ between them. We interpret this by imagining that more than one point of X is mapped to a certain point of Y so f^{-1} cannot be defined set theoretically, so X has 'more' in it than Y. In this section we will justify this notion by showing that X contains at least one rational curve which is mapped to a point in Y. Moreover it will be observed that the canonical divisor K_X of X has negative intersection with this rational curve and we will later investigate the sufficiency of this necessary condition.

We first compare the intersection numbers $[C]\cdot K_X$ and $[f(C)]\cdot K_Y$ for an irreducible curve C in X, where K_X and K_Y are the canonical divisors of X and Y respectively and we use [ ] to denote equivalence class. Let s be a meromorphic section of the canonical sheaf ω_Y of Y. Then the zero set (s) of s defines a divisor which is linearly equivalent to K_Y. Pulling this section back by f defines a section of $f^*\omega_Y$. This then defines a section of ω_X under the natural map

$$f^*\omega_Y \longrightarrow \omega_X.$$

To describe this map locally let $(U, x = (x_1,\dots,x_n))$ be a local coordinate chart on X and $(V, y = (y_1,\dots,y_n))$ a coordinate chart on Y with $f(U)\subset V$. Then in these coordinates $s(y) = h(y)\mathrm{d}y_1\wedge\cdots\wedge\mathrm{d}y_n$ for some meromorphic function h on V. $f^*s = h(f(x))\mathrm{d}y_1\wedge\cdots\wedge\mathrm{d}y_n$ and the induced section on $\omega_X|U$ is $h(f(x))\det\left|\dfrac{\partial f_i}{\partial x_j}\right|\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$, where we take $f(x) = (f_1(x),\dots,f_n(x))$. Since this is a section of $\omega_X|U$, its zero set defines a divisor equivalent to K_X and we have

$$\begin{aligned} K_X|U &= (h(f(x))\det\left|\frac{\partial f_i}{\partial x_j}\right|) \\ &= (h(f(x))) + (\det\left|\frac{\partial f_i}{\partial x_j}\right|). \end{aligned}$$

Here we note that $(h(f(x))$ is the pullback of the divisor $K_Y|V$. On the other hand the above determinant, which is the Jacobian of the map f, vanishes where f is not a local isomorphism. Call this set E. Setting $E = \bigcup E_i$, where the E_i are irreducible codimension one components, we have

$$K_X|U \quad = \quad f^*K_Y|V + \sum a_i E_i \cap U.$$

Here the a_i denote the order of vanishing of the Jacobian and hence are nonnegative. Pasting this local data together we have

$$K_X \quad = \quad f^*K_Y + \sum a_i E_i.$$

Intersecting both sides of this equality with the irreducible curve C we have

$$[C]\cdot K_X \quad = \quad [f(C)]\cdot K_Y + \sum a_i\,[C]\cdot E_i.$$

Assume that $C \not\subset E$. Then C is not one of the E_i so $[C] \cdot E_i$ is nonnegative. Moreover $f(C)$ is a nontrivial curve in Y so that $[f(C)]$ is not identically equal to zero. We then have

$$\begin{aligned}[C] \cdot K_X - [f(C)] \cdot K_Y &= \sum a_i \, [C] \cdot E_i \\ &= \begin{cases} 0 & \text{if } C \cap E = \emptyset \\ > 0 & \text{if } C \cap E \neq \emptyset \end{cases}\end{aligned}$$

We record this for future reference:

Lemma 1 *Let $f\colon X \longrightarrow Y$ be a regular birational surjective map between smooth projective varieties of dimension $n \geq 2$ and C an irreducible curve in X not mapped to a point by f. Then*

$$\begin{aligned}[C] \cdot K_X &= [f(C)] \cdot K_Y \quad \text{if } \; C \cap E = \emptyset, \\ [C] \cdot K_X &> [f(C)] \cdot K_Y \quad \text{if } \; C \cap E \neq \emptyset,\end{aligned}$$

where E is the set consisting of points where f is not a local isomorphism.

Next we analyze the geometry around a point $y_0 \in Y$ where f^{-1} is not defined. First assume that Y is a smooth projective variety of dimension $n \geq 3$. Assume further that Y lies in the projective space $\mathbb{P}^N$. Consider the space G of all hyperplanes in $\mathbb{P}^N$ passing through y_0. G is isomorphic to a copy of $\mathbb{P}^{N-1}$. Let G_1 be the subset of G consisting of hyperplanes not containing $B - \{y_0\}$, where B is the subset of Y where f^{-1} is not defined. If $B = \{y_0\}$ then G_1 is all of G, otherwise it is a proper open subset of G. Further let G_2 be the subset of G consisting of those hyperplanes which intersect Y in a smooth locus. This involves the nonvanishing of certain Jacobians. Since Y is smooth these Jacobians do not vanish identically. Hence G_2 is an open subset of G. Then $G_1 \cap G_2$ is nonempty since any two nonempty open sets intersect in Zariski topology. Hence there is a hyperplane H in $\mathbb{P}^N$ which intersects Y through y_0 in a smooth locus. Note that codim $B \geq 2$ since f^{-1} is a rational map from a projective variety. Consequently codim $B \cap H \geq 2$ since H does not contain B

unless $B = \{y_0\}$. By induction we can construct a smooth surface S in Y such that $S \cap B = \{y_0\}$. Moreover by a similar argument there exist hypersurfaces H_0 and H_1 such that $C_0 = Y \cap H_0$ is a smooth curve in S passing through y_0 and $C_1 = Y \cap H_1$ is another smooth curve in S not passing through y_0. Since they are both hyperplane sections C_0 and C_1 represent the same divisor class in S. If $\dim Y = 2$, then we only construct C_0 and C_1 and set $S = Y$.

We now restrict f^{-1} to S to obtain the rational map

$$f^{-1}: S \dashrightarrow X.$$

Observe that f^{-1} is an isomorphism on $S - \{y_0\}$ into X. Any rational map on a surface can be resolved by a finite sequence of blowups. Let

$$\sigma: S' \longrightarrow S$$

be a composite of these blowups such that the composite

$$f' = f^{-1} \circ \sigma: S' \longrightarrow X$$

is a regular map. Denote the exceptional divisor $\sigma^{-1}\{y_0\}$ of σ by $D = \bigcup D_i$, where as usual the D_i are rational curves.

Our plan at this point involves three stages: **(i)** Pull the curves C_0 and C_1 back to S' by σ^{-1}. **(ii)** Send their preimages to X by f'. **(iii)** Intersect these by canonical classes using Lemma 1 with the setup $f: X \longrightarrow Y$. This plan is carried out below.

(i): Since $y_0 \in C_0$, the pullback of C_0 under σ^{-1} meets the exceptional divisor D but $y_0 \notin C_1$ so its pullback is disjoint from D. As equivalence classes of divisors we can write

$$\begin{aligned} \sigma^*[C_0] &= [C_0'] + \sum a_i[D_i], \quad a_i \geq 0, \\ \sigma^*[C_1] &= [C_1'], \end{aligned}$$

where C_j' is the monoidal transform of C_j under σ, i.e. the closure in S' of $\sigma^{-1}(C_j - \{y_0\})$, $j = 0, 1$. Note at this point that since $[C_0] = [C_1]$ in S we have

$$[C_0'] + \sum a_i[D_i] = [C_1'].$$

(ii): Sending both sides of the above equation by f' we have

$$[f'(C_0')] + \sum a_i[f'(D_i)] \quad = \quad [f'(C_1')]. \qquad (1)$$

Note that $f \circ f'(C_0') = C_0$ so $f'(C_0') \not\subset E$, but since $y_0 \in C_0$ the intersection $f'(C_0') \cap E$ is not empty. On the other hand $f'(C_1') \cap E = \emptyset$.

(iii): We can now apply Lemma 1 to the above class of divisors in X. Intersecting both sides of equation (1) by K_X we have

$$[f'(C_0')] \cdot K_X + \sum a_i[f'(D_i)] \cdot K_X \quad = \quad [f'(C_1')] \cdot K_X. \qquad (2)$$

The key point is to relate this equality with its image under the map $f \colon X \longrightarrow Y$ using Lemma 1:

$$\begin{aligned}
[f'(C_0')] \cdot K_X &> [f \circ f'(C_0')] \cdot K_Y \quad \text{(Lemma 1)} \\
&= [f \circ f^{-1} \circ \sigma(C_0')] \cdot K_Y \\
&= [C_0] \cdot K_Y \\
&= [C_1] \cdot K_Y \\
&= [f \circ f^{-1} \circ \sigma(C_1')] \cdot K_Y \\
&= [f \circ f'(C_1')] \cdot K_Y \\
&= [f'(C_1')] \cdot K_X \quad \text{(Lemma 1).}
\end{aligned}$$

Together with equation (2) this implies that

$$\sum a_i[f'(D_i)] \cdot K_X \quad < \quad 0.$$

This means that at least one of the $f'(D_i)$, say $f'(D_{i_0})$, is not a point and hence is a rational curve, since the D_i are all rational curves, and

$$f'(D_{i_0}) \cdot K_X < 0.$$

Note also that $f(f'(D_{i_0})) = \sigma(D_{i_0}) = y_0$, so $f'(D_{i_0}) \subset E$.

We have thus proved the following theorem:

Theorem 2 *Let X and Y be smooth projective varieties and*

$$f \colon X \longrightarrow Y$$

a regular birational map. Let E be the subset of X where f is not a local isomorphism. Then there exists a rational curve in E whose intersection with the canonical divisor of X is strictly negative.

The converse of this theorem would start a classification program. Suppose that X contains a curve C which has negative intersection with the canonical divisor. If we knew that we could find a regular birational map $f\colon X \longrightarrow Y$ onto some variety Y such that the curve C is collapsed to a point by f, then we could take Y as a 'simpler' representative of the birational equivalence class of X. We also need to know when there are no more curves to collapse. To formalize the results we need to distinguish certain concepts as definition, as we will do in the following section.

References: [3], [13].

3 Curves and Divisors

We are interested in smooth projective threefolds but the following definitions make sense when X is only a normal projective variety. This will be useful later when we discover that we need to work with some singular threefolds too.

Definition 3

$\mathrm{Div}(X)$	:=	*Free Abelian group of Cartier divisors on X.*
$Z^1(X)$	:=	*Free Abelian group of Weil divisors on X.*
$Z_1(X)$	:=	*Free Abelian group of one cycles on X.*
$\mathrm{Pic}(X)$	:=	*Group of line bundles on X, also known as the Picard group.*

When L is a line bundle and C is an irreducible curve in X we can define an intersection number $D \cdot C$, or $C \cdot D$, as the degree of the pullback bundle $f^*(L)$ on $\tilde{C}$ where $f\colon \tilde{C} \longrightarrow C$ is a normalization of C. We then extend this to a product on $\mathrm{Pic}(X) \times Z_1(X)$ by linearity.

There is a natural morphism from $\mathrm{Div}(X)$ into $\mathrm{Pic}(X)$, sending D to its associated line bundle $\mathcal{O}(D)$. When X is projective this map is surjective. This allows us to define a product on $\mathrm{Div}(X) \times Z_1(X)$ by letting $D \cdot C$ be $\mathcal{O}(D) \cdot C$ for a Cartier divisor D.

Later we will be dealing with varieties X for which the canonical divisor K_X does not exist but mK_X is Cartier for some positive integer m. For an irreducible curve C in X we will then define $K_X \cdot C$ as $\frac{1}{m}(mK_X \cdot C)$. This makes it necessary to extend our definitions to divisors with rational coefficients, which we do in the definitions of the following objects.

We say that two Cartier divisors D_1 and D_2 are *numerically equivalent*, and we write $D_1 \equiv D_2$, if $D_1 \cdot C = D_2 \cdot C$ for every curve C in X. We similarly define numerical equivalence on $Z_1(X)$. Taking quotients by numerical equivalence and tensoring with the rationals we obtain the following objects:

Definition 4
$N^1(X) := (\mathrm{Div}(X)/\equiv) \otimes \mathbb{Q}$
$N_1(X) := (Z_1(X)/\equiv) \otimes \mathbb{Q}$

When X is projective $N^1(X)$ is isomorphic with the image of $\mathrm{Pic}(X) \otimes \mathbb{Q}$ in $H^2(X, \mathbb{Z}) \otimes \mathbb{Q}$ where the image is taken as the first Chern class of a line bundle. This image is of finite rank. We denote this rank by $\rho(X)$ and call it the *Picard number* of X. Since $N^1(X)$ and $N_1(X)$ are dual $\mathbb{Q}$-vector spaces, they are both of finite dimension $\rho(X)$.

Since $N_1(X)$ is a finite dimensional vector space we can define a topology on it by choosing a norm, and the topology is independent of the choice of the norm. We now come to the fundamental concept of the subject.

Definition 5

$\mathrm{NE}(X) :=$ *cone generated in* $N_1(X)$ *by irreducible curves in* X.
$\overline{\mathrm{NE}}(X) :=$ *the closure of* $\mathrm{NE}(X)$ *in* $N_1(X)$. *This is known as the closed cone of curves in* X.

A subset F of $\overline{\mathrm{NE}}(X)$ is called *extremal* if for every u and v in $\overline{\mathrm{NE}}(X)$, $u+v$ cannot be in F unless both u and v are in F. A one dimensional extremal subset is called an *extremal ray.*

For any divisor class D in $N^1(X)$ we say that D is *nef* if $D \cdot C \geq 0$ for every C in $\overline{\mathrm{NE}}(X)$. In particular we are interested when K_X is nef.

We have mentioned Weil divisors but did nothing with them so far. Every Cartier divisor defines a Weil divisor but the converse is not necessarily true. If for every Weil divisor D there is a positive integer m such that mD becomes Cartier then we say that X is $\mathbb{Q}$-factorial. This concept will play a vital role later when we start working on singular threefolds.

After this barrage of definitions we are finally ready to do some work!

References: [2], [3], [13].

4 Contraction Map

Going back to Theorem 2 we see that if $f\colon X \longrightarrow Y$ contracts a curve C to a point then **(i)** $K_X \cdot C < 0$ and **(ii)** f contracts every curve which is numerically equivalent to C. We are asking for a converse: If $K_X \cdot C < 0$ for some curve C in X, then can we construct a map $f\colon X \longrightarrow Y$ into some smooth projective variety Y such that f contracts to a point every curve which is numerically equivalent to C?

If D is a Cartier divisor such that $\mathcal{O}(mD)$ is generated by global sections for some $m > 0$, then these sections define a map $\Psi_{mD}: X \longrightarrow \mathbb{P}^N$ into some projective space. Moreover we have $\Psi^*_{mD}(\mathcal{O}(1)) = \mathcal{O}(mD)$. This is expressed by saying that mD is *free* or *base point free* or that D is *eventually free*. If moreover Ψ_{mD} is a closed immersion, then we say that mD is *very ample*, or that D is *ample*.

Assume that D is a nef divisor on X and that mD is base point free. Define a subset of $\overline{\mathrm{NE}}(X)$ as

$$D^\perp = \{C \in \overline{\mathrm{NE}}(X) \mid D \cdot C = 0\}.$$

We then have for any $C \in D^\perp$

$$\begin{aligned} 0 &= D \cdot C \\ &= \frac{1}{m} mD \cdot C \\ &= \frac{1}{m} \Psi^*_{mD}(H) \cdot C \\ &= \frac{1}{m} H \cdot (\Psi_{mD})_*(C) \end{aligned}$$

where H is a hyperplane section in $\Psi_{mD}(X)$. But any nondegenerate curve has positive intersection with a hyperplane section, the intersection being the degree of the curve in general. Since we obtained zero in this case the curve C and all other curves numerically equivalent to it must have been contracted to a point by Ψ_{mD}.

Assume that $K_X \cdot C < 0$. Using the duality of the spaces $N^1(X)$ and $N_1(X)$ it is easy to find a nef divisor D such that $D^\perp = \{C\}$. The question is whether mD will be base point free for some $m > 0$. The affirmative answer was provided by Mori for smooth projective threefolds.

Theorem 6 (Mori) *Let X be a smooth projective threefold and C a curve on X such that C spans an extremal ray of $\overline{\mathrm{NE}}(X)$ and $K_X \cdot C < 0$. Then there is a map $f: X \longrightarrow Y$ onto some projective variety Y such that f contracts C and all other curves numerically equivalent to it. (f is the m-canonical map of some nef divisor D*

with $D^{\perp} = \{C\}$.) We then have the following cases:
(i): $\dim Y < \dim X$. *Then X is a fibred space.*
(ii): $\dim Y = \dim X$.
  (a): *Y is smooth and $\rho(Y) < \rho(X)$.*
  (b): *Y has an isolated hypersurface or quotient singularity.*

The classification program successfully terminates in the case of **(i)** and continues for finitely many more steps before stopping in the case of **(ii-a)**. But we have to stop unsuccessfully in the case of **(ii-b)** since we have reached to a class of varieties outside our accepted category. This is the point where we consent to admit some mild singularities into the theory.

References: [6], [11], [18].

5 Canonical Singularities

The singularities obtained by contracting an extremal ray in a smooth threefold are fully described by Reid. Since these singularities are obtained as the image of an m-canonical mapping associated to an effective divisor Reid called them "canonical singularities". As their surface counterparts are called DuVal singularities, I imagine that the future generations will with equal comfort and justice call them "Reid singularities".

The first obstacle for the program in the singular case is the definition of a suitable K_X. In fact we only need to know if some 'representative' of K_X has negative intersections with some curves. For that matter if K_X fails to exist but mK_X exists for some positive integer m, then we can still define $K_X \cdot C$ as $\frac{1}{m}mK_C \cdot C$. This way we can transport all the terminology about extremal rays into the singular case.

On the other hand the singular varieties obtained by contracting extremal rays are in the same birational class of varieties as the one we started with so any birational invariant defined on the singular space must be the same as the ones before contraction. These expectations are all met in the definition of canonical singularities:

Definition 7 (Reid) *Let X be a normal projective variety. We say that X has* canonical singularities *if*
(i): *rK_X exists as a Cartier divisor for some $r > 0$,*
(ii): *for any resolution of singularities $f\colon Y \longrightarrow X$ with $\{E_i\}$ the family of exceptional divisors, we have*

$$rK_Y = f^*(rK_X) + \sum a_i E_i,$$

where each $a_i \geq 0$. (This is equivalent to saying that the plurigenera of X and Y agree.)

If the requirement that each $a_i \geq 0$ in **(ii)** is replaced by $a_i > 0$, then the singularity is called *terminal*.

Terminal threefold singularities are isolated. If X has only canonical singularities, then there is a partial resolution $f\colon Y \longrightarrow X$ such that $K_Y = f^*K_X$ and Y has only terminal singularities.

Suppose that rK_X is Cartier for some positive integer r. We say that K_X is nef if $rK_X \cdot C \geq 0$ for every $C \in \overline{\text{NE}}(X)$. In general we write $K_X \cdot C$ for $\frac{1}{r} rK_X \cdot C$.

Now we can go back to the considerations of the previous section and ask the same question: Given a nef divisor D on a projective normal threefold with at most canonical singularities, is it true that for some integer $m > 0$, mD is base point free?

By Kawamata's theorem mD is free for some $m > 0$ if $aD - K_X$ is nef for some $a > 0$ and $\dim \Psi_{aD-K_X}(X) = \dim X$ (i.e. $aD - K_X$ is *big*). Since $K_X \cdot C < 0$ the divisor $aD - K_X$ is going to be positive on $\overline{\text{NE}}(X)$ for sufficiently large a. By Kleiman's criteria $aD - K_X$ is then ample. This implies in particular that $aD - K_X$ is big. Thus the map Ψ_{mD} is going to contract the curve C and all those numerically equivalent to it. Denote the contraction map by $f\colon X \longrightarrow Y$.

If rK_Y exists for some $r > 0$ we then need to check if Y has canonical singularities so that we know we are staying inside the category of varieties we started with. To ensure this we need to impose a condition on the canonical varieties: we want to work with $\mathbb{Q}$-factorial varieties, i.e. those in which every Weil divisor becomes Cartier after multiplying by a suitable positive integer.

We then have the following result:

Theorem 8 (Kawamata) *Let X be a $\mathbb{Q}$-factorial threefold with canonical singularities such that $K_X \cdot C < 0$ for some curve C. Then we have a map $f: X \longrightarrow Y$ onto some projective variety Y, $f = \Psi_{mD}$, such that one of the following holds:*
(i): $\dim Y < \dim X$. *Then X is a fibered space.*
(ii): $\dim Y = \dim X$ *and one of the following holds:*
(a): *Y is a $\mathbb{Q}$-factorial threefold with canonical singularities and f contracts a curve iff it is numerically equivalent to C, and we have $\rho(Y) = \rho(X) - 1$. In this case f is called a* divisorial contraction.
(b): *rK_Y does not exist for any $r > 0$. In this case f is called a* small contraction.

If we start with smooth X, then case (ii-b) does not materialize. Case (ii-b) occurs when X is singular and when f contracts a codimension 2 subvariety in X, i.e. when the collection of curves numerically equivalent to C fill out a 1-dimensional subvariety of X. In this case it is easy to check that rK_Y does not exist for any r: Assume that rK_Y exists for some sufficiently large and divisible $r > 0$, (divisibility is mentioned since we also want rK_X to exist as a Cartier divisor). Since the exceptional set E of f has codimension 2, $f^*(rK_Y)$ and rK_X are two extensions of rK_{X-E}. Since X is normal such an extension is unique so $f^*(rK_Y) = rK_X$. But since C is collapsed to a point by f we have $rK_Y \cdot f_*(C) = 0$ whereas $rK_Y \cdot f_*(C) = f^*(rK_Y) \cdot C = rK_X \cdot C < 0$, a contradiction. So rK_Y cannot exist for any $r > 0$.

We once again end up with a variety which is not in our accepted category. We need more magic!

References: [19], [22], [23], [25], [26].

6 Flip

In Section 4 we started with a smooth variety and contracting a negative extremal ray we obtained a singularity. We extended our category of acceptable varieties in Section 5 to include these singularities. However starting with these new set of varieties and contracting negative extremal rays we obtain varieties which are even worse as far as singularities are concerned. It is clear that we cannot continue to accept every stranger. We must find an alternate way of proceeding with the program whenever we meet a variety for which no multiple of the canonical divisor is Cartier. A major strategy for everyday life is to avoid disaster if we are not equipped to handle it. And that is what flip is all about.

Let $f: X \longrightarrow Y$ be a small contraction. Assume that there is an ample Cartier divisor H on Y such that for some sufficiently large integer $m > 0$ the ring

$$R(mf^*H + K_X) := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(n(mf^*H + K_X)))$$

is finitely generated. Then define a new projective variety X^+ as

$$X^+ := \mathbf{Proj}\ R(mf^*H + K_X).$$

We then have the following facts:
(i): X^+ is $\mathbb{Q}$-factorial and has canonical singularities.
(ii): There is a morphism $f^+: X^+ \longrightarrow Y$ contracting an extremal ray R and for any curve C in this ray we have $K_{X^+} \cdot C > 0$.
(iii): There is a birational map $\phi: X \dashrightarrow X^+$ such that $f = f^+ \circ \phi$. In fact ϕ is an isomorphism outside the curves contracted by f and f^+.
(iv): $\rho(X) = \rho(X^+)$.

This set up is known as a *flip*. More often the map $\phi: X \dashrightarrow X^+$ is called a flip. The crucial point is that we have a

Theorem 9 (Mori) *If X is a threefold with at most canonical singularities and $f: X \longrightarrow Y$ is a small contraction which contracts the negative extremal ray R, then a flip $\phi: X \dashrightarrow X^+$ exists.*

Let X be a threefold with canonical singularities and $\pi\colon \tilde{X} \longrightarrow X$ a resolution. We have $K_{\tilde{X}} = \pi^* K_X + \sum a_i E_i$, where $\{E_i\}$ are exceptional divisors and $a_i \geq 0$.

Definition 10 (Shokurov) *The* difficulty $d(X)$ *of* X *is the number of* a_i *which are less than one, i.e.*

$$d(X) = \#\{i \mid a_i < 1\}.$$

This is independent of the resolution chosen and is an invariant of X. Note that if $f\colon X \longrightarrow Y$ is a regular birational map, such as a divisorial contraction, then $d(X) = d(Y)$ since a resolution of X is also a resolution for Y. However the crucial point is what happens to difficulty when we have a flip.

Theorem 11 (Shokurov) *If* $\phi\colon X \dashrightarrow X^+$ *is a flip, then* $d(X^+) < d(X)$.

This says that a threefold with canonical singularities cannot accept infinitely many flips.

We now have an algorithm to apply. Start with a normal projective $\mathbb{Q}$-factorial threefold which has at most canonical singularities. If K_X is nef, we stop. We accept X as a minimal model in the birational equivalence class of X. If however K_X is not nef, then there is a negative extremal ray which we contract to obtain Y. If $\dim Y < 3$ or if K_Y is nef, we stop. In the former case we can still accept X as a minimal model of its class since we can describe it as a fibered space. If Y has canonical singularities and K_Y is not nef, we continue the program with Y observing that $\rho(Y) < \rho(X)$ and $d(Y) = d(X)$. If no multiple of K_Y is Cartier, then we construct a flip X^+ of X and continue with X^+ observing that $\rho(X^+) = \rho(X)$ and $d(X^+) < d(X)$. With these observations we are assured that the algorithm will terminate after finitely many steps and when it terminates we will have at the last step a variety Z which is either of dimension less than 3 or which is a normal projective $\mathbb{Q}$-factorial threefold with at most canonical singularities and with K_Z nef.

When we start with a particular threefold X, then whether the above algorithm stops because we end up with a variety of lower dimension or because we obtain a threefold whose canonical divisor is nef depends only on the birational class X. However if we are going to end up with a threefold at the end of the algorithm, then it is not necessarily unique. The order of extremal rays chosen to contract or flip at each step makes a difference at the end. Suppose that starting with a fixed X but contracting and flipping extremal rays in different orders we obtain the threefolds Z_1 and Z_2 whose canonical divisors are nef. Then there exists a birational map $\psi\colon Z_1 \dashrightarrow Z_2$ which is an isomorphism in codimension one. ψ has the effect of cutting off some 1-cycles from Z_1 and gluing them back again with a different embedding. Observe that this is very much like a flip map except that the canonical divisors here have zero intersection with each of the curves on which ψ and its inverse are not defined. Such a map is called a *flop*.

References: [1], [7], [12], [14], [16], [17], [19], [21], [24], [27], [28], [31], [32].

7 The Minimal Model Program

In this section we summarize the classification algorithm, otherwise known as the Minimal Model Program, MMP for short.

Theorem 12 (Minimal Model Program) *Let X be a normal projective $\mathbb{Q}$-factorial threefold with at most canonical singularities. Assume that K_X is not nef. Let R be a negative extremal ray. Then we can construct a contraction map $f\colon X \longrightarrow Y$ onto a projective variety Y such that a curve C is mapped to a point in Y iff C belong to the ray R. Moreover only one of the following cases holds:*
(i): *f is a Fano contraction, i.e. $\dim Y < 3$.*
(ii): *f is a divisorial contraction, i.e. Y is a normal projective $\mathbb{Q}$-factorial threefold with at most canonical singularities. In this case there is a codimension one subvariety E in X such that $f(E)$ is a point in Y and $f\colon X \setminus E \longrightarrow Y \setminus f(E)$ is an isomorphism. As for the numerical invariants we have $\rho(Y) < \rho(X)$ and $d(Y) = d(X)$.*

(iii): *f is a small contraction, i.e. there is a codimension two subvariety E in X such that $f(E)$ is a point in Y and $f: X \backslash E \longrightarrow Y \backslash f(E)$ is an isomorphism. In this case no multiple of the canonical divisor of Y is Cartier. We then construct a flip $\phi: X \dashrightarrow X^+$, flipping the ray R. Then X^+ is a normal projective $\mathbb{Q}$-factorial threefold with at most canonical singularities. In this case we have $\rho(X) = \rho(X^+)$ and $d(X^+) < d(X)$.*

If case **(i)** is encountered, then we stop. If case **(ii)** or **(iii)** is encountered, we continue the program with X replaced by Y or X^+ respectively. The observation on the numerical invariants assures us that the program will terminate after finitely many steps.

Thus if we start with an acceptable X with K_X not nef, then X is birationally isomorphic to a threefold Y such that either Y admits a Fano contraction or else K_Y is nef.

With this result we can give a full description of the closed cone of curves in X. First we have a

Definition 13 *If X is a normal $\mathbb{Q}$-factorial projective variety and K_X its canonical divisor, define*

$$\overline{\mathrm{NE}}_{K_X}(X) := \{R \in \overline{\mathrm{NE}}(X) | K_X \cdot R \geq 0\}.$$

We can now describe the structure of $\overline{\mathrm{NE}}(X)$.

Theorem 14 (Cone Theorem) *Let X be as in Theorem 12. Then there are a finite number of extremal rays R_i such that*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}_{K_X}(X) + \sum \mathbb{Q}_+ R_i,$$

where $\mathbb{Q}_+$ denotes the positive rational numbers. (If we had chosen to define N^1 and N_1 as $\mathbb{R}$-vector bundles from the beginning (see Definition 4), then we would replace $\mathbb{Q}_+$ with $\mathbb{R}_+$.)

The choice of singularities to admit seems to vary with taste but the category of terminal singularities is the smallest category within which the MMP works.

It is time now to check if the reader is comfortable with these new ideas:

Exercise 15 *Classify smooth projective surfaces within the framework of MMP.*

References: [2], [4], [8], [9], [10], [11], [13], [15], [20], [25], [29], [30], [33].

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