GENERALIZED SHIODA - INOSE STRUCTURES

ON K3 SURFACES

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This work concerns algebraic K3 surfaces admitting generalized Shioda-Inose structures (Definition 1 below). To generalize the classical Shioda-Inose structure ([S-I], [M]), one needs to determine finite groups with suitable actions both on K3 surfaces and on abelian surfaces. To this end, finite groups with symplectic actions on K3 surfaces were completely determined in ([Mu2]) and ([X]) and in the latter article the configurations of singularities on the quotients were also listed. On the complementary side, Katsura's article ([K]) contains the classification of all finite groups acting on abelian surfaces so as to yield generalized Kummer surfaces (cf. [B] for related lattice theoretic discussion).

In this paper, using the results of ([K], [X]) we show that a K3 surface X admitting a Shioda-Inose structure with $G \neq \mathbf{Z}_2$ has $\rho(X) \geq 19$ in general and $\rho(X) = 20$ if G is noncyclic. We also show that on a singular K3 surface X, all Shioda-Inose structures are induced by a unique abelian surface.

Throughout the paper we will consider only algebraic K3 surfaces over $\mathbf{C}$.

Our notation will be as follows :

A (resp. X) denotes an abelian (resp. an algebraic K3) surface.
A_G is the Kummer surface constructed from A/G for a suitable finite group G.
$K_?$ denotes the canonical class of ?.
$T_?$ = the transcendental lattice of ?.
$\rho(?)$ is the Picard number of ?.
We use the standard notation A_k, D_k, E_k to denote the rational singularities on surfaces.
$|G|$ denotes the order of the group G.

We begin with giving a precise definition of Shioda-Inose structures on K3 surfaces. For this, we first recall that a generalized Kummer surface A_G is a K3 surface which is the minimal resolution of the quotient A/G of an abelian surface A by some finite group G ([K], Definition 2.1).

Definition 1 : *A K3 surface X admits a Shioda-Inose structure with group G if G acts on X symplectically and the quotient X/G is birational to a generalized Kummer surface A_G.*

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We note that generalized Kummer surfaces (in characteristic 0) arise only if G is isomorphic to one of the following groups ([K], Corollary 3.17) :

$\mathbf{Z}_k, k = 2, 3, 4, 6,$
binary dihedral groups Q_8, Q_{12} and
binary tetrahedral group T_{24}.

All of these possibilities occur ([K], Examples).

Comparing this list with the list of finite groups acting symplectically on K3 surfaces ([X], Table 2), we see that all such G appear as a group of symplectic automorphisms of some K3 surface. Hence the question of the existence of K3 surfaces admitting Shioda-Inose structures with group G makes sense for each of these groups G.

In the classical case of $G = \mathbf{Z}_2$, Morrison obtained the following lattice theoretic characterization of K3 surfaces admitting Shioda-Inose structure ([M], Corollary 6.4).

Theorem [M] : *An algebraic K3 surface X admits a Shioda-Inose structure if and only if X satisfies one of the following conditions :*
(i) $\rho(X) = 19$ or 20,
(ii) $\rho(X) = 18$ and $T_X = U \oplus T'$,
(iii) $\rho(X) = 17$ and $T_X = U^2 \oplus T'$
where U is the standard hyperbolic lattice.

To contrast this case with the general situation, we include the following elementary observation.

Lemma 2 : *Given an abelian surface A, there exists a K3 surface X with $\rho(X) = 16+\rho(A)$ admitting a classical Shioda-Inose structure induced by A.*

Proof :

Given A, we have $T_A \hookrightarrow U^3$ with $signature(T_A) = (2, 4 - \rho(A))$. Therefore, taking $\rho = 16 + \rho(A)$, by the surjectivity of the period map for K3 surfaces (cf. [M], Corollary 1.9 (ii)) we have a K3 surface X with $\rho(X) = \rho$ and T_X is isometric to T_A. Applying ([M], Theorem 6.3) the conclusion follows. $\square$

We will see that for generalized Shioda-Inose structures one has $\rho(X) \geq 19$. This bound on the Picard number follows from the configuration of the exceptional curves on A_G for which we will need the following result on the singularities of the quotient A/G for noncyclic G.

Proposition 3 : *If G is a non-cyclic group acting on an abelian surface A to yield a generalized Kummer surface, then the singularities of A/G are given as follows :*

$3A_1 + 4D_4$ *for* $G = Q_8$,
$A_1 + 2A_2 + 3A_3 + D_5$ *for* $G = Q_{12}$ *and*
$4A_2 + 2A_3 + A_5$ *or* $A_1 + 4A_2 + D_4 + E_6$ *for* $G = T_{24}$.

Proof :

As is the case with analysis of this type, the proof is combinatorial in essence and is quite standard (cf. [B], [K], [X]). We know that as we have only quotient singularities, the possible types of singularities are :

$A_k, k = 1, 2, 3, 5$ corresponding to stabilizer groups of type $\mathbf{Z}_k, k = 2, 3, 4, 6$ respectively and D_4 (resp. D_5, resp. E_6) corresponding to Q_8 (resp. Q_{12}, resp. T_{24}). We index these types in this order with $i = 1, .., 7$ and we let n_i be the number of singular points of type i on A/G.

Comparing the topological Euler characteristic of $A - \{\text{fixed points of } G\}$ to that of $A_G - \{$ exceptional curves $\}$, we obtain

$0 = \chi_{top}(A) = |G|(24 - \sum \chi_i n_i) + n$

where χ_i is the topological Euler characteristic of the configuration corresponding to the singularity of type i and n is the total number of fixed points of G on A. Clearly we have $n = \sum m_i n_i$ where m_i is the index in G of the stabilizer group corresponding to i. Furthermore, as the lattice generated by (-2)-curves on A_G has rank ≤ 19, in all cases we have

$n_1 + 2n_2 + 3n_3 + 5n_4 + 4n_5 + 5n_6 + 6n_7 \leq 19.$

Using these restrictions together with the subgroup structure of each G, the result follows.$\square$

Corollary 4 : *If* X *admits a Shioda-Inose structure with* $G \neq \mathbf{Z}_2$*, then* $\rho(X) \geq 19$ *and* $\rho(X) = 20$ *if* G *is noncyclic.*

Proof :

By ([I2], Corollary 1.2), we know that the Picard number of X is equal to the Picard number of the associated generalized Kummer surface A_G. Therefore, if $G = \mathbf{Z}_k$ for $k = 3, 4, 6$, it follows from ([K], p. 17) that $\rho(X) \geq 19$. In case G is noncyclic, we apply Proposition 3 to see that $\rho(X) = 20$.$\square$

Next, we consider the variation of Shioda-Inose structures with respect to the isogenies of abelian surfaces.

Given a K3-surface X which admits a Shioda-Inose structure with group G and associated abelian surface A, we denote by π_A (resp. π_X) the rational covering map $A \to A_G$ (resp. $X \to A_G$) into the corresponding generalized Kummer surface A_G.

The following results follow by exactly the same proofs as in the case of classical Shioda-Inose structures (cf. [S-I], [I2], [M]):

(1) $\pi_A^*(K_{A_G}) = K_A$ and $\pi_X^*(K_{A_G}) = K_X$,
(2) $\pi_A^* : T_{A_G} \to T_A$ (resp. π_X^*) gives an isomorphism of lattices $T_{A_G} \cong T_A(n)$ (resp. $T_{A_G} \cong T_X(n)$) where $n = |G|$,
(3) T_A and T_X are isometric.

Using these elementary observations we prove

Lemma 5 : *If X is a singular K3 surface, then each and every Shioda-Inose structure on X is induced only by A.*

Proof :

We let p_A, p_{A_G}, p_X denote the period maps of A, A_G, X respectively.

From (1) above, it follows that the isometry $\phi : T_X \to T_A$ satisfies $p_A \circ \phi = cp_X$ for some $c \in \mathbf{C}$.
If we have another abelian surface A' inducing some Shioda-Inose structure on X, with corresponding isometry $\psi : T_X \to T_{A'}$ satisfying $p_{A'} \circ \psi = c'p_X$ for some $c' \in \mathbf{C}$, then we get an isometry $\phi \circ \psi^{-1} : T_{A'} \to T_A$. As $\rho(A) = 4$, $\phi \circ \psi^{-1}$ extends to an isometry $\alpha : H^2(A', \mathbf{Z}) \to H^2(A, \mathbf{Z})$ ([S-M], Theorem 1 in Appendix) to give $p_A \circ \alpha = p_A \circ (\phi \circ \psi^{-1}) = cc'^{-1} p_{A'}$, and it follows that $A' \cong A$ or the dual $\hat{A}$ of A ([S], Theorem 1). This completes the proof because X admits a classical Shioda-Inose structure for which the associated abelian surface $A^{'}$ is self-dual (Theorem [S-I]). □

Remark :

If A_1 and A_2 are two abelian surfaces which are isogeneous, then we have $T_{A_1} \otimes \mathbf{Q} \cong T_{A_2} \otimes \mathbf{Q}$. Therefore two K3 surfaces X_1, X_2 are isogeneous in the sense of ([Mu1], Definition 1.8) if they admit Shioda-Inose structures (not necessarily with the same group) induced from A_1, A_2 respectively ([Mu1], Remark 1.11). In case X_1, X_2 are singular K3 surfaces, the stronger form of isogeny follows from Lemma 5 using ([I1]); that is, we have rational maps $X_1 \to X_2$, $X_2 \to X_1$ of finite degree.

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