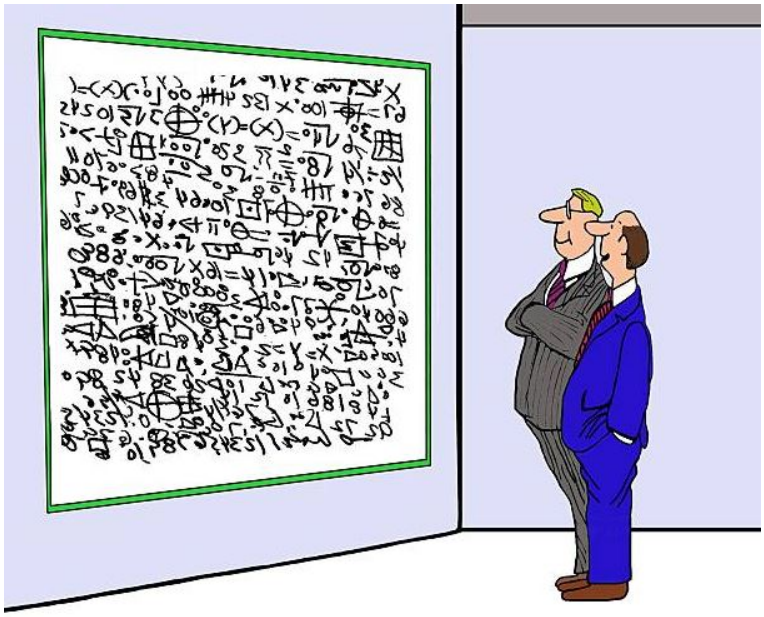
Math 503 Complex Analysis I
Midterm Exam 1
November 6, 2026
10:30 - 12:20
Solutions



"When you put it like that, it makes complete sense."

1. In the following simplify your answer as much as possible.

(i) Find the absolute value and conjugate of $\omega = \dfrac{3-i}{\sqrt{2}+3i}$.

(ii) Evaluate the cross-ratio $(2, 1-i, 1, 1+i)$.
Hint: The cross-ratio (z_1, z_2, z_3, z_4) is the image of z_1 under the Möbius transformation which takes z_2 to 1, z_3 to 0, and z_4 to ∞.

Solution (i):

$$\omega = \frac{3-i}{\sqrt{2}+3i} = \frac{3-i}{\sqrt{2}+3i}\,\frac{\sqrt{2}-3i}{\sqrt{2}-3i} = \frac{(3\sqrt{2}-3)-i(\sqrt{2}+9)}{11}$$

$$|\omega| = \frac{\sqrt{10}}{\sqrt{11}}$$

$$\overline{\omega} = \frac{(3\sqrt{2}-3)+i(\sqrt{2}+9)}{11}$$

Solution (ii): Note that $(z, z_2, z_3, z_4) = T(z) = \dfrac{z-z_3}{z-z_4}\,\dfrac{z_2-z_4}{z_2-z_3}$. Then in our case

$$(2, 1-i, 1, 1+i) = T(z) = \frac{z-1}{z-(1+i)}\,\frac{(1-i)-(1+i)}{(1-i)-1} = (2)\,\frac{z-1}{z-1-i},$$

and

$$T(2) = (2)\,\frac{2-1}{2-1-i} = 1+i.$$

(i) is on page 2, (ii) is on page 54.

2. Let f be an entire function and suppose that there is a constant M, an $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for all $|z| > R$. Show that f is a polynomial of degree $\leq n$.

Hint: You may consider using Cauchy's Estimate: Let f be analytic in $B(a;R)$ and suppose $|f(z)| \leq M$ for all z in $B(a;R)$. Then $|f^{(n)}(a)| \leq \dfrac{n!M}{R^n}$. (as given on page 73 of Conway's book.)

Solution: Let $k > n$ be any positive integer, and $r > R$.

$$\begin{aligned} |f^{(k)}(0)| &\leq \frac{k!}{r^k} \max_{|z|\leq r} |f(z)|, \text{ Cauchy's Estimate for } z \in B(0;r) \\ &= \frac{k!}{r^k} \max_{|z|= r} |f(z)|, \text{ by the Maximum Modulus Principle} \\ &\leq \frac{k!}{r^k} Mr^n, \quad \text{by the given data, since } |z| = r > R. \end{aligned}$$

Since $r > R$ is arbitrary, by sending r to infinity we conclude that $f^{(k)}(0) = 0$ for all $k > n$.

Since f is entire, its Taylor series at $z = 0$ converges everywhere.

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k,$$

and by the above result all derivatives of degree greater than n are zero, so f is a polynomial of degree n.

This is exercise 1 on page 80

3. Let $\lambda > 1$ and show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : \text{ Re } z > 0\}$. Show that this solution must be real. What happens to the solution as $\lambda \to 1$?

Solution: Consider the half disc

$$D_R = \{z \ : \ \text{Re } z > 0, |z| = r\}, \ R > \lambda + 1.$$

Compare the functions

$$u(z) = -e^{-z}, \ v(z) = \lambda - z$$

on the boundary of D_R.

On the vertical part, $z = iy$, so we have

$$|u(z)| = 1 < \lambda \leq |\lambda - iy| = |v(z)|.$$

On the semicircle, $|z| = R$ and Re $z \geq 0$, so

$$|u(z)| = e^{-\text{Re } z} \leq 1 < R - \lambda \leq |\lambda - z| = |v(z)|.$$

Thus $|u(z)| < |v(z)|$ on the boundary of D_R. By a version of Rouché's theorem $u(z) + v(z) = \lambda - z - e^{-z}$ and $v(z) = \lambda - z$ have the same number of zeros inside D_R. And that number is clearly one since $\lambda - z = 0$ has a single solution inside D_R.

Here is a proof of this version of Rouché, using Conway's version:

Let $f(z) = u(z) + v(z) = \lambda - z - e^{-z}$, and $g(z) = -v(z) = z - \lambda$. On the boundary of D_R we have

$$|f(z) + g(z)| = |u(z)| < |v(z)| = |g(z)| \leq |f(z)| + |g(z)|.$$

Then by Conway's version $f(z) = \lambda - z - e^{-z}$ and $g(z) = z - \lambda$ have the same number of zeros inside D_R.

Using the real function $\phi(x) = \lambda - x - e^{-x}$, we note that $\phi(0) > 0$ since $\lambda > 1$, and $\phi(\lambda) < 0$. Then by the Intermediate Value Theorem, $\phi(x)$ has a zero on the real interval $(0, \lambda)$.

For the last part, consider the Taylor expansion of $\lambda - z - e^{-z}$ around zero.

$$\lambda - z - e^{-z} = (\lambda - 1) - \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

As λ approaches to 1, the zeros of $\lambda - z - e^{-z}$ approach to zero, giving us a double root at the origin.

This is exercise 9 on page 127.

4. Using the residue theory show that

$$\int_0^\infty \frac{x^2\,dx}{(x^2+\pi^2)^2} = \frac{1}{4\pi^2}.$$

Solution: Consider the function

$$f(z) = \frac{z^2}{(z^2+\pi^2)^2} = \frac{z^2}{(z-i\pi)^2(z+i\pi)^2}.$$

For $R > \pi$, integrate counterclockwise along the segment from $-R$ to R and then along the upper semicircle Γ_R from R back to $-R$. The only enclosed singularity is the double pole at $z = i\pi$.

The residue at $i\pi$ is

$$\begin{aligned}\operatorname{Res}(f, i\pi) &= \frac{d}{dz}\frac{z^2}{(z+i\pi)^2}\bigg|_{z=i\pi} \\ &= \left(\frac{2z}{(z+i\pi)^2} - \frac{2z^2}{(z+i\pi)^3}\bigg|_{z=i\pi}\right) = \frac{1}{4\pi i}. \quad \textit{(See Conway page 113.)}\end{aligned}$$

The residue theorem gives

$$\int_{-R}^{R} \frac{dx}{(x^2+\pi^2)^2} + \int_{\Gamma_R} f(z)\,dz = 2\pi i\,\operatorname{Res}(f, i\pi) = \frac{1}{2}.$$

On the arc we have

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \le \frac{\pi R}{(R^2-\pi^2)^2} \longrightarrow 0 \qquad (R \to \infty).$$

Passing to the limit, we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+\pi^2)^2} = \frac{1}{2}.$$

Finally, the integrand is even, so

$$\int_0^\infty \frac{dx}{(x^2+\pi^2)^2} = \frac{1}{4}.$$

This is exercise 2.a on page 121

Bonus: Prove the Casorati-Weierstrass Theorem: If f has an essential singularity at $z = a$, then for every $\delta > 0$, the closure of the set $\{\ f(z)\ \ :\ \ 0 < |z - a| < \delta\ \}$ is all of $\mathbb{C}$.

Hint: There are different proofs of this result. All proofs start by assuming to the contrary that the result does not hold, and then find a way of showing that a cannot be an essential singularity under that assumption.

Solution: Assume to the contrary that there is a point $c \in \mathbb{C}$ such that $|f(z) - c| > \delta$ for some $\delta > 0$ and for all $0 < |z - a| < R$ for some $R > 0$. Then

$$\lim_{z \to a} \frac{f(z) - c}{z - a} = \infty,$$

and hence $z = a$ is a pole for $\dfrac{f(z) - c}{z - a}$. Then there is a function $g(z)$ analytic around $z = a$ and a positive integer m such that

$$\frac{f(z) - c}{z - a} = \frac{g(z)}{(z - a)^m}. \quad \text{(Conway page 105)}$$

This gives us

$$f(z) = \frac{g(z)}{(z - a)^{m-1}} + c,$$

which is a contradiction since the function on the right hand side either has a removable singularity, $m = 1$ case, or a pole, $m > 1$ case.

Casorati-Weierstrass Theorem is on page 109 of Conway. The proof given above is adobted from Ahlfors.