A TRIGONOMETRIC PROOF OF MORLEY'S THEOREM

[H. DEMIR, M.E.T.U., Ankara, Turkey]

Frank Morley has discovered some 80 years ago a remarkable theorem [16] and many proofs and extensions have been obtained (see references) since then. We notice in particular the two synthetic proofs, one direct [3] and the other indirect [16]. We offer here a trigonometric proof.

THEOREM. If the trisectors of the interior angles of a triangle are drawn so that those adjacent to each side intersect, the intersections are vertices of an equilateral triangle.

Proof. Let the trisectors adjacent to the side BC (CA, AB) of ABC intersect at A' (B', C'). We show that A'B'C' is an equilateral triangle (Fig. 1). Setting

(1) $$<BAC = 3\alpha, \quad <CBA = 3\beta, \quad <ACB = 3\gamma, \quad \alpha + \beta + \gamma = 60^{\circ},$$

and denoting the circumradius by R, we have

(2) $$\begin{aligned} a &= 2R \sin 3\alpha = 2R \sin\alpha\,(3\cos^2\alpha - \sin^2\alpha), \\ b &= 2R \sin 3\beta = 2R \sin\beta\,(3\cos^2\beta - \sin^2\beta), \\ c &= 2R \sin 3\gamma = 2R \sin\gamma\,(3\cos^2\gamma - \sin^2\gamma). \end{aligned}$$

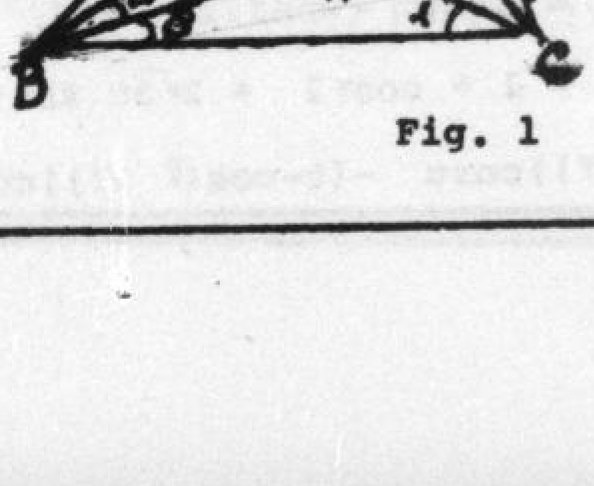

Fig. 1

The proof consists of evaluating the side $a' = B'C'$ of $A'B'C'$ in terms of α, β, γ and R, showing that the expression for a' is symmetric in the variables. This symmetry establishes the equalities $a' = b' = c'$ for $A'B'C'$, and hence the theorem.

To this end, we apply cosine law to $AB'C'$, the sides $x = AB'$, $y = AC'$ of which are obtained by sine laws applied to $AB'C$ and ABC':

(3) $$x = b \sin \gamma/\sin(\alpha+\gamma), \quad y = c \sin \beta/\sin(\alpha+\beta).$$

Using (1) and (2), we have

$$x = 2R \sin \beta(3\cos^2\beta-\sin^2\beta) \sin \gamma/\sin(60^\circ- \beta)$$
$$= 4R \sin \beta\sin \gamma(3\cos^2\beta-\sin^2\beta)/(\sqrt{3}\cos \beta- \sin\beta)$$
$$= 4R \sin \beta\sin \gamma(\sqrt{3}\cos\beta + \sin\beta).$$

Setting

(4) $$k_a = 4R \sin \beta\sin \gamma$$
$$x = k_a x_1, \quad y = k_a y_1$$

where

(6) $$x_1 = \sqrt{3} \cos \beta+ \sin \beta, \quad y_1 = \sqrt{3}\cos \gamma+ \sin \gamma$$

we have, by cosine law applied to $AB'C'$:

(7) $$a'^2 = k_a^2(x_1^2+ y_1^2 - 2x_1y_1\cos \alpha) = k_a^2u^2$$

with

$$u^2=(3\cos \beta+ \sin\beta)^2+ (\sqrt{3}\cos \gamma+ \sin\gamma) - 2(\sqrt{3}\cos \beta+\sin\beta)\cdot(\sqrt{3} \cos \gamma+\sin\gamma)\cos \alpha$$
$$=3(\cos^2\beta +\cos^2\gamma) + (\sin^2\beta +\sin^2\alpha) +2\sqrt{3}(\cos\beta \sin\beta +\sin\gamma \cos\gamma)$$
$$-2[3\cos\beta\cos \gamma+\sin\beta\sin \gamma+ \sqrt{3}(\cos\beta\sin\gamma + \sin\beta\cos\gamma] \cos \alpha$$
$$=3(\cos^2\beta +\cos^2\gamma) +2 - (\cos^2\beta +\cos^2\gamma) + \sqrt{3}(\sin 2\beta - \sin 2\gamma)$$
$$-3(2\cos\beta\cos\gamma)\cos\alpha - (2\sin\beta\sin\gamma)\cos\alpha - 2\sqrt{3}\sin(\beta +\gamma)\cos\alpha.$$

Introducing now $t = \cos(\beta - \gamma)$,

$$u^2= 2 + 1 + \cos 2\beta + 1 + \cos\gamma 2 + 2\sqrt{3}t \sin(60^\circ -\alpha)$$
$$-3(t + \cos(\beta+\gamma))\cos\alpha -(t-\cos(\beta +\gamma))\cos\alpha-2\sqrt{3}\cos\alpha\sin(60^\circ-\alpha)$$

$= 4 + (\cos 2\beta + \cos 2\gamma) + [2\sqrt{3}\sin(60^\circ-\alpha)]t$

$- 4t \cos\alpha - 2\cos\alpha \cos(60^\circ-\alpha) - 2\sqrt{3}\cos\alpha \sin(60^\circ-\alpha)$

$= 4 + 2t \cos(\beta + \gamma) + 2\sqrt{3}t \sin(60^\circ- \alpha) - 4t \cos\alpha$

$-2\cos\alpha \cos(60^\circ- \alpha) - 2\sqrt{3}\cos\alpha \sin(60^\circ- \alpha)$

$=[2(1/2\cos\alpha + 1/2\sqrt{3}\sin\alpha) + 2\sqrt{3}(1/2\sqrt{3}\cos\alpha - 1/2\sin\alpha)-4\cos\alpha]t$

$+4- 2\cos\alpha(1/2\cos\alpha+ 1/2\sqrt{3}\sin\alpha) -2\sqrt{3}\cos\alpha(1/2\sqrt{3}\cos\alpha-1/2\sin\alpha)$

$=[\cos \alpha+\sqrt{3}\sin\alpha +3\cos\alpha -\sqrt{3}\sin\alpha -4\cos\alpha]t$

$+4 - \cos^2\alpha -\sqrt{3}\cos\alpha \sin\alpha -3\cos^2\alpha +\sqrt{3}\cos\alpha \sin\alpha$

$= 4 - 4\cos^2\alpha = 4\sin^2\alpha$.

Hence by (7)

$$a'^2 = k_a^2 \cdot 4 \sin^2\alpha = 64R^2 \sin^2\alpha\sin^2\beta\sin^2\gamma ,$$

and since a', $\sin\alpha$, $\sin\beta$, $\sin\gamma > 0$, we have

$$a' = 8R \sin\alpha \sin\beta \sin\gamma$$

which is symmetric in the variables and hence

$$a' = b' = c' = 8R \sin\alpha \sin\beta \sin\gamma$$

and this completes the proof of the theorem.

References

1 Taylor, F. G., Marr, W.L., Proceedings of the Edinburgh Math. Soc. 32 (1914) 119-150.

2 Morley, F., American Journal of Math. 51 (1929) 465-472.

3 Johnson, R.A., Modern Geometry, Houghton Mifflin, New York, 1929, pp. 253-254.

4 Haarbleicher, A., De l'emploie des droites isotropes comme axes de coordinnees; la nouvelle geometrie du triangle, Gauthier Villars, Paris, 1931.

5 Hoffmann, J.E., Ein neuer Beweiss des Morleyschen Satzes, Deutsche Math. 43 (1939) 589-590.

6 Loria, G., Triangles equilateraux derives d'un triangle quelconque, Math. Gaz. 23 (1939) 364-372.

7 Lebesque, H., Sure les n-sectrices d'un triangle, Enseignement Math. 38 (1940) 39-58.

8 Banning, T., On a generalization of F. Morley's theorem, Mathematica, Zutphen. B. 9 (1940) 17-33.

9 Peters, J.W., The theorem of Morley, Nat. Math. Mag. 16 (1941) 119-126.

10 Gibbins, N.M., The non-equilateral Morley triangles, Math. Gaz. 26 (1942) 81-

11 Bottema, O., Hoofdstukken uit de Elementaire Merrkunde, v.V. Servire, the Hague, 1944.

12 van Buuren, C.L., The Theorem of Morley, Mathematica Autphen. B. 8 (1949) 3

13 Roger, P., Eureka, 16 (1953) 6-7.

14 Morley, F., Morley, F.V., Inversite Geometry, Chelsea, New York, 1954, pp. 243-244.

15 Venkatachaliengar, K., Am. Math. Monthly, 64 (1958) 612-613.

16 Coxeter, H.S.M., Introduction to Geometry, Wiley, New York, 1961, pp. 23-25.

17 Demir, H. A Theorem Analogous to Morley's Theorem, Math. Mag. 38 (1965)