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ON THE GEOMETRY OF n DIMENSIONS AND EXTENSIONS IN E^2 OF ERDÖS - MORDELL - OPPENHEIM INEQUALITIES

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Abstract. In this paper several properties in the n dimensional Euclidean space E^n are obtained by the use of cartesian, hypervolume and $(n+1)$ – polar coordinates, and also some extensions of Erdös-Mordell-Oppenheim inequalities for triangles to polygons are given.

This paper includes the following results : The equation in volume coordinates of the circumhypersphere of the simplex of reference (2.5), the expression for the power of a point relative to the circumhypersphere (2.10) and that for circumradius (2.15); the equation of the generalized Euler hypersphere (3.4); various relations for general hyperspheres (4) including the representation of a hypersphere by a finite sequence of powers (4.1); equations of some radical hyperplanes (5) and as application the obtention of hypervolume coordinates of the circumcenter (5.12); a proof of the general Pythagorean theorem (6.2). The paper is concluded by Section 7 in which is stated a conjecture (7.12) as an extension of Erdös-Mordell-Oppenheim inequalities to convex polygons, and the proof is provided (7.18) for regular polygons.

1. Introduction. We recall here some definitions of the geometry of n dimensions. An n-simplex in the Euclidean space E^n is obtained in a manner as explained in [1, *p*. 120] : The only figure in E^0 is a point, denoted by $S_0 \equiv A_0$, called 0 – simplex. In E^1 two distinct points A_0 and A_1 define a line segment called 1 – simplex $S_1 \equiv A_0 A_1$. In E^2 taking A_2 exterior to $A_0 A_1$ one gets a triangle $A_0 A_1 A_2$ called a 2 – simplex $S_2 \equiv A_0 A_1 A_2$. Continuing this way one obtains a 3 – simplex $S_2 \equiv A_0 A_1 A_2 A_3$ (a tetrahedron) in E^3 and so on, and finally an n – simplex $S_n \equiv A_0 A_1 \dots A_n$ in E^n.

We consider the simplex $S_n \equiv A_0 A_1 \ldots A_n$ where the vectors $\overrightarrow{A_0A_1}, \overrightarrow{A_0A_2}, \ldots, \overrightarrow{A_0A_n}$ are linearly independent as the *simplex of reference.* The $n+1$ $(n-1)$-simplexes obtained by removing from S_n a vertex at a time will be denoted by

$$(1.1) \qquad T_0 \equiv A_1 \ldots A_n\,, \ T_i \equiv A_0 \ldots A_{i-1}\, A_{i+1} \ldots A_n\,, \\ T_n \equiv A_0 \ldots A_{n-1}\,, \ [1 \leqslant i \leqslant n-1]$$

and the sides of S_n by

$$(1.2) \qquad a_{ij}\,(= a_{ji}) = A_i\, A_j\ (= A_j\, A_i\,)\,, \ [\,0 \leqslant i,\, \hat{\imath} \leqslant n\,]\,, \ a_{ii} = 0$$

The oriented hypervolume $\overline{A_0 A_1 \ldots A_n}$ of S_n is defined [2, *p.* 124] by the determinant

$$(1.3) \qquad v = \overline{A_0 A_1 \ldots A_n} = \frac{1}{n!} \begin{vmatrix} \alpha_{01} & \cdots & \alpha_{0n} & 1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \alpha_{n1} & \cdots & \alpha_{nn} & 1 \end{vmatrix} \neq 0$$

which is invariant under the Euclidean motions.

We agree to denote the hypervolumes of the bases (1.1) by

$$(1.4) \qquad a_i = Vol\ (T_i)\,, \ [0 \leqslant i \leqslant n].$$

Taking a rectangular coordinate system $0x_1 x_2 \ldots x_n$ in E^n, the *cartesian coordinates* of a point P is the n-tuple

$$(1.5) \qquad P = (x_1\,, x_2\,, \ldots, x_n)$$

and those of the vertices of the simplex of reference S_n will be denoted by

$$(1.6) \qquad A_i = (\alpha_{i1}\,, \alpha_{i2}\,, \ldots, \alpha_{in})\,, \ [0 \leqslant i \leqslant n]\,.$$

Definition. By the *volume coordinates* [3, *p.* 122] of a point P relative to the simplex of reference S_n are meant

$$(1.7) \qquad v_0 : v_1 : \ldots : v_n = \mathrm{Vol}\ (S^0) : \mathrm{Vol}\ (S^1) : \ldots : \mathrm{Vol}\ (S^n)$$

with

$$(1.8) \qquad S^i \equiv A_0 \ldots A_{i-1}\, P\, A_{i+1} \ldots A_n\,, \ [0 \leqslant i \leqslant n]$$

where P occupies the first place for $i = 0$ and last place for $i = n$.

For convenience the hypervolume coordinates of P will be represented by using square brackets as

(1.9) $$P = [v_0, v_1, \ldots, v_n]$$

so that we have

(1.10) $$A_0 = [1, 0, \ldots, 0], A_1 = [0, 1, 0, \ldots, 0], \ldots, A_n = [0, \ldots, 0, 1]$$

DEFINITION. If $m_0, m_1, \ldots, m_n$ are not all zero, then the vectorial equality

(1.11) $$m\,\overrightarrow{OP} = \sum_{i=0}^{n} m_i\,\overrightarrow{OA_i}, \quad m = \sum_{i=0}^{n} m_i \neq 0$$

defines a single point P independently of the point 0. The numbers $m_0, m_1, \ldots, m_n$ are referred to as the *barycentric coordinates* of the point P relative to the simplex of reference $S_n \equiv A_0 A_1 \ldots A_n$.

The barycentric coordinates of the vertices A_i of S_n are the same as their hypervolume coordinates (1.10), and indeed we will show in the next section (2.3) that the same is true for any point P in E^n.

(1.12) DEFINITION. By the $(n+1)$-*polar coordinates* of a point P relative to the simplex of reference $S_n \equiv A_0 A_1 \ldots A_n$ are meant the distances $p_0, p_1, \ldots, p_n$ of P from the vertices, or the numbers proportional to them.

To make a distinction, the $(n+1)$-polar coordinates of P will be denoted by

(1.13) $$P = \langle p_0, p_1, \ldots, p_n \rangle, \quad p_i = PA_i, \quad [0 \leqslant i \leqslant n].$$

In particular the polar coordinates of the vertices of the simplex of reference are

(1.14) $$\begin{aligned} A_0 &= \langle 0, a_{01}, a_{02}, \ldots, a_{0n} \rangle \\ A_1 &= \langle a_{10}, 0, a_{12}, \ldots, a_{1n} \rangle \\ &\cdot \\ &\cdot \\ &\cdot \\ A_n &= \langle a_{n0}, a_{n1}, a_{n2}, \ldots, 0 \rangle \end{aligned}$$

In the Euclidean space E^n the *hypersphere* (Ω, r) with center $\Omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and radius r is defined as the locus of points

(1.15) $$\{X : \|X - \Omega\| = r\}$$

with cartesian equations

$$f(X) = \sum_{i=0}^{n} (x_i - \omega_i)^2 - r^2 = 0$$

$$(1.16) \quad f(X) = \sum_{i=0}^{n} x_i^2 - 2 \sum_{i=0}^{n} \omega_i x_i + \omega_0 = 0 , \ \omega_0 = \sum_{i=0}^{n} \omega_i^2 - r^2 .$$

The *power* of the point P with respect to the hypersphere (1.16) is given by

$$(1.17) \qquad \pi = f(P) = p^2 - r^2 , \qquad p = P\Omega .$$

and hence the power of P relative to the hypersphere

$$g(x) = \lambda \sum_{i=0}^{n} x_i^2 - 2 \sum_{i=0}^{n} \lambda_i x_i + \lambda_0 = 0 , \quad (\lambda \neq 0)$$

is

$$(1.18) \qquad \pi = g(p) / \lambda , \qquad (\lambda \neq 0)$$

2. Circumhypersphere of S_n. Proving two lemmas we obtain the equation in volume coordinates of the circumhypersphere (S_n) of $S_n \equiv A_0 A_1 \ldots A_n$, the expression for the power of a point with respect to this hypersphere, and for its circumradius R in terms of the sides a_{ij} of S_n.

(2.1) LEMMA. *For any point $P = [v_0 , v_1 , \ldots , v_n]$ with $v_i = Vol\ (S^i)$ and $v = Vol\ (S_n)$ we have the equality*

$$\sum_{i=0}^{n} v_i = v .$$

Proof. Denoting the left hand side of this equality by V, we have in view of (1.71), (1.8) and (1.3),

$$(2.2) \quad n!\sum_{i=0}^{n} v_i = \sum_{i=0}^{n} \begin{vmatrix} \alpha_{01} & \cdots & \alpha_{0j} & \cdots & \alpha_{0n} & 1 \\ \cdot & & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & & \cdot & \cdot \\ \alpha_{i-1\,1} & \cdots & \alpha_{i-1\,j} & \cdots & \alpha_{i-1\,n} & 1 \\ x_1 & \cdots & x_j & \cdots & x_n & 1 \\ \alpha_{i+1\,1} & \cdots & \alpha_{i+1\,j} & \cdots & \alpha_{i+1\,n} & 1 \\ \cdot & & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & & \cdot & \cdot \\ \alpha_{n1} & \cdots & \alpha_{n} & \cdots & \alpha_{nn} & 1 \end{vmatrix} = n!\ V$$

where the row $x_1, x_2, \ldots, x_n$ occupies the first row for $i = 0$ and the last row for $i = n$. The coefficient of x_j in (2.2) is equal to the sum

$$A_{0j} + A_{1j} + \ldots + A_{nj}$$

of the $n + 1$ cofactors. But this sum is nothing but the Laplace expansion of (1.3) when its (j) th column is replaced by the last column consisting of l's. Hence it is zero, showing that $n!V$ is independent of the point $P = (x_1, x_2, \ldots, x_n)$. To evaluate $n!\,V$ it will then suffice to take in (2.2) the point P at the origin $(0, 0, \ldots, 0)$, and in that case $n!V$ equals the sum of the cofactors of the elements of the last column, that is, the value $n!v$ of (1.3). Hence $n!V = n!v$ and therefore (2.1) is true.

(2.3) LEMMA. *The hypervolume and barycentric coordinates of a point are identical.*

Proof. These coordinates coincide at $n + 1$ linearly independent points, namely, at the vertices of S_n (1.10), and by their linear characters we conclude that they coincide at every point of E^n.

As a consequence of (2.3) and (1.1) we have

$$v\,\overrightarrow{OP} = \sum_{i=0}^{n} v_i\,\overrightarrow{OA_i}\,, \quad v = \sum_{i=0}^{n} v_i\,. \tag{2.4}$$

(2.5) THEOREM. *The equation in volume coordinates of the circumhypersphere of the simplex of reference $S_n \equiv A_0 A_1 \ldots A_n$ is*

$$\sum_{i,j=0}^{n} a_{ij}^2\, v_i\, v_j = 0\,, \qquad (a_{ij} = A_i A_j)$$

Proof (by induction). For $n = 2$ the equation of the circumcircle of a triangle ABC being [4, *p.* 168]

$$a^2\,\Delta_b\,\Delta_c + b^2\,\Delta_c\,\Delta_a + c^2\,\Delta_a\,\Delta_b = 0$$

where a, b, c are the sides of the triangle and Δ_a, Δ_b, Δ_c are the areal coordinates [3, *p.* 119], and the theorem is true for this case.

Assuming that the theorem is true for $n - 1$ $(n \geqslant 3)$, that is, for $(n - 1)$-simplex $T_0 \equiv A_1 \ldots A_n$ in the space E^{n-1}, let the equation be

$$\sum a_{ij}^2\, u_i\, u_j = 0 \tag{2.6}$$

where u_i's are volume coordinates of P relative to $A_1 \ldots A_n$.

The equation of (S_n) being a quadratic form

$$\Sigma\, \alpha_i^2\, v_i^2 + 2\, \Sigma\, v_i\, v_j = 0 \tag{2.7}$$

and satisfying $A_i = [0, \ldots, 0, 1, 0, \ldots, 0]$ we obtain $\alpha_i = 0$, $[0 \leqslant i \leqslant n]$ and (2.7) reduces to

$$\Sigma\, \alpha_{ij}\, v_i\, v_j = 0\,. \tag{2.8}$$

We want (2.8) to be the equation of the circumhypersphere. Then it should contain the circumhyperspheres of the bases T_i of S_n, in particular, that of $T_0 = A_1 \ldots A_n$. This means that (2.8) should be reduced to (2.6) upon the substitution $v_0 = 0$. In case $v_0 = 0$ the coordinates $v_1, \ldots, v_n$ of P should be equal to $u_1, \ldots, u_n$ in (2.6) and therefore for α_{ij} $(i \neq 0)$ one has

$$\alpha_{12} : \ldots : \alpha_{ij} = a_{12} : \ldots : a_{ij}\,.$$

Setting then in turn $v_1 = 0, \ldots, v_n = 0$, and repeating the argument for each case we conclude that α_{ij}'s are proportional to a_{ij}'s, and therefore (2.5) is established.

The foregiong proof rests on intuition at some stage. So to justify the result (2.5) another proof is supplied using vector algebra :

Proof II. Taking the circumcenter Ω as coincident with the origin of cartesian coordinates and denoting the circumradius by R, and by (2.4) letting the point

$$P = \left(\sum_{i=0}^{n} v_i \right) P = \sum_{i=0}^{n} v_i\, A_i\,, \qquad \sum_{i=0}^{n} v_i = v = 1\,, \tag{2.9}$$

be on the hyrcumhypersphere (Ω), we get by inner product

$$R^2 = P\,.\,P = \Sigma\, v_i\, v_j\, A_i\,.\,A_j$$

where $A_i\,.\,A_j$ sarisfies the relation

$$a_{ij}^2 = (A_i - A_j)^2 = 2\, A_i\,.\,A_i - 2\, A_i\,.\,A_j = 2\, R^2 - 2\, A_i\,.\,A_j\,.$$

Hence

$$\begin{aligned} 2\, R^2 &= \Sigma\, v_i\, v_j\, (2\, R^2 - a_{ij}^2) \\ &= 2\, R^2\, \Sigma\, v_i\, v_j - \Sigma\, a_{ij}^2\, v_i\, v_j \\ &= 2\, R^2\, (\Sigma\, v_i)^2 - \Sigma\, a_{ij}^2\, v_i\, v_j \\ &= 2\, R^2 - \Sigma\, a_{ij}^2\, v_i\, v_j \end{aligned}$$

and (2.5) holds.

(2.10) THEOREM. *The power of the point* $P = [v_0, v_1, \ldots, v_n]$ *with respect to the circumhypersphere of* $A_0 A_1 \ldots A_n$ *is given by*

$$\pi = -v^{-2} \sum_{0 \leq i < j}^{n} a_{ij}^2 v_i v_j, \qquad (a_{ij} = A_i A_j).$$

Proof (by induction). We first prove the theorem for $n = 2$. Let ABC be a triangle and let P have areal coordinates $\Delta_a, \Delta_b, \Delta_c$ and trilinear (normal) coordinates [3, *p.* 112] x, y, z so that we have

$$\text{(2.11)} \qquad \Delta_a = \frac{1}{2} ax, \quad \Delta_b = \frac{1}{2} by, \quad \Delta_c = \frac{1}{2} cz.$$

Denoting the pedal triangle of P by XYZ and its area by $\overline{XYZ}$ we have [5, *p.* 140] :

$$\text{(2.12)} \qquad \overline{XYZ} = \frac{1}{2}(R^2 - OM^2) \sin A \sin B \sin C$$

where R is the circumradius and 0 the circumcenter. Then solving (2.12) for $\pi = OM^2 - R^2$ we get

$$\pi = -\frac{2\,\overline{XYZ}}{\sin A \sin B \sin C} = -\frac{16 R^3}{abc} \cdot \overline{XYZ}$$

$$= -\frac{16 R^3}{4R\Delta} \cdot \overline{XYZ} = -\frac{4R^2}{\Delta} \sum \overline{PYZ}, \quad (abc = 4R\Delta)$$

$$= -\frac{2R^2}{\Delta} \sum yz \sin A$$

$$= -\frac{2R^2}{\Delta} \sum \frac{2\Delta_b}{b} \frac{2\Delta_c}{c} \frac{a}{2R} \qquad (\textit{from } 2.11)$$

$$= -\frac{2R^2}{\Delta} \sum \frac{4a^2 \Delta_b \Delta_c}{abc \,.\, 2R} = -\frac{\sum a^2 \Delta_b \Delta_c}{\Delta^2}$$

This result proves (2.10) for $n = 2$.

Now we suppose that the theorem is true for $n - 1$, that is, let the power of $P = [u_1, \ldots, u_n]$ with respect to $T_0 \equiv A_1 \ldots A_n$ have the form

$$\text{(2.13)} \qquad \pi = -u^{-2} \sum_{0 \leq i < j}^{n} a_{ij}^2 u_i u_j, \quad u = \sum_{i=0}^{n} u_i.$$

For $n = n$ the expression for the power π should admit the left hand side of equality in (2.5) as a factor and we have

$$\pi = \lambda\, v^{-2} \sum a^2_{ij}\, v_i\, v_j\,, \quad v = \sum_{i=0}^{n} v_i\,. \tag{2.14}$$

Since (2.14) is homogeneous of degree zero, in the variables v_i, the factor λ is independent of v_i's and is a function of a_{ij}'s alone, hence it is a constant. Since for $v_0 = 0$ (2.14) should be reduced to (2.13), it follow that $\lambda = 1$, and this establishes (2.10).

(2.15) THEOREM. *The circumradius of the simplex of reference $A_0 A_1 \ldots A_n$ is given by*

$$R^2 = \frac{(-1)^2}{2^{n+1}\,(n\,!)^2}\,\frac{|a^2_{ij}|}{v^2}\,, \quad v = Vol\ (S_n)\,, \quad |a^2_{ij}| = det\ [\,a^2_{ij}\,]\,.$$

Proof. Consider the Cayley's formula [2, *p.* 124] for the volume of the simplex in terms of its sides :

$$v^2 = \frac{(-1)^{n+1}}{2^n\,(n\,!)^2} \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & a^2_{01} & \ldots & a^2_{0n} \\ 1 & a^2_{10} & 0 & \ldots & a_{1n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & a^2_{n1} & a^2_{n1} & & 0 \end{vmatrix} = k_n \begin{vmatrix} 0 & 1 \ldots 1 \\ 1 & \\ \cdot & \\ \cdot & \\ \cdot & a^2_{ij} \\ \cdot & \\ 1 & \end{vmatrix} \tag{2.16}$$

where $a_i = A_i A_j$ and $k_n = (-1)^n / 2^{n+1}\,(n\,!)^2$.

Taking a point Ω in E^{n+1} and applying (2.16) to the $(n+1)$-simplex $\Omega\, A_0\, A_1 \ldots A_n$ we have for its volume v_{n+1} :

$$v^2_{n+1} = k_{n+1} \begin{vmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & \Omega A^2_0 & \Omega A^2_1 & \ldots & \Omega A^2_n \\ 1 & A_0 \Omega^2_1 & 0 & A_0 A^2_1 & \ldots & A_0 A^2_n \\ 1 & A_1 \Omega^2 & A_1 A^2_0 & 0 & \ldots & A_1 A^2_n \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ 1 & A_n \Omega^2 & A_n A^2_0 & A_n A^2_1 & \ldots & 0 \end{vmatrix}$$

If now Ω is taken in space E^n of S_n the hypervolume v_{n+1} in E^{n+1} vanishes in E^n, and if furthermore Ω is taken at the circumcenter of S_n, that is, if $\Omega A_i = R$ $[0 \leqslant i \leqslant n]$, we get the relation for R :

$$\begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & R^2 & \ldots & R^2 \\ 1 & R^2 & & & \\ \cdot & \cdot & & a_{ij}^2 & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ 1 & R^2 & & & \end{vmatrix} = 0 , \qquad \begin{vmatrix} -2 & 1 & 0 & \ldots & 0 \\ 1 & 0 & R^2 & \ldots & R^2 \\ 0 & 1 & & & \\ \cdot & \cdot & & a_{ij}^2 & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ 0 & 1 & & & \end{vmatrix} = 0$$

where the second equality is obtained from the first by elemementary row operations. Expanding the latter with respect to the first row we get

$$2\,R^2\; v^2 / k_n + |\, a_{ij}^2 \,| = 0$$

proving (2.15).

3. Euler hypersphere. The Euler circle of a triangle [5, *p.* 195] can be extended to simplexes in several directions. If this circle is defined as the circumcircle of the medial triangle of the triangle, and the mid-points of the sides are interpreted as the centroids of the sides, then one extension (E) is obtained as the circumhypersphere of the n-simplex $G_0\, G_1 \ldots G_n$ where G_i is the centroid of the base T_i of S_n. This will be taken as the definition of the *Euler hypersphere* of the simplex of reference S_n. To obtain the equation in volume coordinates of (E) we give two lemmas.

(3.1) LEMMA. *For any point O*

$$\begin{bmatrix} \overrightarrow{OG_0} \\ \overrightarrow{OG_1} \\ \cdot \\ \cdot \\ \cdot \\ \overrightarrow{OG_n} \end{bmatrix} = \begin{bmatrix} 0 & 1/n & \ldots & 1/n \\ 1/n & 0 & \ldots & 1/n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1/n & 1/n & \ldots & 0 \end{bmatrix} \cdot \begin{bmatrix} \overrightarrow{OA_0} \\ \overrightarrow{OA_1} \\ \cdot \\ \cdot \\ \cdot \\ \overrightarrow{OA_n} \end{bmatrix}$$

This follows directly from the definition of G_i above.

The square matrix Γ in (3.1) is non singular, since $\det \Gamma = (-1)^n n^{-n}$. Hence Γ^{-1} exists so that (3.1) can be solved for $\overrightarrow{OA_i}$:

$$(3.2)\qquad \begin{bmatrix} \overrightarrow{OA_0} \\ \overrightarrow{OA_1} \\ \cdot \\ \cdot \\ \cdot \\ \overrightarrow{OA_n} \end{bmatrix} = \begin{bmatrix} -(n-1) & 1 & \dots & 1 \\ 1 & -(n-1) & \dots & 1 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & 1 & \dots & -(n-1) \end{bmatrix} \cdot \begin{bmatrix} \overrightarrow{OG_0} \\ \overrightarrow{OG_1} \\ \cdot \\ \cdot \\ \cdot \\ \overrightarrow{OG_n} \end{bmatrix}$$

(3.3) LEMMA. *If v_0, v_1, ..., v_n are the volume coordinates of a point P relative to the n-simplex $A_0 A_1 \dots A_n$, then the volume coordinates of P relative the n-simplex $G_0 G_1 \dots G_n$, is the n-tuple*

$$[v - nv_0\,,\ n - nv_1\,,\ \dots\,,\ v - nv_n]\,,\qquad v = \sum_{i=0}^{n} v_i\,.$$

Proof. Setting the values of OA_i from (3.2) into (2.4) we get

$$v\,\overrightarrow{OP} = (v - nv_0)\,\overrightarrow{OG_0} + (v - nv_1)\,\overrightarrow{OG_1} + \dots + (v - nv_n)\,\overrightarrow{OG_n}$$

which gives the required result.

(3.4) THEOREM. *The equation in volume coordinates of the Euler hypersphere (E) of the the n-simplex $A_0 A_1 \dots A_n$ is given by*

$$\sum_{i,j=0}^{n} a_{ij}^2\,(v - nv_i)\,(v - nv_j) = 0$$

Proof. If we denote the sides of $G_0 G_1 \dots G_n$ by b_{ij}, then the equation of (E) is, in view of (2.5),

$$(3.5)\qquad \sum_{i,j=0}^{n} b_{ij}^2\,(v - nv_i)\,(v - nv_j) = 0$$

since $v - nv_i$'s are the volume coordinates relative to $G_0 G_1 \dots G_n$ by (3.3). One may show that $b_{ij} = a_{ij} / n$. The latter, set in (3.5), proves (3.4).

For $n = 2$, the equation (3.4) reduces to one given in [4, *p.* 171].

4. General hyperspheres. The content of this section rests essentially on the idea of representing a given hypersphere by a finite sequence of powers. Let (Ω, r) be a given hypersphere (1.16) and let π_i be the power (1.17) of the vertex A_i of S_n relative to (Ω, r). Then the following holds :

(4.1) THEOREM. *Given a simplex of reference $A_0 A_1 \ldots A_n$, a hypersphere (Ω) determines a finite sequence $< \Pi_i >_{i=0}^{n}$ of powers, where Π_i is the power of A_i relative to (Ω), and conversely every finite sequence determines a hypersphere, and one writes $(\Omega) \equiv < \Pi_i >_{i=0}^{n}$.*

Proof. The necessity follows from the definition of Π_i's. To prove the sufficiency we show that there exists a unique sphere

$$(\Omega) : f(X) = \Sigma x_i^2 - 2 \Sigma \omega_i x_i + \omega_0 = 0$$

such that the powers of A_i relative to (Ω) are the terms of the given sequence $\Pi_1, \Pi_2, \ldots, \Pi_n$. In view of (1. 16) and (1.17) we write

$$f(A_i) = \sum_{j=1}^{n} \alpha_{ij}^2 - 2 \sum_{j=1}^{n} \omega_i \alpha_{ij} + \omega_0 = \Pi_i, \quad [0 \leqslant i \leqslant n]$$

which is a system of $n+1$ equations in the unknowns $\omega_0, \omega_1, \ldots, \omega_n$ admitting $n!v$ (1.3) as Cramer determinant. Since $v \neq 0$, the system has a unique solution. ω_0 gives the radius and $(\omega_1, \ldots, \omega_n)$ the center.

In particular $< 0, 0, \ldots, 0 >$ is the representative of the circumhypersphere of S_n.

(4.2) THEOREM. *If π is the power of a point $P = [v_0, v_1, \ldots, v_n]$ relative to the circumhypersphere (S_n) of $S_n \equiv A_0 A_1 \ldots A_n$, and if π_i denotes the power of A_i relative to the circumhypersphere (S^i) of $S^i \equiv A_0 \ldots A_{i-1} PA_{i-1} \ldots A_n$, $[0 \leqslant i \leqslant n]$, then*

$$\pi_i v_i = -\pi v, \quad [0 \leqslant i \leqslant n].$$

Proof. The cartesian equation of (S_n) is given by

$$f(X) = \begin{vmatrix} \Sigma x_j^2 & x_1 & \ldots & x_n & 1 \\ \Sigma \alpha_{0j}^2 & \alpha_{01} & \ldots & \alpha_{0n} & 1 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \Sigma \alpha_{nj}^2 & \alpha_{n1} & \ldots & \alpha_{nn} & 1 \end{vmatrix}$$

where the cofactor of $\sum x_j^2$ is $n!v$. Hence by (1.18),

(4.3) $$\pi = f(P) / (n!v), \quad \text{or}$$

(4.4) $$\pi v = f(P) / n! .$$

From the determinantal equation $f_i(X) = 0$ of the hypersphere (S^i) we get a similar result, namely

(4.5) $$\alpha_i v_i = f_i(A_i) / n! .$$

But since $f_i(A_i) = -f(P)$ we have our required result.

(4.6) Corollary. *If in E^n are taken $n+2$ points $A_0, \ldots, A_n, A_{n+1}$ no $n+1$ of them lie on the same hyperplane, and if π_i is the power of A_i $[0 \leqslant i \leqslant n+1]$ relative to the circumhypersphere of the simplex formed by the remaining $n+1$ points, then*

$$\sum_{i=0}^{n+1} \pi_i^{-1} = 0 .$$

Proof. If A_{n+1} is considered as P in (4.2) and thus $\pi_{n+1} = \pi$, then from (4.2) and (2.1),

$$\frac{v}{-\pi_{n+1}^{-1}} = \frac{v_0}{\pi_0^{-1}} = \frac{v_1}{\pi_1^{-1}} = \ldots = \frac{v_n}{\pi_n^{-1}} = \frac{\sum_{i=0}^{n} v_i}{\sum_{i=0}^{n} \pi_i^{-1}} = \frac{v}{\sum_{i=0}^{n} \pi_i^{-1}}$$

or

$$-\pi_{n+1}^{-1} = \sum_{i=0}^{n} \pi_i^{-1} .$$

(4.7) Corollary. *Among $n+2$ points taken in E^n no $n+1$ of which lie on the same hyperplane, there is at least one point lying inside its corresponding hypersphere (as defined in 4.6).*

Proof. In the equality (4.6) the left side must contain at least one term with negative sign.

(4.8) Corollary. *If P is the barycenter of S_n, that is, if $P = [1, 1, \ldots, 1]$, then*

$$\pi_0 = \pi_1 = \ldots = \pi_n = -\pi .$$

Proof. It follows directly from (4.2).

(4.9) THEOREM. *If π and Π are the powers of the point $P=[v_0, v_1, \ldots, v_n]$ with respect to the circumhypersphere (S_n) and the hypersphere $<\Pi_i>$ respectively, then*

$$\sum_{i=0}^{n} (\Pi_i - \Pi)\, v_i = -\pi\, v\,.$$

Proof. If the determinant $f(P)$, $P=(\alpha_1, \alpha_2, \ldots, \alpha_n)$, in (4.3) is expanded in the first column, and if we set

$$\sum_{j=0}^{n} \alpha_j^2 = \Omega P^2 = p^2\,, \quad \sum_{j=1}^{n} \alpha_{ij}^2 = \Omega A_i^2 = p_i^2\,, \quad [0 \leqslant i \leqslant n]$$

where Ω denotes the circumcenter taken at the center of coordinate axes, we have

$$\pi\, v = p^2\, v - p_0^2\, v_0 - p_1^2\, v_1 - \ldots - p_n^2\, v_n\,.$$

Subtracting from it the quantity

$$r^2\, v - r^2\, v_0 - r^2\, v_1 - \ldots - r^2\, v_n$$

which is zero by (2.1), we have, in view of (1.17)

$$\pi\, v = (p^2 - r^2)\, v - \sum_{i=0}^{n} (p_i^2 - r^2)\, v_i$$

$$= \Pi\, v - \sum \Pi_i\, v_i = -\sum (\Pi_i - \Pi)\, v_i$$

or (4.8).

(4.10) COROLLARY. If P is on (Ω), then $\sum \Pi_i\, v_i = -\pi\, v$.

(4.11) COROLLARY. If P is on (S_n), then $\sum (\Pi_i - \Pi)\, v_i = 0$.

(4.12) COROLLARY. If (Ω) is tangent to (S_n) at P, then $\sum \Pi_i\, v_i = 0$.

(4.13) THEOREM. *The equation in volume coordinates of the hypersphere $(\Omega) \equiv <\Pi_i>_{i=0}^{n}$ is given by*

$$\sum_{i,\, j=0}^{n} (\Pi_i - \Pi_j = a_{ij}^2)\, v_i\, v_j = 0\,.$$

Proof. Setting in (4.9) the value of π given by (2.10) we obtain

$$\Sigma\,(\Pi_i - \Pi)\,v_i = v^{-1} \sum_{i<j} a_{ij}^2\, v_i\, v_j\,.$$

Taking then P on (Ω) we have $\Pi = 0$, and

$$v \,\Sigma\, \Pi_i\, v_i - \sum_{i<j} a_{ij}^2\, v_i\, v_j = 0$$

$$2 \sum_0 v_i \sum_0 \Pi_i\, v_i - 2 \sum_{i<j} a_{ij}^2\, v_i\, v_j = 0$$

$$\sum_{0,0} (\Pi_i + \Pi_j)\, v_i\, v_j - \sum_{0,0} a_{ij}^2\, v_i\, v_j = 0\,.$$

5. Radical hyperplanes. We shall obtain the equation in volume coordinates of the radical hyperplane of two given hyperspheres and those of ones having certain relations to some given elements.

(5.1) THEOREM. *In E^n the locus of points P having the same power relative to two given hyperspheres $(\Omega_1) \equiv \overset{n}{\underset{0}{<\Pi_{1i}>}}$, $(\Omega_2) \equiv \overset{n}{\underset{0}{<\Pi_{2i}>}}$ is a hyperplane having the equation*

$$\sum_{i=0}^{n} (\Pi_{1i} - \Pi_{2i})\, v_i = 0\,.$$

Proof. For the given hyperspheres the raletions (4.9) hold :

$$\Sigma\,(\Pi_{1i} - \Pi_1)\, v_i = -\pi\, v\,, \quad \Sigma\,(\Pi_{2i} - \Pi_2)\, v_i = -\pi\, v$$

where Π_1, Π_2 are powers of $P = [v_0, v_1, \ldots, v_n]$ and π is the power of P relative to the circumhypersphere (S_n) of the simplex of reference Since $\Pi_1 = \Pi_2$ we obtain the equation (5.1) of the locus, and since it is linear in the $n+1$ coordinates v_i, it represents a hyperplane in E^n and its equation is as given in (5.1).

(5.2) COROLLARY. *The equation of the hyperplane tangent to the hypersphere $(\Omega) \equiv \overset{n}{\underset{0}{<\Pi_i>}}$ at the point $P = <p_0, p_1, \ldots, p_n>$ is given by*

$$\Sigma\,(\Pi_i - p_i^2)\, v_i = 0\,.$$

Proof. Thinking of P as the hypersphere $(P, r = 0)$, it defines the sequence $\overset{n}{\underset{0}{<p_i^2>}}$ and by (5.1) we have (5.2).

(5.3) COROLLARY. *The equation of the hyperplane passing through a point A and being perpendicular to the segment joining $P = <p_i>$ and $Q = <q_i>$, with $p = PA$ and $q = QA$, is given by*

$$\Sigma\ [p_i^2 - q_i^2 - (p^2 - q^2)]\ v_i = 0\ .$$

Proof. For suitable radii r_1 and r_2, the point A will lie on the radical hyperplane of (P, r_1) and (Q, r_2). So setting in (5.1)

$$\Pi_{1i} = p_i^2 - r_1^2\ , \qquad \Pi_{2i} = q_i^2 - r_2^2$$

we have (5.3), since $p^2 - r_1^2 = q^2 - r_2^2$.

(5.4) COROLLARY. *The perpendicular bisector hyperplane of the line segment* PQ, *with* $P = < p_i >_0^n$, $Q = < q_i >_0^n$ *is given by*

$$\sum_{i=0}^{n} (p_i^2 - q_i^2)\ v_i = 0\ .$$

Using the last corollary (5.4) we can obtain the hypervolume coordinates of the circumenter Ω of the simplex of reference S_n. This center is the intersection of the perpendicular bisector hyperplanes π_{ij} of the sides $A_i A_j$ $(0 \leqslant i,\ j < n)$, in particular, of $\pi_{01}, \pi_{02}, \ldots, \pi_{0n}$.

The equation of π_{01}, say, is obtained by (1.14) and (5.4):

$$(5.5)\quad \pi_{01} : \left(-a_{01}^2\right) v_0 + \left(a_{01}^2\right) v_1 + \left(a_{02}^2 - a_{12}^2\right) v_2 + \cdots + \left(a_{0n}^2 - a_{1n}^2\right) v_n = 0$$

in which setting $v_0 = v - v_1 - v_2 - \ldots - v_n$ from (2.1), we get

$$(5.6)\quad (2a_{01})\ v_1 + \left(a_{01}^2 + a_{02}^2 - a_{12}^2\right) v_2 + \cdots + \left(a_{01}^2 + a_{0n}^2 - a_{1n}^2\right) v_n = a_{01}^2\ v\ .$$

Now setting

$$(5.7)\qquad \theta_{ij} = \theta_{ji} = \text{ angle } (A_0 A_i,\ A_0 A_j)$$

and applying cosine laws to triangles $A_0 A_i A_j$, the equation (5.6) reduces to

$$(5.8)\quad (a_{01})\ v_1 + (a_{02} \cos \theta_{12})\ v_2 + \cdots + (a_{0n} \cos \theta_{1n})\ v_n = \frac{1}{2}\ a_{01}\ v.$$

Doing the same for the hyperplanes $\pi_{02}, \ldots, \pi_{0n}$ we have the system

$$(5.9)\qquad \begin{bmatrix} \cos\theta_{11} & \cos\theta_{12} & \dots & \cos\theta_{1n} \\ \cos\theta_{21} & \cos\theta_{22} & \dots & \cos\theta_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cos\theta_{n1} & \cos\theta_{2n} & \dots & \cos\theta_{nn} \end{bmatrix} \cdot \begin{bmatrix} a_{01}\ v_1 \\ a_{02}\ v_2 \\ \cdot \\ \cdot \\ a_{0n}\ v_n \end{bmatrix} = \frac{1}{2} \begin{bmatrix} a_{01} \\ a_{02} \\ \cdot \\ \cdot \\ a_{0n} \end{bmatrix} v$$

The non-singularity of the square matrix follows from

$$(5.10)\qquad v = \frac{1}{n!}\, a_{01}\, a_{02} \dots a_{0n} \left| \cos\theta_{ij} \right|^{\frac{1}{2}}, \qquad [2, p. 122]$$

So multiplying (5.9) by the inverse of the square matrix, we have the

(5.11) THEOREM. *Let $S_n \equiv A_0 A_1 \dots A_n$ be the simplex of reference, and let θ_{ij} be as defined in* (5.7). *Then the hypervolume coordinates $v_1, \dots, v_n$, and $v_0 = v - (v_1 + \dots + v_n)$ of the circumphypersphere (S_n) are given by*

$$\begin{bmatrix} a_{01}\ v_1 \\ a_{02}\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{0n}\ v_n \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \cos\theta_{11} & \cos\theta_{12} & \dots & \cos\theta_{1n} \\ \cos\theta_{21} & \cos\theta_{22} & \dots & \cos\theta_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cos\theta_{1n} & \cos\theta_{2n} & \dots & \cos\theta_{nn} \end{bmatrix}^{-1} \cdot \begin{bmatrix} a_{01} \\ a_{02} \\ \cdot \\ \cdot \\ \cdot \\ a_{0n} \end{bmatrix} v.$$

6. Generalized Pythagorean theorem. The Pythagorean theorem for right 2-simplexes (triangles) has been generalized to E^3 by De Gua [9] :

(6.1) THEOREM. *The sum of the squares of the lateral areas of a trirectangular tetrahedron $A_0 A_1 A_2 A_3$ is equal to the square of the area of the base $A_1 A_2 A_3$, that is,*

$$a_1^2 + a_2^2 + a_3^2 = a^2 .$$

The next generalization is obtained by Lepteva [6] for right 4-simplexes in E^4 proved also by Zirakadeh [7] who states that his proof for E^4 can be extended to establish the general Pythagorean theorem for right n - simplexes in the space E^n.

Here we give a complete proof of this general extension :

(6.2) THEOREM. *Let* $A_0 A_1 \ldots A_n$ *be an* n*-simplex in* E^n *such that* $A_0 A_1, A_0 A_2, \ldots, A_0 A_n$ *are mutually perpendicular. If* $a_i = Vol\,(T_i)$ *with* $a = a_0$, *then*

$$a_1^2 + a_2^2 + \ldots + a_n^2 = a^2 .$$

Proof. Let A_0 be taken at the origin of cartesian coordinate system and $A_0 A_1, A_0 A_2, \ldots, A_0 A_n$ as axes with $A_i = (0, \ldots, 0, h_i, 0, \ldots, 0)$, $h_i > 0$, so that the equation of the hyperplane $A_1 A_2 \ldots A_n$ is

$$x_1/h_1 + x_2/h_2 + \ldots + x_n/h_n - 1 = 0, \quad h_i > 0. \tag{6.3}$$

Denoting the elementary symmetric functions of $h_1^2, h_2^2, \ldots, h_n^2$ by $s_1, s_2, \ldots, s_n$, then (6.3) assumes its normal form

$$(x_1/h_1 + x_2/h_2 + \ldots + x_n/h_n - 1)\sqrt{s_n}\,/\sqrt{s_{n-1}} = 0 \tag{6.4}$$

from which we obtain

$$h = \sqrt{s_n/s_{n-1}} \tag{6.5}$$

as the distance (altitude) of $A_0 = (0, \ldots, 0)$ from the hyperplane.

Applying the formula (5.10) where $a_{0i} = h_i$, $\theta_{ij} = \frac{1}{2}\pi$ we have

$$v = \frac{1}{n!} h_1 h_2 \ldots h_n = \sqrt{s_n}\,/\,n! \tag{6.6}$$

and using the formula $v = ah/n$ [2, p. 123] in (6.6) we get

$$a = \frac{nv}{h} = \frac{n\sqrt{s_n}\,/\,n!}{\sqrt{s_n/s_{n-1}}} = \sqrt{s_{n-1}}\,/\,(n-1)!$$

At the other hand for the $(n-1)$-simplex T_i

$$a_i = h_1 \ldots h_{i-1} \cdot h_{i+1} \ldots h_n/(n-1)! = h_1 h_2 \ldots h_n/h_i\,(n-1)!$$
$$= \sqrt{s_n}\,/\,h_i\,(n-1)!$$

and we have

$$\sum_{i=0}^{n} a_i^2 = \sum_{i=0}^{n} \frac{s_n}{(n-1)!^2} \frac{1}{h_i^2} = \frac{s_n}{(n-1)!^2} \sum_{i=1}^{n} \frac{1}{h_i^2}$$

$$= \frac{s_n}{(n-1)!^2} \frac{s_{n-1}}{s_n} = \left(\frac{\sqrt{s_{n-1}}}{(n-1)!} \right)^2 = a^2 .$$

(6.7) COROLLARY. $1/h_1^2 + 1/h_2^2 + \dots + 1/h_n^2 = 1/h^2$.

7. Extension of Erdös - Mordell and Oppenheim inequalities for triangles to regular polygons. Here we state a conjecture as an extension of Erdös-Mordell and Oppenheim inequalities (for triangles) to convex polygons and a proof is provided in the special case where the polygon is regular.

The Erdös - Mordell inequality for a triangle is given by the following theorem [11] :

(7.1) THEOREM. *If P is an interior or a boundary point of a triangle $A_1 A_2 A_3$ and if p_i and x_i are the distances of P from the vertices A_i and the sides opposite A_i respectively, then*

$$p_1 + p_2 + p_3 \geqslant 2\,(x_1 + x_2 + x_3)$$

where equality holds only if the triangle is equilateral and P is the center of it.

Several proofs of this theorem are known [8], [9].

Other inequalities between the same quantities are discovered by A. Oppenheim [9], [11], four of which are taken into consideration here, namely,

(7.2) $p_2 p_3 + p_3 p_1 + p_1 p_2 \geqslant 4\,(x_2 x_3 + x_3 x_1 — (x_1 x_2)$,

(7.3) $p_1 p_2 p_3 \geqslant 8\,(x_1 x_2 x_3)$,

(7.4) $p_2 p_3 + p_3 p_1 + p_1 p_2 \geqslant (x_3 + x_1)\,(x_1 + x_2) + (x_1 + x_2)\,(x_2 + x_3) + (x_2 + x_3)\,(x_3 + x_1)$

(7.5) $p_1 p_2 p_3 \geqslant (x_2 + x_3)\,(x_3 + x_1)\,(x_1 + x_2)$.

Noticing that some symmetric functions enter, by introducing

(7.6) $$y_i = x_{i-1} + x_i ,\ [1 \leqslant i \leqslant 3] , \quad x_0 \equiv x_3$$

and the elementary symmetric functions

(7.7) $$\sigma_1 ,\ \sigma_2 ,\ \sigma_3 ;\quad s_1 ,\ s_2 ,\ s_3 ;\quad t_1 ,\ t_2 ,\ t_3$$

of p_i, x_i, y_i respectively, the inequalities (7.1), (7.2), (7.3) may be condensed into a single form, namely

(7.8) $$\sigma_k \geqslant 2^k s_k, \quad [1 \leqslant k \leqslant 3],$$

and the inequalities (7.1), (7.4), (7.5) into

(7.9) $$\sigma_k \geqslant t_k, \quad [1 \leqslant k \leqslant 3]$$

where both forms contain the Erdös-Mordell inequality (7.1) for $k = 1$.

We can go still further into condensation by the following lemma :

(7.10) LEMMA. (7.9) *implies* (7.8), *that is*, $t_k \geqslant 2^k s_k$, $[1 \leqslant k \leqslant 3]$.

Proof. If $k = 1$ the inequality is, in view of (7.6), an identity. If $k = 2$,

$$2t_2 = 2 \sum y_1 y_2 = (\sum y_1)^2 - \overset{2}{y_1} = 4 (\sum x_1)^2 - \sum (x_1 + x_2)^2$$
$$= 2 \sum \overset{2}{x_1} + 6 \sum x_1 x_2 = \sum (x_1 - x_2)^2 + 8 \sum x_1 x_2 \geqslant 8 \sum x_1 x_2 .$$

For $k = 3$,

$$t_3 = y_1 y_2 y_3 = (x_2 + x_3)(x_3 + x_1)(x_1 + x_2)$$
$$= (s_1 - x_1)(s_1 - x_2)(s_1 - x_3) = \overset{3}{s_1} - \overset{2}{s_1} \sum x_1 + s_1 \sum x_1 x_2 - x_1 x_2 x_3$$
$$= s_1 s_2 - s_3 = s_3 \left[(x_1 + x_2 + x_3)(x_1^{-1} + x_2^{-1} + x_3^{-1}) - 1 \right]$$
$$\geqslant s_3 (9 - 1) = 8 s_3 \qquad \text{by [8, p. 95]}$$

Using this lemma, the Erdös-Mordell and the four Oppenheim inequalities assume the final form

(7.11) $$\sigma_k \geqslant t_k, \quad [1 \leqslant k \leqslant 3] .$$

We call (7.11) as the *Erdös - Mordell - Oppenheim inequalities* for triangles.

These inequalities hold true for any triangle which is a convex figure. So we expect that the same is true for any convex polyon and we state this as a conjectiure :

(7.12) CONJECTURE. *Let P be an interior or a boundary point of a convex polygon $A_1 A_2 \ldots A_n$. Let p_i, x_i be the distances of P from the vertices A_i and the sides $A_i A_{i-1}$ (ind mod n) respectively. Consider*

(7.13) $$y_i = x_{i-1} + x_i, \quad [1 \leqslant i \leqslant n], \quad x_0 \equiv x_n$$

and the elementary symmetric functions

(7.14) $\sigma_k = \sum p_1 \ldots p_k, \quad s_k = \sum x_1 \ldots x_k, \quad t_k = \sum y_1 \ldots y_k, \quad [1 \leqslant k \leqslant n]$.

Then the inequalities

(7.15) $$\sigma_k \geqslant \left(sec \frac{\pi}{n} \right)^k s_k, \quad [1 \leqslant k \leqslant n]$$

(7.16) $$\sigma_k \geqslant \left(\frac{1}{2} sec \frac{\pi}{n} \right)^k t_k, \quad [1 \leqslant k \leqslant n]$$

hold, and equalities occur only when the polygon is regular and P is the center of the polygon.

REMARK. If

(7.17) $$t_k \leqslant 2^k s_k, \quad [1 \leqslant k \leqslant n]$$

can be proved, admitting (7.10) as a special case, then the conjecture will consist of (7.16) alone. But the proof of (7.17) being not at hand, the latter stays as an open problem. However its proofs for $n = 1$, 2 and $n - 1$ offer no difficulty.

Now we write the conjecture for regular polygons as a theorem and prove it.

(7.18) THEOREM. *For a regular poygon $A_1 A_2 \ldots A_n$*

$$\sigma_k \geqslant \left(\frac{1}{2} sec \frac{\pi}{n} \right)^k t_k, \quad [1 \leqslant k \leqslant n]$$

is true, and equality holds only if P is at the center of the regular polygon.

Proof. We attempt to prove the conjecture (7.12), and at a stage we obtain the proof of the theorem (7.18). We follow the method of proof given by Mordell [9].

Let the vertical projection of P on the side $A_i A_{i+1}$ $(A_{n+1} \equiv A_1)$ be H_i and set

(7.19) $$b_i = H_{i-1} H_i, \quad [1 \leqslant i \leqslant n].$$

Since $PH_{i-1} A_i H_i$ is an inscribable quadrilateral admitting PA_i as diameteter, we have

(7.20) $$p_i = b_i / sin\ A_i, \quad [1 \leqslant i \leqslant n]$$

and from cosine law we get

$$b_i^2 = x_{i-1}^2 + x_i^2 + 2\, x_{i-1}\, x_i \cos A_i \,.$$

Introducing the angles α_i, β_i such that

$$\alpha_i + \beta_i = \pi - A_i\,, \qquad 0 < \alpha_i\,,\ \beta_i < \pi\,, \tag{7.21}$$

the last relation reduces to

$$b_i^2 = (\, x_{i-1} \sin \alpha_i + x_i \sin \beta_i\,)^2 + (\, x_{i-1} \cos \alpha_i - x_i \cos \beta_i\,)^2$$
$$+ 2\, x_{i-1}\, x_i\, [\cos A_i + \cos\, (x_i + \beta_i)\,]$$

where the expression in the square bracket vanishes by (7.21). Then setting

$$U_i = x_{i-1} \cos \alpha_i - x_i \cos \beta_i \tag{7.22}$$

we have

$$b_i^2 = (x_{i-1} \sin \alpha_i + x_i \sin \beta_i)^2 + U_i^2 \geqslant (x_{i-1} \sin \alpha_1 + x_i \sin \beta_i)^2 \tag{7.23}$$

where the expression in the last bracket is positive by (7.21). Hence taking square root we obtain

$$b_i \geqslant x_{i-1} \sin \alpha_i + x_i \sin \beta_i$$

or, using (7.20)

$$p_i \geqslant (x_{i-1} \sin \alpha_i + x_i \sin \beta_i) \,/\, \sin A_i \,.$$

The equality occurs only if $U_i = 0$ in (7.23), or by (7.22), only if

$$x_i : x_{i-1} = \cos \alpha_i : \cos \beta_i\,, \qquad [1 \leqslant i \leqslant n]\,. \tag{7.24}$$

Now taking $\alpha_i = \beta_i$, or by (7.21)

$$\alpha = \alpha_i = \beta_i = \frac{1}{2}\,\pi - \frac{1}{2}\,A_i \tag{7.25}$$

again for convex polygons we have

$$p_i \geqslant (x_{i-1} + x_i) \sin \alpha \,/\, \sin A_i = y_i \cos \frac{1}{2}\,A_i \,/\, 2 \sin \frac{1}{2}\,A_i \,.\, \cos \frac{1}{2}\,A_i$$

or

$$p_i \geqslant \left(\frac{1}{2} \sec \frac{1}{2}\,A_i\right) y_i\,, \quad [1 \leqslant i \leqslant n]\,. \tag{7.26}$$

Now at this stage we leave the general case of convex polygons, and consider the special case of regular polygons, and set $A_i = \left(1 - \frac{2}{n}\right)\pi$.

Then

$$p_i \geqslant \left(\frac{1}{2} \sec \frac{\pi}{n}\right) y_i \qquad [1 \leqslant i \leqslant n]. \tag{7.27}$$

Forming the product from 1 to k followed by summation we have our required result, namely

$$\Sigma\, p_1 p_2 \cdots p_k \geqslant \left(\frac{1}{2} \text{ sed } \frac{\pi}{n}\right)^k \Sigma\, y_1 y_2 \cdots y_k$$

The equality sign holds, by (7.24) and (7.25), only if

$$x_i : x_{i-1} = \cos\alpha : \cos\alpha = 1 ,$$

or only if P is equidistant from all the vertices of the regular polygon, that is, only if P is the center of the regular polygon, and the proof of (7.18) is complete.

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n BOYUTLU GEOMETRİYE VE ERDÖS - MORDELL - OPPENHEIM EŞİTSİZLİKLERİNİN DÜZLEMDE TEŞMİLİNE DAİR

Bu makalede karteziyen, hiperhacım ve $(n+1)$ - kutupsal koordinatlarının kullanılmasiyle n boyutlu E^n Euclid uzayına ait bazı özellikler ile üçgenlere dair olan Erdös-Mordell-Oppenheim eşitsizliklerinin çokgenlere olan bazı teşmilleri elde edilmiştir.

Makale şu neticeleri içine almaktadır : Bir esas simpleksin çevrel hiperküresinin hiperhacım koordinatları türünden denklemi (2.5), noktanın çevrel hiperküreye göre kuvvetinin ifadesi (2.10) ile çevrel yarıçapın ifadesi (2.15); genel Euler hiperküresinin denklemi (3.4), genel hiperkürelere dair çeşitli bağıntılar (4) ile bir hiperkürenin sonlu bir kuvvet dizisi ile temsili (4.1); bazı kuvvet hiperdüzlemlerinin denklemleri (5) ve bunların bir uygulaması olarak çevrel hiperkürenin merkezinin hiperhacım koordinatlarının elde edilmesi (5.12); genel Pythagoras teoreminin bir ispatı (6.2). Makale, Madde 7 ile son bulmakta olup burada Erdös-Mordell-Oppenheim eşitsizliklerinin konveks çokgenlere teşmiline dair bir Tahmin (7.12) ve bunun düzgün çokgenler için bir ispatı (7.18) yer almaktadır.

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