

HÜSEYİN DEMİR

A life dedicated to problem composing
and problem solving

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Revised collected edition

Editorial reconstruction of the expanded September 2026 collection

October 2026

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Abstract

This volume presents a revised, newly typeset edition of the mathematical problems and solutions associated with Hüseyin Demir, based on the expanded September 2026 collection compiled by Ali Sinan Sertöz with the collaboration of Kürşat Aker and Alp Eden. Its thirteen chapters bring together material from the *American Mathematical Monthly*, *Mathematics Magazine*, the Quickies column, *Mathesis*, and other publications, together with a complete list of 117 problems in the source collection. The material ranges across Euclidean geometry, algebra, number theory, combinatorics, inequalities, and elementary analysis. Historical problem numbers, journal references, solution attributions, and the original English and French texts are retained. The present editorial reconstruction replaces reproduced journal pages with native LaTeX text and redraws mathematical figures in TikZ. An appendix documents 95 corrections, clarifications, and source discrepancies, distinguishing published corrections from new editorial observations. Incomplete source passages and unresolved mathematical issues are identified explicitly. This is an edited historical problem collection; it does not claim an exhaustive verification of every original solution.

2020 Mathematics Subject Classification. Primary 00A07; Secondary 01A75, 51M04, 00A08.

Keywords. Hüseyin Demir; mathematical problems and solutions; Euclidean geometry; recreational mathematics; collected works; editorial corrections.

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Editorial reconstruction and typesetting. Mete Han Karakaş.

Attribution. The authors and proposers of individual contributions are credited at their entries. Editorial additions are identified separately from the historical texts.

About this edition

Scope of this edition. All thirteen chapters have been transcribed into native LaTeX. Source omissions and unresolved mathematical questions remain explicitly documented. Chapter numbers are preserved so that the progress of the reconstruction can be compared with the original volume. Appendix A records 95 corrections or clarifications identified so far; it is not an exhaustive mathematical audit of the volume.

This edition reconstructs the supplied 233-page collection as editable LaTeX. The original languages, problem numbers, attributions, and historical wording are retained, with documented corrections of clear mathematical, typographical, and bibliographic errors. Journal references, display punctuation, names with diacritics, and the presentation of headings and solutions are made consistent. Original journal editorial notes are explicitly identified. New editorial observations and substantive corrections appear in the notes at the end of the volume; a superscript reference connects each observation to the relevant passage. Doubtful readings that cannot be resolved from the source are identified rather than guessed.

Source-page references identify the physical page of the supplied PDF, including its cover and preliminary pages. They remain stable when the new typesetting changes pagination. For the main text, the original printed page is seven less than the physical PDF page. Figures are newly drawn in TikZ and numbered within chapters. The portrait is retained as a photograph.

This edition preserves the credits for the original compilation and the attribution of individual problems and solutions.

Editorial assistance. GPT-6 Astra was used to assist in the preparation of this edition, including LaTeX transcription, mathematical typesetting, TikZ figures, and editorial notes.

References. Journal references are given at each entry and in the complete problem list. The source for the present edition is Ali Sinan Sertöz's expanded September 2026 compilation, available on the Hüseyin Demir page of his personal website (sertoz.bilkent.edu.tr).

Mete Han Karakaş
October 2026
Santa Cruz, CA, USA

Contents

Abstract	ii
About this edition	iii
Preface	xii
Preface to the Second Edition	xiv
1 Advanced Problems for the Monthly	1
Advanced Problem 4102	1
Advanced Problem 4125	1
Advanced Problem 4134	1
Advanced Problem 4193	2
Advanced Problem 4215	2
Advanced Problem 4679	2
Advanced Problem 4695	2
Advanced Problem 4710	3
Advanced Problem 4735	3
Advanced Problem 4818	3
2 Solutions of Advanced Problems	4
4102. A Special Triangle Transversal	4
4125. Trigonometric Determinants	6
4134. Angle Trisection	7
4193. Concurrent Lines in a Pentagon	8
4215. Hermite Polynomials	9
4679. Ten Conccyclic Orthopoles	10
4695. Simson Line and Euler Line	11
4710. Isotomic Lines	11
4735. Concurrent Lines	12
4818. A Property of the Miquel Circle of a Complete Quadrilateral	12
3 Elementary Problems for the Monthly	14
Elementary Problem E1134	14
Elementary Problem E1160	14
Elementary Problem E1197	14
Elementary Problem E1209	14
Elementary Problem E1217	15
Elementary Problem E1778	15

Elementary Problem E1779	15
Elementary Problem E1877	15
Elementary Problem E1878	16
Elementary Problem E2100	16
Elementary Problem E2101	16
Elementary Problem E2109	16
Elementary Problem E2110	17
Elementary Problem E2124	17
Elementary Problem E2160	17
Elementary Problem E2213	17
Elementary Problem E2311	18
Elementary Problem E2312	18
Elementary Problem E2363	18
Elementary Problem E2462	18
Elementary Problem E2625	19
Elementary Problem E3135	19
Elementary Problem E3164	19
Elementary Problem E3422	19
Elementary Problem E3469	19
4 Solutions of Elementary Problems	21
E1134. Squares and Perfect Numbers	21
E1160. A Property of the Newton Line of a Complete Quadrilateral	21
E1197. A Rich Configuration	22
E1209. A Cross Ratio Associated with Any Triangle	24
E1217. A Property of Euler's Function	24
E1778. The Radii of a Triangle	25
E1779. The Altitudes and Exradii of a Triangle	26
E1877. Convex Pentagon Inscribed in a Semicircle	27
E1878. Incenters Associated with a Regular Polygon	27
E2100. Six Relations	28
E2101. Three Parabolas and a Triangle	28
E2109. Triangle Construction	30
E2110. Similar Triangles	30
E2124. All Triangles Generate Right Triangles	31
E2160. Some New Triangle Inequalities	31
E2213. Quadrilaterals with the Nagel Property	32
E2311. The Compleat Cyclic Quadrilateral	34
E2312. An Application of Ceva's Theorem	36
E2363. On Spherical Triangles	36
E2462. The Extended Erdős–Mordell Inequality	37
E2625. A Property of Conics	46

E3135. Matching Distances to Vertices	47
E3164. Elliptical Tangents	48
E3422. Tangents Intersect on the Axis of Involution	48
E3469. Six Barycenters in Search of a Conic	50
5 Contributed Solutions to Monthly Problems	52
4057. Euler Line	52
E1107. A Pencil of Planes Associated with a Tetrahedron	52
E1142. Semi-vertical Angle of a Right Circular Cone	53
E1148. Two Equiareal Triangles	53
E1166. Chain of Circles in a Segment	54
E1687. An Application of Menelaus' Theorem	55
E2122. An Extension of Napoleon's Theorem	56
E2398. A Result Known to Johnson	57
6 Proposals for Mathematics Magazine	58
208	58
217	58
227	58
234	59
242	59
248	59
258	59
266	59
298	60
304	60
334	60
349	61
372	61
380	61
384	61
398	62
407	62
415	62
419	62
425	63
437	63
440	63
458	63
472	64
487	64
498	64

509	64
517	65
529	65
537	65
544	65
563	66
572	66
587	66
599	66
600	67
609	67
628	67
639	67
649	68
680	68
724	68
738	68
743	69
756	69
763	69
775	69
806	70
839	70
859	70
916	70
963	71
998	71
1197	71
1206	71
1211	72
1298	72
1305	72
1327	72
1356	73
1371	73
1377	73
1405	73
7 Solutions of Proposals	74
208	74
217	75
227: Forces in Equilibrium	75

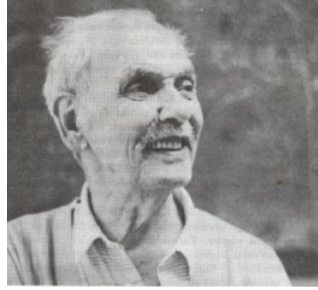
234	76
242	77
248	78
258: An Orthocentric Locus	79
266: Points Inverse in a Circumcircle	80
298: An Invariant Curve	81
304: Circles Connected with a Triangle	82
349: A Bisector	82
334: An Irregular Area	83
372: A Trigonometric Identity	83
380: A System of Equations	84
384: An Infinite Group	85
398: Simultaneous Quadratics	86
407: Resistance in a Cube	87
415: A Trigonometric Sum	87
419: Constant Speed Curve	89
425: Euler's Phi-function	90
437: A Well Known Problem	91
E1288: Odd Binomial Coefficients	92
440: Circle Packing	93
458: De Moivre's Theorem	94
472: Symmetric Conics	95
487: The Square Root of a Matrix	96
498: A Property of Multiplicative Functions	97
509: An American Alphametic	98
517: Parabolic Areas	99
529: Center of Curvature	101
537: Extreme Overlap	102
544: A Conditional Alphametic	103
563: Angles in a Hexagon	104
572: A Memorial Cryptarithm	105
587: A Trigonometric Inequality	105
599: Linearly Dependent Vectors	107
600: Related Triangles	108
609: Crypta-Equivalence	108
628: Pythagorean Alphametic	109
639: A Convex Quadrilateral Inequality	110
649: An Alphametic	111
680: A Circular Locus	112
724: Triangle Probability	113
738: A Ferry and a River Current	114

743: Tetrahedral Inequality	115
756: Centrally Symmetric Curves	116
763: Quasi Zeta Functions	117
775: Inverse Functions	118
806: Symmetry About a Line	120
839: Probability of No Change	120
859: A Non-Unique Cryptarithm	121
916: Trilinear Coordinates	122
963: Convex Quadrilaterals	123
998: A 120° Triangle	124
1197: Collinear Mid-Altitudes	124
1206: Sum of Inradii of a Dissected Triangle	126
1211: Isoptic of an Ellipse	128
1298: Circumscribable Quadrangle	130
1305: Inradii Identity	131
1327: Diagonals of Exscribed Quadrangles	132
1356: Collinearity and Symmetry	133
1371: A Triangle Invariant	134
1377: A Triangle Invariant	135
1405: Isogonally Related Circles	136
8 Quickies for Mathematics Magazine	138
Q117	138
Q138	138
Q141	138
Q166	138
Q188	139
Q227	139
Q234	139
Q242	139
Q266	139
Q281	140
Q284	140
Q341	140
Q343	140
Q710	140
9 Solutions of Quickies	141
Q117	141
Q138	141
Q141	141
Q166	141

Q188	142
Q227	142
Q234	142
Q242	143
Q266	143
Q281	143
Q284	143
Q341	143
Q343	144
Q710	144
10 Contributed Solutions in Mathematics Magazine	145
192	145
203	146
204	147
221	149
226	149
270: Circles in a Crescent	150
273: A Harmonic Range	150
332: A Polynomial Factor	151
353: Tangent Circles	151
374: Equivalent Triangles	152
383: Dissecting a Square	153
387: An Induction Proof	154
400: A Triangle Construction	155
401: A Fibonacci Series	156
412: Projective Correspondence	157
427: A Cevian Relation	158
428: Permuted Digits	158
432: Cevian Lines	159
435: Triangular Extrema	160
445: Correction to a Published Proof	161
476: Perimeter and Area Bisector	161
646: The Complete Quadrilateral	163
653: Exponential Derivative	163
1199: Perpendicular Lines in an Isosceles Triangle	164
1256: Cyclic Quadrilateral	165
11 Problems and Solutions in Mathesis	167
Question d'examen 10	167
Question d'examen 11	168
Question d'examen 12	168

Question 3954	169
Question 3957	169
12 Other Publications	170
13 Complete List of Problems Composed by Hüseyin Demir	171
A Errors and Corrections in the Previous Edition	175

Preface



1916–1995

Source PDF p. 5–6.

Hüseyin Demir was a prolific problem composer who composed more than one hundred problems in his lifetime. He almost exclusively published in *American Mathematical Monthly* and in *Mathematics Magazine*. His first publication appeared in 1943 in *American Mathematical Monthly* as “Advanced Problem 4102”. His last publication was in 1993 in *Mathematics Magazine* as the “Solution by the proposer” to his “Proposal 1405” which had appeared the year before.

I have used JSTOR’s search engine to find publications of Hüseyin Demir in *American Mathematical Monthly* and *Mathematics Magazine*. Here I not only collected problems proposed by Demir but also the solutions supplied to his problems. In addition to these I also collected published solutions contributed by Demir to other composers’s problems.

Hüseyin Demir was an alumnus of Darüşşafaka High School.

Darüşşafaka Cemiyeti was established in 1863 by five young Ottoman gentlemen whose ages were 38, 35, 31, 27 and 24. Sultan Abdülaziz who gave his consent for this establishment was also only 33 years old at the time.

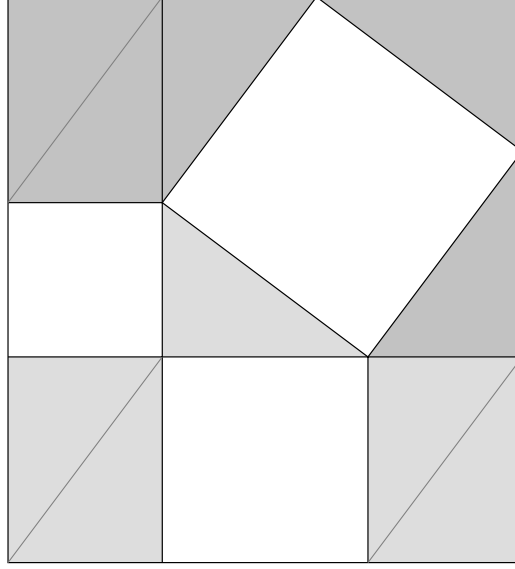
Darüşşafaka is the first non-governmental educational organization in Turkish history. During the last decades of the Empire most muslim Ottoman men who were recruited for the army were lost as the result of long and frequent battles. Thus most families were losing their fathers, and consequently their children were then forced to leave their education and start working at *Kapalıçarşı*.

On the other hand non-muslim Ottoman children did not suffer from this mishap. Hence the motto of Darüşşafaka Cemiyeti, at that time was and still today is “Equal opportunity in education”. Children who have lost one of their parents and are financially not able to pursue a proper education are accepted to Darüşşafaka after a competitive entrance exam. Darüşşafaka is a boarding high school and all the daily and educational expenses of students are provided by Darüşşafaka Cemiyeti which is supported to this day by donations. Today Darüşşafaka is recognized as one of the best educational establishments of Turkey.

Salih Zeki, another of our famous mathematicians, was an 1882 alumnus of Darüşşafaka. Hüseyin Demir is 1935 alumnus and according to his telling he read books of Salih Zeki when he was in school. While he was a middle school student at Darüşşafaka he came up with a novel proof of Pythagoras theorem, which is in the genre of “proof without words” and I am reproducing it here.

I found it my responsibility to my school to compile this collection of Hüseyin Demir's problems. To continue the tradition under which we grew, this collection is meant to be used freely for educational purposes.

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December 2021 Ankara



Preface to the Second Edition

Source PDF p. 7.

Hüseyin Demir published several problems and an article in *Mathesis*, a journal that ceased publication in 1965. These materials were kindly provided by Kürşat Aker.

Demir also authored an article in the mathematics journal *Delta*, which ended publication in 1977. That article was obtained through the valuable assistance of the Bilkent Library Interlibrary Loan System.

Finally, Alp Eden located an additional article by Demir, published in the *METU Journal*, but not recorded elsewhere. This article was likewise supplied by the Bilkent Library.

The *Mathesis* problems are included in this booklet. The articles mentioned above can be found on my Hüseyin Demir page.

I extend my sincere thanks to the Bilkent Library for their ready assistance, and I gratefully acknowledge the contributions of my friends Kürşat and Alp in bringing this collection of Hüseyin Demir's work to completion.

Ali Sinan Sertöz
September 2026 Ankara

Advanced Problems for the Monthly

Advanced Problem 4102

American Mathematical Monthly **50** (1943), 638

Source PDF p. 8.

Proposed by Hüseyin Demir, Columbia University.

Let O and I be respectively the circumcenter and incenter of a given triangle ABC . Let A_0, B_0, C_0 be points taken respectively on BC, CA, AB so that the sums of the algebraic distances of each point to two other sides are equal to a given length l . Prove synthetically that: (1) the points A_0, B_0, C_0 are collinear; (2) the sum of distances to the sides of ABC of points on $A_0B_0C_0$ is the constant l ; (3) the line $A_0B_0C_0$ is perpendicular to the line OI .

Advanced Problem 4125

American Mathematical Monthly **51** (1944), 252

Source PDF p. 9.

Proposed by Hüseyin Demir, Columbia University.

Prove that

$$\begin{vmatrix} \sin \theta_1 & -e^{-i\theta_1} & 0 & 0 & \cdots & 0 & 0 \\ \sin \theta_2 & e^{i\theta_2} & -e^{-i\theta_2} & 0 & \cdots & 0 & 0 \\ \sin \theta_3 & 0 & e^{i\theta_3} & -e^{-i\theta_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \sin \theta_n & 0 & 0 & 0 & \cdots & 0 & e^{i\theta_n} \end{vmatrix} = \sin(\theta_1 + \theta_2 + \cdots + \theta_n).$$

Advanced Problem 4134

American Mathematical Monthly **51** (1944), 475

Source PDF p. 9.

Proposed by Hüseyin Demir, Columbia University.

Let $C_1^1 C_2^1 C_3^1$ be the inscribed triangle of a reference triangle $A_1 A_2 A_3$, and $C_1^2 C_2^2 C_3^2$ be that of $C_1^1 C_2^1 C_3^1$, and so on, obtaining a triangle $C_1^n C_2^n C_3^n$ after n steps. Denoting the angles of the n th triangle by C_i^n , prove that

1. $(C_i^n - \pi/3)/(A_i - \pi/3) = (-1)^n 2^{-n}$.
2. The limit of the direction of $C_2^n C_3^n$ as $n \rightarrow \infty$ is the direction of one of the trisectrices of the angle $(A_2 A_3, C_2^1 C_3^1)$, and from that observe a method of trisecting an angle by ruler and compass in infinitely many steps.

Advanced Problem 4193

American Mathematical Monthly **53** (1946), 160

Source PDF p. 9.

Proposed by Hüseyin Demir, Columbia University.

If on the sides of an arbitrary pentagon $A_1A_2A_3A_4A_5$ the triangles $B_iA_{i+2}A_{i+3}$ (with indices reduced mod 5) are constructed such that $B_iA_{i+2} \parallel A_iA_{i+1}$ and $B_iA_{i+3} \parallel A_iA_{i+4}$, then the lines A_iB_i concur in a point C .

Advanced Problem 4215

American Mathematical Monthly **53** (1946), 470

Source PDF p. 10.

Proposed by Hüseyin Demir, Columbia University.

Prove that the Hermite polynomials defined as follows,

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},$$

have the property

$$n! \sum_{p=0}^n \frac{H_p^2(x)}{p!} = H_{n+1}^2(x) - H_n(x)H_{n+2}(x).$$

Advanced Problem 4679

American Mathematical Monthly **63** (1956), 191

Source PDF p. 10.

Proposed by Hüseyin Demir, Zonguldak, Turkey.

If $A_1A_2A_3A_4A_5$ is a cyclic pentagon and if Ω_{ij} denotes the orthopole of the line A_iA_j with respect to the triangle formed by the remaining three vertices, then prove that the ten points Ω_{ij} all lie on a circle.

Advanced Problem 4695

American Mathematical Monthly **63** (1956), 426

Source PDF p. 10.

Proposed by Hüseyin Demir, Zonguldak, Turkey.

Prove that if in a cyclic quadrangle the Simson line of one vertex with respect to the triangle formed by the other three is perpendicular to the Euler line of that triangle, then the same property holds for the other vertices of the quadrangle.

Advanced Problem 4710

American Mathematical Monthly **63** (1956), 669

Source PDF p. 10.

Proposed by Hüseyin Demir, Zonguldak, Turkey.

Prove that if in a complete quadrangle inscribed in a circle (O) one pair of opposite sides are isotomic lines with respect to a triangle inscribed in (O), then the remaining pairs of opposite sides are also isotomic lines with respect to the same triangle.

Advanced Problem 4735

American Mathematical Monthly **64** (1957), 277

Source PDF p. 11.

Proposed by Hüseyin Demir, Zonguldak, Turkey.

Let $A_1A_2A_3A_4A_5$ be a simple 5-point plane figure, and let d be any line in the plane of the figure. Let the common point of the line d and the side a_i opposite to A_i be denoted by B_i , and the common point of the lines A_iB_{i+1} , B_iA_{i+1} by C_{i+3} . Then the five lines A_iC_i have a point D in common.

Advanced Problem 4818

American Mathematical Monthly **65** (1958), 779

Source PDF p. 11.

Proposed by Hüseyin Demir, Zonguldak, Turkey.

Let d_i be the sides of a complete quadrilateral, and A_{ij} be the vertex on d_i, d_j . Let t_i be the triangle formed by the sides other than d_i , and (O_i) denote the circumcircle of t_i . Denote the Simson line of a point S_i of (O_i) with respect to t_i by D_i .

Then prove that, if D_i and d_i are parallel for all i , (1) the line S_iO_p passes through the vertex A_{qr} ($i, p \neq q, r$), and (2) the points S_i all lie on the Miquel circle (O).

Solutions of Advanced Problems

4102. A Special Triangle Transversal

American Mathematical Monthly **52** (1945), 103–104

Source PDF p. 12–13.

4102 [1943, 638]. Proposed by Hüseyin Demir, Columbia University.

Let O and I be respectively the circumcenter and incenter of a given triangle ABC . Let A_0, B_0, C_0 be points taken respectively on BC, CA, AB so that the sums of the algebraic distances of each point to two other sides are equal to a given length l . Prove synthetically that: (1) the points A_0, B_0, C_0 are collinear; (2) the sum of distances to the sides of ABC of points on $A_0B_0C_0$ is the constant l ; (3) the line $A_0B_0C_0$ is perpendicular to the line OI .

Solution by the proposer. (1) The locus of points whose sum of distances to the sides CA, AB is l , is a straight line passing through A_0 , and perpendicular to AI . Let B_c, C_b be points where this locus cuts CA, AB . Similarly we consider two other loci corresponding to B_0, C_0 . Let $A'B'C'$ be the triangle formed by these three loci. We shall prove that the last triangle is in perspective with ABC , I being the center of perspective. This is obvious, because since A' is the intersection of two loci, its distances to CA, AB are equal, that is, A' belongs to AI . Similarly B', C' belong respectively to BI, CI . Thus applying Desargues' theorem we have collinearity of A_0, B_0, C_0 .

(2) Let M be a point of $A_0B_0C_0$ with x, y, z its distances to BC, CA, AB . We shall prove that $x + y + z = l$. Consider the locus of points with $y + z = \text{Cst}$. This locus MQ (see figure) is parallel to B_cC_b , and $QQ_1 = y + z$. Now, C_b, A_b having equal distances l to CA (see (1)) what we have to prove is that $QQ_2 = x$. Draw MP parallel to BC , then $x = MX = PP_1$. Since $A'A_b$ is the bisector of $A_0A_bC_b$, we have $x = PP_1 = PP_2$. It remains to prove that $QP \parallel C_bA_b$. This is true because the two triangles $QMP, C_bA_0A_b$ have two sides parallel, namely QM, C_bA_0 and MP, A_0A_b , and they are in perspective, with C_0 as center of perspective. Therefore their third sides QP, C_bA_b must be parallel, that is $x = PP_2 = QQ_2$.

(3) We shall prove two things: (a) $A_0B_0C_0$ is the radical axis of circles (ABC) and $(A'B'C')$. (b) The center O' of $(A'B'C')$ lies on OI , thus property (3) will be proved.^[E001]

(a) For, observe the relation $C_0A \cdot C_0B = C_0A' \cdot C_0B'$. This is true because the quadrilateral $ABB'A'$ is cyclic. (Note the equality of angles $A_bB'B = A'AB = \frac{1}{2}A$.) Thus C_0 has equal powers with respect to the two circles. A similar property holds for A_0, B_0 .

(b) To prove that O' belongs to OI we shall remark that the locus of O' is a straight line when $A'B'C'$, whose sides are perpendicular to AI, BI, CI , varies, and since $A'B'C'$ is always in perspective with ABC , with I the center of perspective, O' will describe a straight line passing through I . It also passes through O . For, let A' be taken at the point where AI meets the circle (ABC) . It is easy to see that B', C' will be similar points on the same circle. Thus O' , the center of $(A'B'C')$, coincides with O , the center of (ABC) . Therefore the radical axis $A_0B_0C_0$ of (ABC) and $(A'B'C')$ is perpendicular to the line OI passing through the centers O and O' .

Editorial note (original journal). The first two theorems follow from similar triangles. The case of an isosceles ABC may be discarded. For, if say the sides AB, AC have equal lengths, then in consequence of symmetry about AI the same is true for AB_0, AC_0 , B_0 and C_0 being respectively on AC and AB . It then follows that B_0C_0 is perpendicular to OI ; the converse is

true as well as parts (1) and (2), but A_0 has an exceptional position. The points A_0, B_0, C_0 are uniquely determined by the given constant l . The distances x_b, y_b, z_b for B_0 are such that $y_b = 0$, $x_b + z_b = l$, etc. Let P be a point on the straight line of C_0B_0 and let it divide this segment in the ratio $\lambda : 1$. Then we have

$$(\lambda + 1)x = x_c + \lambda x_b, \quad (\lambda + 1)y = y_c, \quad (\lambda + 1)z = \lambda z_b,$$

where x, y, z are the distances for P . By addition we have

$$(\lambda + 1)(x + y + z) = (x_c + y_c) + \lambda(x_b + z_b) = (\lambda + 1)l;$$

and, if P is a finite point $x + y + z = l$. The straight line C_0B_0 meets BC in a finite point for which $x_a = 0$ and $y_a + z_a = l$; hence this point is A_0 . The two straight lines $A_0B_0C_0$ for different values of l are parallel; for, if they meet in a finite point, this point would have the sum of its distances equal to two different values. If P is a point not on B_0C_0 the line through it parallel to the latter meets the two sides in points different from B_0 and C_0 . Hence the sum of its distances must be different from l ; and this proves that the locus of points for a given l is the straight line $A_0B_0C_0$ for that value of l .

In the special case where $A'B'C'$ is inscribed in (O) the polar of C_0 passes through $(AA', BB') = I$, similarly, the polar of B_0 passes through I . Hence the polar B_0C_0 of I is perpendicular to OI .

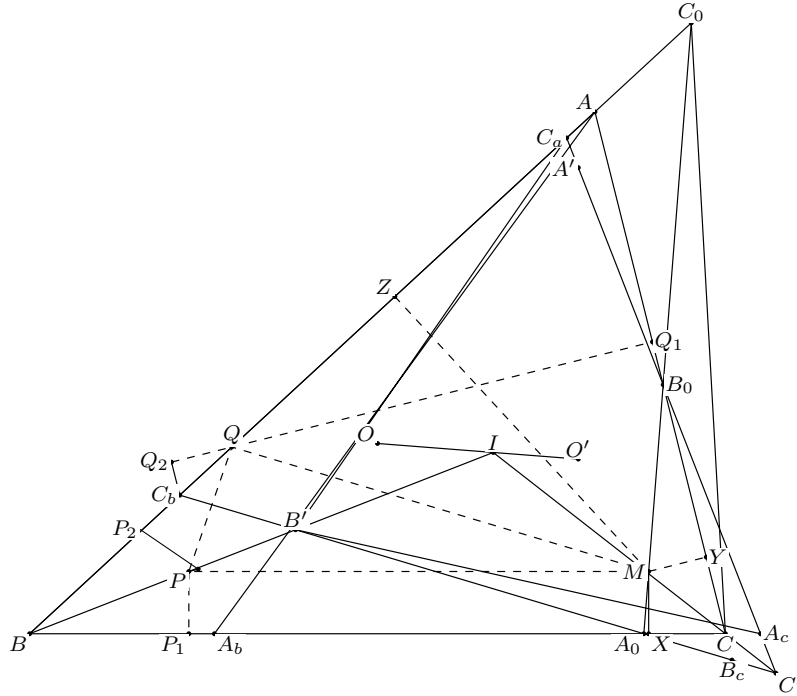


Figure 2.1: The transversal construction for Problem 4102.

4125. Trigonometric Determinants

American Mathematical Monthly **52** (1945), 523

Source PDF p. 14–15.

4125 [1944, 252].^[E002] Proposed by Hüseyn Demir, Columbia University.

Prove that

$$\begin{vmatrix} \sin \theta_1 & -e^{-i\theta_1} & 0 & 0 & \cdots & 0 & 0 \\ \sin \theta_2 & e^{i\theta_2} & -e^{-i\theta_2} & 0 & \cdots & 0 & 0 \\ \sin \theta_3 & 0 & e^{i\theta_3} & -e^{-i\theta_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \sin \theta_n & 0 & 0 & 0 & \cdots & 0 & e^{i\theta_n} \end{vmatrix} = \sin(\theta_1 + \theta_2 + \cdots + \theta_n).$$

Solution by Mary L. Boas, Tufts College. Put each $\sin \theta = (e^{i\theta} - e^{-i\theta})/(2i)$ and remove the factor $1/(2i)$ outside the determinant. Subtract from each element of the first column the sum of all the other elements in its row. The determinant then becomes

$$\frac{1}{2i} \begin{vmatrix} e^{i\theta_1} & -e^{-i\theta_1} & 0 & \cdots & 0 & 0 \\ 0 & e^{i\theta_2} & -e^{-i\theta_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{i\theta_{n-1}} & -e^{-i\theta_{n-1}} \\ -e^{-i\theta_n} & 0 & 0 & \cdots & 0 & e^{i\theta_n} \end{vmatrix}.$$

Expand by elements of the first column. The minor of $e^{i\theta_1}$ is $e^{i(\theta_2+\theta_3+\cdots+\theta_n)}$ since all elements below the main diagonal of this minor are zero. The minor of $-e^{-i\theta_n}$ is $(-1)^{n-1}e^{-i(\theta_1+\theta_2+\cdots+\theta_{n-1})}$ since all elements above its main diagonal are zero. Therefore the determinant equals

$$\frac{1}{2i} [e^{i(\theta_1+\theta_2+\cdots+\theta_n)} - e^{-i(\theta_1+\theta_2+\cdots+\theta_n)}] = \sin(\theta_1 + \theta_2 + \cdots + \theta_n).$$

Also solved by E. F. Allen, Murray Barbour, C. B. Barker, Jr., Shepard Bartnoff, R. P. Boas, Jr., Mrs. R. C. Buck, Howard Eves, Clifford Gardner, P. C. Hammer, R. Hamming, J. F. Hofmann, L. M. Kelly, E. Lukacs, Norman Miller, Henry Nelson, Ivan Niven, H. N. Shapiro, Robert Steinberg, R. H. Wilson, Jr., and the proposer.

Editorial note (original journal). About half of the solutions used induction proofs and about the same number used simple determinant transformations without induction. Hammer considered the transformation of the determinant and its value by replacing θ_j by $\pi/2 - \theta_j$ which gives after reduction a determinant with $a_{j1} = \cos \theta_j$ in the first column and the principal diagonal $\cos \theta_1, e^{-i\theta_2}, e^{-i\theta_3}, \dots$ and the parallel above it $e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, \dots$ with zeros in the remaining places. He found for the value of the determinant $\cos \sum \theta_j$ if n is odd, and $-i \sin \sum \theta_j$ if n is even. A simpler procedure is to make the same change in the first column but to alter the principal diagonal to $\cos \theta_1, -e^{i\theta_2}, -e^{i\theta_3}, \dots, -e^{i\theta_n}$ and leave the rest of the original determinant unaltered. The value of this determinant is the same as that for Hammer's determinant.

4134. Angle Trisection

American Mathematical Monthly **52** (1945), 587

Source PDF p. 15–17.

4134 [1944, 475]. Proposed by Hüseyin Demir, Columbia University.

Let $C_1^1 C_2^1 C_3^1$ be the inscribed triangle of a reference triangle $A_1 A_2 A_3$, and $C_1^2 C_2^2 C_3^2$ be that of $C_1^1 C_2^1 C_3^1$, and so on, obtaining a triangle $C_1^n C_2^n C_3^n$ after n steps. Denoting the angles of the n th triangle by C_i^n , prove that

1. $(C_i^n - \pi/3)/(A_i - \pi/3) = (-1)^n 2^{-n}$.
2. The limit of the direction of $C_2^n C_3^n$ as $n \rightarrow \infty$ is the direction of one of the trisectrices of the angle $(A_2 A_3, C_2^1 C_3^1)$, and from that observe a method of trisecting an angle by ruler and compass in infinitely many steps.

[E003]

Solution by Howard Eves, College of Puget Sound. 1. Designating the incenter of $A_1 A_2 A_3$ by I we have $C_2^1 I C_3^1 = 2C_1^1$. Therefore $A_1 + 2C_1^1 = \pi$. Similarly, $A_i + 2C_i^1 = \pi$, or

$$\frac{C_i^1 - \pi/3}{A_i - \pi/3} = -\frac{1}{2}.$$

By the same process

$$\frac{C_i^j - \pi/3}{C_i^{j-1} - \pi/3} = -\frac{1}{2}, \quad j = 2, 3, \dots, n.$$

By multiplication we then get

$$\frac{C_i^n - \pi/3}{A_i - \pi/3} = (-1)^n 2^{-n}.$$

2. Now

$$\begin{aligned} \lim_{n \rightarrow \infty} [\text{angle}(A_2 A_3, C_2^n C_3^n)] &= \sum_{n=1}^{\infty} \text{angle}(C_2^{2n-2} C_3^{2n-2}, C_2^{2n} C_3^{2n}), \quad C_i^0 \equiv A_i \\ &= \sum_{n=1}^{\infty} (C_3^{2n-1} - C_1^{2n}) \\ &= \sum_{n=1}^{\infty} [(C_3^{2n-1} - \pi/3) - (C_1^{2n} - \pi/3)] \\ &= \sum_{n=1}^{\infty} [-2^{-2n+1}(A_3 - \pi/3) - 2^{-2n}(A_1 - \pi/3)], \end{aligned}$$

by the relations of part 1 above. The infinite sum on the right reduces to

$$\begin{aligned} \sum_{n=1}^{\infty} [\pi - A_1 - 2A_3] 2^{-2n} &= (A_2 - A_3) \sum_{n=1}^{\infty} 2^{-2n} = \frac{1}{3}(A_2 - A_3) \\ &= \frac{2}{3} \left(\frac{A_2 - A_3}{2} \right) \\ &= \frac{2}{3} \left[\frac{1}{2}(A_2 - \pi/3) - \frac{1}{2}(A_3 - \pi/3) \right] \\ &= \frac{2}{3} [-(C_2^1 - \pi/3) + (C_3^1 - \pi/3)] \\ &= \frac{2}{3} (C_3^1 - C_2^1) = \frac{2}{3} \text{angle}(A_2 A_3, C_2^1 C_3^1). \end{aligned}$$

This establishes part 2. The method suggested here for asymptotically obtaining one of the angle trisectors of a given acute angle ($A_2A_3, C_2^1C_3^1$) is apparent. It is needless to say, however, that there are many better euclidean asymptotic constructions for trisecting an angle.

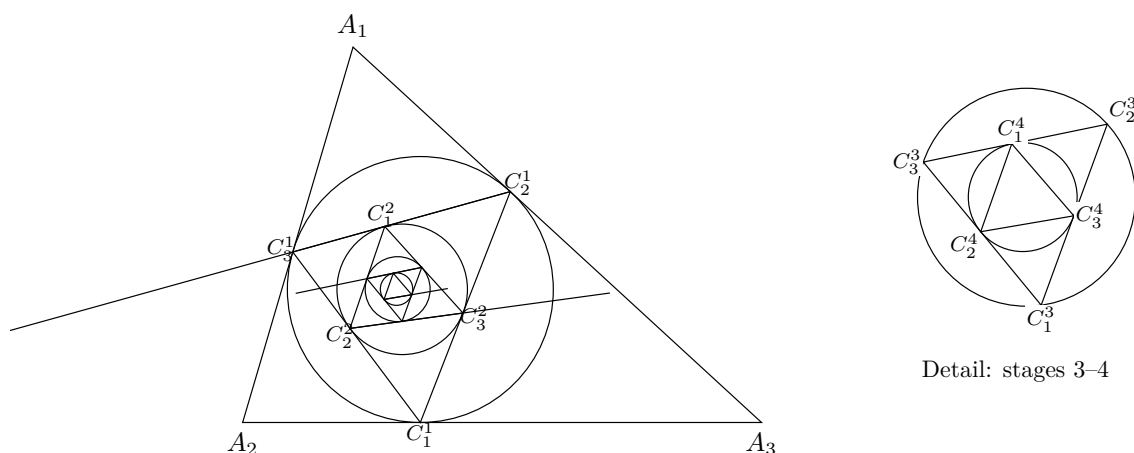


Figure 2.2: Successive contact triangles in the angle-trisection construction.

Also solved by the proposer.

Editorial note (original journal). Below are some references to this *American Mathematical Monthly* regarding approximate methods of angle trisection with limits for the error:

1932, 478, *Angle Division*, article by E. C. Kennedy; 2972 [1925, 95]; 3114 [1925, 483]; 3522 [1933, 303]; 3563 [1934, 113]; the method of Pappus using conics 3490 [1932, 243].

4193. Concurrent Lines in a Pentagon

American Mathematical Monthly **54** (1947), 349

Source PDF p. 17.

4193 [1946, 160]. Proposed by Hüseyin Demir, Columbia University.

If on the sides of an arbitrary pentagon $A_1A_2A_3A_4A_5$ the triangles $B_iA_{i+2}A_{i+3}$ (with indices reduced mod 5) are constructed such that $B_iA_{i+2} \parallel A_iA_{i+1}$ and $B_iA_{i+3} \parallel A_iA_{i+4}$, then the lines A_iB_i concur in a point C .

Solution by J. W. Clawson, Ursinus College, Collegeville, Pennsylvania. Take the triangle $A_1A_2A_4$ as the triangle of reference for a system of homogeneous trilinear coördinates. Let A_1 be $(1, 0, 0)$, A_2 $(0, 1, 0)$, A_3 (d, e, f) , A_4 $(0, 0, 1)$, A_5 (k, l, m) .

Then the equations of the line through A_3 parallel to A_1A_2 and of the line through A_4 parallel to A_1A_5 are

$$afx + bfy - (ad + be)z = 0, \quad alx + (bl + cm)y = 0.$$

Thus B_1 has the coördinates

$$(bl + cm)(ad + be), \quad -al(ad + be), \quad acfm;$$

and the equation of A_1B_1 is

$$cfmy + l(ad + be)z = 0.$$

In the same way the equations of A_2B_2 and A_4B_4 are found to be

$$cfmx + d(ak + bl)z = 0, \quad l(ad + be)x - d(ak + bl)y = 0.$$

These three lines are easily seen to be concurrent in a point C which has the coördinates

$$d(ak + bl), \quad l(ad + be), \quad -cfm.$$

Using the triangles $A_1A_3A_4$ and $A_2A_4A_5$ we can prove in the same way that A_3B_3 and A_5B_5 also pass through the point C .

Also solved by the proposer.

Editorial note (original journal). Clawson gave a second proof using the converse of Ceva's Theorem. The proposer employed the pencils of lines A_3B_3, A_4B_4 formed when the side A_4A_5 rotates about A_5 , other sides remaining fixed; since the correspondence between A_3B_3 and A_4B_4 is homographic, the locus of the intersection C of the rays A_3B_3, A_4B_4 is a conic; this conic decomposes into A_1B_1 and A_3A_4 , thus giving the proof.

4215. Hermite Polynomials

American Mathematical Monthly **55** (1948), 34

Source PDF p. 18–19.

4215 [1946, 470]. Proposed by Hüseyin Demir, Columbia University.

Prove that the Hermite polynomials defined as follows,

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},$$

have the property

$$n! \sum_{p=0}^n \frac{H_p^2(x)}{p!} = H_{n+1}^2(x) - H_n(x)H_{n+2}(x).$$

Solution by Hsien-yü Hsü, Yenching University, Peiping, China. In Pólya–Szegő, *Aufgaben und Lehrsätze II*, pp. 294–295, Hermite polynomials are defined as follows

$$h_n(x) = \frac{1}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},$$

and satisfy the difference equation

$$nh_n(x) = -xh_{n-1}(x) - h_{n-2}(x), \quad n = 2, 3, \dots$$

We notice that

$$H_n \equiv H_n(x) = (-1)^n n! h_n(x),$$

whence the difference equation is

$$H_n = xH_{n-1} - (n-1)H_{n-2}, \quad n = 2, 3, \dots \quad (1)$$

Upon eliminating x from this equation and the analogous equations for H_{n+1} and H_{n+2} we obtain immediately

$$\begin{aligned} H_{n+1}^2 - H_n H_{n+2} &= n(H_n^2 - H_{n+1} H_{n-1}) + H_n^2 \\ &= n(n-1)(H_{n-1}^2 - H_n H_{n-2}) + nH_{n-1}^2 + H_n^2 \\ &= \dots \\ &= n! \left\{ (H_1^2 - H_2 H_0) + \frac{1}{1!} H_1^2 + \frac{1}{2!} H_2^2 + \dots + \frac{1}{n!} H_n^2 \right\} \\ &= n! \sum_{p=0}^n H_p^2 / p!, \end{aligned}$$

since $H_1^2 - H_2H_0 = 1$.

Also solved by F. E. Cothran, A. B. Farnell, L. M. Kelly, Norman Miller, S. T. Parker, W. A. Pierce, W. H. Spragens, M. S. Webster, M. Wyman, Professor Otto Szász's class, and the proposer.

Editorial note (original journal). Several solvers mentioned that equation (1) of the above solution is found in Dunham Jackson, *Fourier Series and Orthogonal Polynomials*, p. 176, ff. The solution by members of Professor Szász's class in Orthogonal Developments proceeds from the (so-called) Christoffel's formula

$$\sum_{p=0}^n \frac{H_p(x)H_p(y)}{p!} = \frac{H_{n+1}(x)H_n(y) - H_n(x)H_{n+1}(y)}{n!(x-y)}.$$

See Szegő, *Orthogonal Polynomials*, p. 102. Webster's solution employs the relation

$$H_m(x)H_n(x) = n! \sum_{r=0}^n \binom{m}{n-r} \frac{H_{m-n+2r}}{r!}, \quad (m \geq n),$$

established by E. Feldheim, *Journal of the London Mathematical Society*, 1938, pp. 22–29.

4679. Ten Conccyclic Orthopoles

American Mathematical Monthly **63** (1956), 191

Source PDF p. 20.

4679 [1956, 191]. Proposed by Hüseyin Demir, Zonguldak, Turkey.

If $A_1A_2A_3A_4A_5$ is a cyclic pentagon and if Ω_{ij} denotes the orthopole of the line A_iA_j with respect to the triangle formed by the remaining three vertices, then prove that the ten points Ω_{ij} all lie on a circle.

Solution by Chih-yi Wang, University of Minnesota. We make use of the following known

Theorem. If a line meets the circumcircle of a triangle, the Simson lines of the points of intersection with the circle meet in the orthopole of the line for the triangle.

(See Court, *College Geometry*, 2nd ed., p. 289.)

Let the circumcircle $A_1A_2A_3A_4A_5$ be the unit circle, and the coordinates of A_i be $(\cos \theta_i, \sin \theta_i)$, $i = 1, 2, \dots, 5$. For definiteness let us find the coordinates of Ω_{12} . The equations of the Simson lines of A_1 and of A_2 are given by

$$\begin{aligned} y - \frac{1}{2}(\sin \theta_j + \sin \theta_3 + \sin \theta_4) + \frac{1}{2} \sin(\theta_3 + \theta_4 - \theta_j) \\ = \tan \frac{1}{2}(\theta_3 + \theta_4 + \theta_5 - \theta_j) \\ \times \left[x - \frac{1}{2}(\cos \theta_3 + \cos \theta_4 + \cos \theta_j) + \frac{1}{2} \cos(\theta_3 + \theta_4 - \theta_j) \right], \end{aligned}$$

for $j = 1, 2$.^[E004] By solving the simultaneous equations we obtain

$$\Omega_{12} = \left(\alpha + \frac{1}{2} \cos(\theta_3 + \theta_4 + \theta_5 - \theta_1 - \theta_2), \beta + \frac{1}{2} \sin(\theta_3 + \theta_4 + \theta_5 - \theta_1 - \theta_2) \right),$$

where $\alpha = \frac{1}{2} \sum \cos \theta_k$, $\beta = \frac{1}{2} \sum \sin \theta_k$, $k = 1, 2, \dots, 5$. Since α and β are symmetric functions, by interchanging the subscripts we see readily that the ten points Ω_{ij} all lie on the circle of radius $\frac{1}{2}$ with center (α, β) .

Also solved by J. W. Clawson, R. Goormaghtigh, O. J. Ramler, Sister M. Stephanie, and the proposer.

Editorial note (original journal). Goormaghtigh gave this theorem in *Mathesis*, 1939, p. 312. Ramler gives an extension to the cyclic heptagon. If Ω_{ijk} denotes the Kantor point of a triangle $A_iA_jA_k$ with respect to the quadrangle formed by the remaining four vertices, then the thirty-five points Ω_{ijk} all lie on a circle one-half as large as the circumcircle of the heptagon.

4695. Simson Line and Euler Line

American Mathematical Monthly **64** (1957), 437

Source PDF p. 20–21.

4695 [1956, 426]. Proposed by Hüseyin Demir, Zonguldak, Turkey.

Prove that if in a cyclic quadrangle the Simson line of one vertex with respect to the triangle formed by the other three is perpendicular to the Euler line of that triangle, then the same property holds for the other vertices of the quadrangle.

Solution by Sister M. Stephanie, Georgian Court College, Lakewood, N. J. Using complex coordinates and taking the circle to be the unit circle, let the vertices of the quadrangle be t_1, t_2, t_3, t_4 , $|t_i| = 1$. The Simson line of t_1 , with respect to the triangle formed by t_2, t_3, t_4 , has the equation

$$2t_1^2z - 2t_1s_3\bar{z} + s_3 + t_1s_2 - t_1^3 - t_1^2s_1 = 0,$$

where s_1, s_2, s_3 are the elementary symmetric functions of t_2, t_3, t_4 . The Euler line of triangle t_2, t_3, t_4 has the equation $s_2z - s_1s_3\bar{z} = 0$. If the two lines are perpendicular, one clinant is the negative of the other, whence $s_3/t_1 = -s_1s_3/s_2$, or upon simplifying,

$$t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = 0.$$

The symmetry of this result guarantees that the property holds equally for any vertex.

Also solved by J. W. Clawson, G. W. Courter, R. Deaux, Beckham Martin, O. J. Ramler, Robert Sibson, Chih-yi Wang, and the proposer.

4710. Isotomic Lines

American Mathematical Monthly **64** (1957), 601

Source PDF p. 21.

4710 [1956, 669]. Proposed by Hüseyin Demir, Zonguldak, Turkey.

Prove that if in a complete quadrangle inscribed in a circle (O) one pair of opposite sides are isotomic lines with respect to a triangle inscribed in (O), then the remaining pairs of opposite sides are also isotomic lines with respect to the same triangle.

I. Solution by O. J. Ramler, Catholic University of America. Using a system of conjugate coordinates we take the circle (O) as the unit circle, and on it points whose vector coordinates are T_1, T_2, T_3 as the inscribed triangle and t_1, t_2, t_3, t_4 as the inscribed complete quadrangle. Then the line t_2t_3 intersects the side T_2T_3 of the triangle in a point whose vector coordinate is

$$z = \frac{T_2T_3(t_2 + t_3) - (T_2 + T_3)t_2t_3}{T_2T_3 - t_2t_3}.$$

Similarly t_1t_4 meets side T_2T_3 where

$$z' = \frac{T_2T_3(t_4 + t_1) - (T_2 + T_3)t_1t_4}{T_2T_3 - t_1t_4}.$$

The hypothesis implies $T_3 - z = z' - T_2$ which becomes, upon making proper substitutions and simplifying,

$$T_2^2T_3^2(T_2 + T_3 - s_1) + T_2T_3s_3 - (T_2 + T_3)s_4 = 0,$$

where s_1, s_3, s_4 are elementary symmetric functions of t_1, t_2, t_3, t_4 . The symmetry of this result establishes the proposed theorem.

II. Solution by Roland Deaux, Faculté Polytechnique, Mons, Belgium. Circle (O) may be replaced by any conic Γ . Let ABC and $PQRS$ be the triangle and the quadrangle inscribed in Γ . By virtue of Desargues' theorem, the four conics Γ , (PQ, RS) , (PR, QS) , (PS, QR) determine on each side of ABC four pairs of an involution. This involution, defined by Γ and the isotomic lines PQ, RS , has for double points the midpoint and the point at infinity of the side. Hence the property.

Also solved by J. W. Clawson, N. A. Court, R. Goormaghtigh, Josef Langr, and the proposer.

4735. Concurrent Lines

American Mathematical Monthly **65** (1958), 128

Source PDF p. 22.

4735 [1957, 277]. *Proposed by Hüseyin Demir, Zonguldak, Turkey.*

Let $A_1A_2A_3A_4A_5$ be a simple 5-point plane figure, and let d be any line in the plane of the figure. Let the common point of the line d and the side a_i opposite to A_i be denoted by B_i , and the common point of the lines A_iB_{i+1} , B_iA_{i+1} by C_{i+3} . Then the five lines A_iC_i have a point D in common.

Solution by E. J. F. Primrose, The University, Leicester, England. There is a unique polarity P for which each A_i is the pole of the opposite side a_i (Coxeter, *The Real Projective Plane*, 5.65). We consider the 4-point $A_1C_1B_3B_4$. The pole of A_1B_3 for P is A_3 , so A_1B_3 and C_1B_4 are conjugate lines, and similarly A_1B_4 and C_1B_3 are conjugate lines. By the dual of Hesse's theorem (Coxeter, 5.55), A_1C_1 and B_3B_4 are conjugate lines, so A_1C_1 passes through D , the pole of d for P . By a similar argument, all the lines A_iC_i pass through D .

Also solved by W. B. Carver, J. W. Clawson, R. Deaux, and the proposer.

4818. A Property of the Miquel Circle of a Complete Quadrilateral

American Mathematical Monthly **66** (1959), 732

Source PDF p. 22–23.

4818 [1958, 779]. *Proposed by Hüseyin Demir, Zonguldak, Turkey.*

Let d_i be the sides of a complete quadrilateral, and A_{ij} be the vertex on d_i, d_j . Let t_i be the triangle formed by the sides other than d_i , and (O_i) denote the circumcircle of t_i . Denote the Simson line of a point S_i of (O_i) with respect to t_i by D_i .

Then prove that, if D_i and d_i are parallel for all i , (1) the line S_iO_p passes through the vertex A_{qr} ($i, p \neq q, r$), and (2) the points S_i all lie on the Miquel circle (O) .

Solution by the proposer. (1) Let the projections of S_i and O_p on d_r be denoted by U_{ir}, V_{pr} . Then, using directed angles, we have

$$\begin{aligned}\angle V_{pr}O_pA_{qr} &= \angle A_{ir}A_{qi}A_{qr} && \text{(from the circle } (O_p)), \\ &= \angle U_{ir}U_{iq}A_{qr} && \text{(since } D_i \text{ is parallel to } d_i), \\ &= \angle U_{ir}S_iA_{qr} && \text{(from the circle } S_iU_{ir}A_{qr}U_{iq}).\end{aligned}$$

Now, since O_pV_{pr} is parallel to S_iU_{ir} , we get the required collinearity of S_i, O_p, A_{qr} .

(2) Since the line of centers O_pO_q is perpendicular to the radical axis FA_{pq} , where F is the Miquel point, we have successively

$$\begin{aligned}\angle O_pO_rO_q &= \angle A_{qi}FA_{pi}, \\ &= \angle A_{qr}A_{pq}A_{pi} && \text{(from the circle } (O_r)), \\ &= \angle A_{qr}A_{pq}A_{pr}, \\ &= \angle A_{qr}S_iA_{pr} && \text{(from the circle } (O_i)), \\ &= \angle O_pS_iO_q && \text{(from property (1)),}\end{aligned}$$

and S_i lies on the Miquel circle (O) .

Also solved by A. E. Landry.

Elementary Problems for the Monthly

Elementary Problem E1134

American Mathematical Monthly **61** (1954), 568

Source PDF p. 24.

Proposed by Hüseyin Demir, Zonguldak, Turkey.

Prove that a square integer is not a perfect number.

Elementary Problem E1160

American Mathematical Monthly **62** (1955), 182

Source PDF p. 25.

Proposed by Hüseyin Demir, Zonguldak, Turkey.

Prove that in a complete quadrilateral the isotomic line of any side with respect to the triangle formed by the other three is parallel to the Newton line of the quadrilateral.

Elementary Problem E1197

American Mathematical Monthly **63** (1956), 39

Source PDF p. 25.

Proposed by Hüseyin Demir, Zonguldak, Turkey.

Let ABC be a right triangle and CH the altitude on the hypotenuse AB . Show that the sum of the radii of the inscribed circles of triangles ABC, HCA, HCB is equal to CH .

Elementary Problem E1209

American Mathematical Monthly **63** (1956), 186

Source PDF p. 25.

Proposed by Hüseyin Demir, Zonguldak, Turkey.

Let ABC be any triangle and (I) its incircle. Let (I) touch BC, CA, AB at D, E, F , and intersect the cevians BE, CF at E', F' respectively. Show that the anharmonic ratio $D(E, F, E', F')$ is the same for all triangles ABC .

Elementary Problem E1217*American Mathematical Monthly* **63** (1956), 342

Source PDF p. 25.

*Proposed by Hüseyin Demir, Zonguldak, Turkey.**Evaluate $\prod_{d|n} d^{\phi(n/d)+\phi(d)}$.***Elementary Problem E1778***American Mathematical Monthly* **72** (1965), 420

Source PDF p. 26.

*Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.**If R, r, r_1, r_2, r_3 are the circumradius, inradius and exradii of a triangle, prove that*

$$\frac{1}{r^3} - \frac{1}{r_1^3} - \frac{1}{r_2^3} - \frac{1}{r_3^3} = \frac{12R}{r \cdot r_1 \cdot r_2 \cdot r_3}.$$

Elementary Problem E1779*American Mathematical Monthly* **72** (1965), 420

Source PDF p. 26.

*Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.**If h_i and r_i are the altitudes and exradii of a triangle prove that*

$$\frac{r_1}{h_1} + \frac{r_2}{h_2} + \frac{r_3}{h_3} \geq 3.$$

Elementary Problem E1877*American Mathematical Monthly* **73** (1966), 410

Source PDF p. 26.

*Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.**Let $ABCDE$ be a convex pentagon inscribed in a unit circle with AE as diameter, and let $AB = a$, $BC = b$, $CD = c$, $DE = d$. Then prove that*

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4.$$

Elementary Problem E1878

American Mathematical Monthly **73** (1966), 410

Source PDF p. 26.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Let $A_1A_2\cdots A_n$ be a regular polygon inscribed in circle (O) of radius R . Denote the in-center of the triangle $A_{i-1}A_iA_{i+1}$ (indices mod n) by I_i , and that of the triangle formed by $A_iA_{i+2}, A_{i+2}A_{i+1}, A_{i+1}A_{i+3}$ by J_i . Then show that the points $I_1, \dots, I_n$ and $J_1, \dots, J_n$ all lie on the same circle of radius $R' = R \cos(3\pi/2n) / \cos(\pi/2n)$.

Elementary Problem E2100

American Mathematical Monthly **75** (1968), 670

Source PDF p. 27.

Proposed by H. Demir, Middle East Technological University, Ankara, Turkey.

Show that any five of the relations

$$\begin{aligned} (1) \quad \frac{x-a_1}{a_1-a_2} &= \frac{a-b}{b-c}, & (2) \quad \frac{x-b_1}{b_1-b_2} &= \frac{b-c}{c-a}, & (3) \quad \frac{x-c_1}{c_1-c_2} &= \frac{c-a}{a-b}, \\ (4) \quad x+a &= b_2+c_1, & (5) \quad x+b &= c_2+a_1, & (6) \quad x+c &= a_2+b_1 \end{aligned}$$

imply the sixth. Interpret this set of consistent relations geometrically letting a, b, c be the affixes, in the complex plane, of a triangle of reference ABC and other numbers be those of other points.

Elementary Problem E2101

American Mathematical Monthly **75** (1968), 670

Source PDF p. 27.

Proposed by H. Demir, Middle East Technological University, Ankara, Turkey.

ABC is a triangle. Let P_a denote the parabola tangent to the sides AB, AC at B, C respectively. The parabolas P_b and P_c are similarly defined. Let these parabolas intersect to the points A', B', C' inside ABC . Denote the areas of the (curvilinear) triangular regions $ABC, A'B'C', AB'C', BC'A', CA'B', A'BC, B'CA, C'AB$, by $\Delta, \Delta_0, \Delta'_a, \Delta'_b, \Delta'_c, \Delta''_a, \Delta''_b, \Delta''_c$. Then prove

$$\begin{aligned} (1) \quad \Delta'_a &= \Delta'_b = \Delta'_c = (\Delta_1), & \Delta''_a &= \Delta''_b = \Delta''_c = (\Delta_2), \\ (2) \quad \Delta_0 : \Delta_1 : \Delta_2 : \Delta &= 15 : 17 : 5 : 81. \end{aligned}$$

Elementary Problem E2109

American Mathematical Monthly **75** (1968), 780

Source PDF p. 27.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Let ABC be a triangle and A' be any fixed point on the side BC . Construct the inscribed triangle $A'B'C'$ which is directly similar to a given triangle XYZ .

Elementary Problem E2110

American Mathematical Monthly **75** (1968), 780

Source PDF p. 27.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

If, in a plane, the triangles AUV, VBU, UVC are directly similar to a given triangle, then so is ABC , provided A, B, C are not all coincident.^[E006]

Elementary Problem E2124

American Mathematical Monthly **75** (1968), 899

Source PDF p. 28.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Construct on the sides BC, CA, AB of a triangle ABC , exteriorly, the squares $BCDE, ACFG, BAHK$ and build the parallelograms $FCDQ, EBKP$. Show that APQ is an isosceles right triangle.

Elementary Problem E2160

American Mathematical Monthly **76** (1969), 300

Source PDF p. 28.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Let p_i, x_i be the distances of an interior or a boundary point P of a triangle $A_1A_2A_3$ from the vertex A_i and the side opposite to A_i , $i = 1, 2, 3$, with r the inradius. Prove the inequalities

$$(a) \quad \sum_{i=1}^3 p_i \left(\frac{1}{2} \sin A_i \right) \leq \sum_{i=1}^3 x_i \leq \sum_{i=1}^3 p_i \sin \left(\frac{1}{2} A_i \right),$$

$$(b) \quad p_2 p_3 + p_3 p_1 + p_1 p_2 \geq 8x_1 x_2 x_3 / r.$$

Elementary Problem E2213

American Mathematical Monthly **77** (1970), 79

Source PDF p. 28.

Proposed by H. Demir, Middle East Technical University, Ankara, Turkey.

Let us say that a (planar) polygon has the Nagel property if the lines through the vertices of the polygon and bisecting the perimeter of the polygon are concurrent. It is known that all triangles have the Nagel property and that not all quadrilaterals have the property. Determine the simple nondegenerate quadrilaterals that have the Nagel property.

Elementary Problem E2311*American Mathematical Monthly* **78** (1971), 793

Source PDF p. 28.

*Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.**Prove that, if a quadrilateral $A_1A_2A_3A_4$ can be inscribed in a circle, then the (six) lines drawn from the midpoints of A_pA_q perpendicular to A_rA_s (p, q, r, s are distinct) are concurrent.***Elementary Problem E2312***American Mathematical Monthly* **78** (1971), 793

Source PDF p. 29.

*Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.**Let D be a point in the plane of a positively oriented triangle ABC and let AD, BD, CD intersect the respective opposite sides in A_1, B_1, C_1 . If the oriented segments BA_1, CB_1, AC_1 are equal ($= \delta$), then D is uniquely determined and lies in the interior of ABC . (Notice the analogy between D and the Brocard point Ω .)***Elementary Problem E2363***American Mathematical Monthly* **79** (1972), 662

Source PDF p. 29.

*Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.**Characterize pairs of spherical triangles ABC and $A'B'C'$ for which $A' = a, B' = b, C' = c, A = a', B = b', C = c'$.***Elementary Problem E2462***American Mathematical Monthly* **81** (1974), 281

Source PDF p. 29.

*Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.**Let P be a point interior to the triangle $A_1A_2A_3$. Denote by R_i the distance from P to the vertex A_i , and denote by r_i the distance from P to the side a_i opposite to A_i . The Erdős–Mordell inequality asserts that*

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3).$$

Prove that the above inequality holds for every point P in the plane of $A_1A_2A_3$ when we make the interpretation $R_i \geq 0$ always and r_i is positive or negative depending on whether P and A_i are on the same side of a_i or on opposite sides.

Elementary Problem E2625

American Mathematical Monthly **83** (1976), 812

Source PDF p. 29.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Let A_i ($i = 0, 1, 2, 3 \pmod{4}$) be four points on a circle Γ . Let t_i be the tangent to Γ at A_i and let p_i and q_i be the lines parallel to t_i passing through the points A_{i-1} and A_{i+1} , respectively. If $B_i = t_i \cap t_{i+1}$, $C_i = p_i \cap q_{i+1}$, show that the four lines $B_i C_i$ have a common point.

Elementary Problem E3135

American Mathematical Monthly **93** (1986), 215

Source PDF p. 30.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

For a scalene triangle ABC inscribed in a circle, prove that there is a point D on the arc of the circle opposite to some vertex whose distance from this vertex is the sum of its distances from the other two vertices. Show how D may be constructed with straightedge and compass.

Elementary Problem E3164

American Mathematical Monthly **93** (1986), 566

Source PDF p. 30.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Let s, t be the lengths of the tangent line segments to an ellipse from an exterior point. Find the extreme values of the ratio s/t .

Elementary Problem E3422

American Mathematical Monthly **98** (1991), 158

Source PDF p. 30.

Proposed by H. Demir and C. Tezer, Middle East Technical University, Ankara, Turkey.

Suppose F and F' are points situated symmetrically with respect to the center of a given circle, and suppose S is a point on the circle not on the line FF' . Let P and P' be the second points of intersection of SF and SF' respectively with the circle. If the tangents to the circle at P and P' intersect at T , prove that the perpendicular bisector of FF' passes through the midpoint of the line segment ST .

Elementary Problem E3469

American Mathematical Monthly **98** (1991), 955

Source PDF p. 30.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Suppose P is a point in the interior of triangle ABC and suppose AP, BP, CP meet the lines BC, CA, AB respectively at the points D, E, F . Prove that the centroids of the six triangles

$PBD, PDC, PCE, PEA, PAF, PFB$ lie on a conic if and only if P lies on at least one of the three medians of the triangle.

Solutions of Elementary Problems

E1134. Squares and Perfect Numbers

American Mathematical Monthly **62** (1955), 257

Source PDF p. 32.

E1134 [1954, 568]. Proposed by Hüseyin Demir, Zonguldak, Turkey.

Prove that a square integer is not a perfect number.

I. Solution by C. F. Pinzka, Princeton, N. J. If $N = \prod p^{2a}$, the sum of the divisors of N is

$$\prod \frac{p^{2a+1} - 1}{p - 1}.$$

Since the latter is always odd, it cannot equal $2N$ as required for a perfect number.

II. Solution by C. D. Olds, San Jose State College. Euler proved that an odd perfect number must have the form $r^{4k+1}P^2$, where r is a prime of the form $4n + 1$. An even perfect number must be of Euclid's type, that is, of the form $2^{p-1}(2^p - 1)$ where $2^p - 1$ is a prime. Thus a square cannot be perfect.

E1160. A Property of the Newton Line of a Complete Quadrilateral

American Mathematical Monthly **62** (1955), 658

Source PDF p. 33.

E1160 [1955, 182]. Proposed by Hüseyin Demir, Zonguldak, Turkey.

Prove that in a complete quadrilateral the isotomic line of any side with respect to the triangle formed by the other three is parallel to the Newton line of the quadrilateral.

I. Solution by the proposer. Let d be one of the four sides of the quadrilateral and let ABC be the corresponding triangle. Denote the intersections of d with the sides BC, CA, AB of triangle ABC by α, β, γ . The isotomic line IKJ of $\alpha\beta\gamma$ with respect to triangle ABC is obtained by taking the symmetrics I, J, K of the points α, β, γ with respect to the midpoints A', B', C' of the sides BC, CA, AB of triangle ABC . Let the midpoints of $A\alpha, B\beta, C\gamma$ be denoted by I', J', K' . These points of the Newton line of the quadrilateral are evidently on the sides of the medial triangle $A'B'C'$ of triangle ABC . It is easy to see that the complete quadrilateral formed by triangle ABC and line IKJ is similar to that formed by triangle $A'B'C'$ and line $I'J'K'$, for, firstly, triangles ABC and $A'B'C'$ are similar, and are in the ratio $2 : 1$, and secondly,

$$BI = C\alpha = 2(B'I'), \quad AJ = C\beta = 2(A'J'), \quad AK = B\gamma = 2(A'K').$$

This proves that the lines IKJ and $I'J'K'$ are parallel.

II. Solution by Sister M. Stephanie, Georgian Court College, Lakewood, N. J. Since there is one and only one parabola tangent to four lines, let us consider the complete quadrilateral as

tangent to the parabola (referred to rectangular coordinates) $y^2 = 4ax$. Then $y = m_i x + a/m_i$, $i = 1, 2, 3, 4$, may be taken as the equations of the four sides 1, 2, 3, 4 of the quadrilateral. Point

$$\left(\frac{a}{m_1 m_2}, \frac{a}{m_1} + \frac{a}{m_2} \right)$$

is the intersection of sides 1 and 2; other intersections are similarly given. The midpoint of the side 2 of triangle 123 has coordinates

$$\frac{a}{2} \left[\frac{m_1 + m_3}{m_1 m_2 m_3}, \frac{2}{m_2} + \frac{1}{m_1} + \frac{1}{m_3} \right].$$

If (x, y) is the point on side 2 isotomic to the intersection of side 4 with side 2, then

$$y + \frac{a}{m_2} + \frac{a}{m_4} = \frac{2a}{m_2} + \frac{a}{m_1} + \frac{a}{m_3},$$

whence

$$y = a \left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} - \frac{1}{m_4} \right),$$

a result which is symmetric in m_1, m_2, m_3 . This proves that the isotomic line of side 4 with respect to triangle 123 is parallel to the axis of the parabola. But the Newton line is also parallel to the axis of the parabola, for it is the locus of centers of all conics inscribed in the quadrilateral, and this locus contains the center of the parabola.

E1197. A Rich Configuration

American Mathematical Monthly **63** (1956), 493

Source PDF p. 34–36.

E1197 [1956, 39]. Proposed by Hüseyin Demir, Zonguldak, Turkey.

Let ABC be a right triangle and CH the altitude on the hypotenuse AB . Show that the sum of the radii of the inscribed circles of triangles ABC, HCA, HCB is equal to CH .

Solution by Leon Bankoff, Los Angeles, Calif. I. The diameter of the circle inscribed in a right triangle is equal to the sum of the legs minus the hypotenuse. Applying this relation in triangles ABC, HCA, HCB , we get

$$\frac{(AC + CB - AB) + (AH + CH - AC) + (CH + HB - CB)}{2} = CH.$$

II. In similar right triangles, the ratios of inradius to hypotenuse are equal. We may therefore write

$$r/c = r_1/b = r_2/a = (r + r_1 + r_2)/(a + b + c),$$

where r, r_1, r_2 are the inradii of triangles ABC, HCA, HCB . Since

$$r/c = CH/(a + b + c)$$

it follows that $r + r_1 + r_2 = CH$.

Additional selected properties of the configuration. R, S, T are the incenters of triangles AHC, CHB, ABC , respectively, and H, R', S', T' the orthogonal projections of C, R, S, T on AB . Let r, r_1, r_2 denote the inradii of triangles ABC, AHC, CHB . P and Q are the feet of the cevians CR and CS . RS cuts AC in U and CB in V . K is the intersection of PS and RQ .

- (1) T' is the circumcenter of triangle RST ; Q, S, T, R, P are concyclic; $PT' = TT' = T'Q$.
- (2) T is the orthocenter of triangle CRS and the circumcenter of triangle CPQ .
- (3) $r_1^2 + r_2^2 = r^2$.
- (4) $RS = CT = PT = QT = r\sqrt{2}$.
- (5) Triangles HSR, ABC, AHC, HCB are similar.
- (6) A, R, S, B are concyclic. Triangles RST and ABT are inversely similar.
- (7) R, T', H, S are concyclic. (RS is a diameter of the circle.)
- (8) $T'S$ is parallel to AC ; RT' is parallel to CB .
- (9) Triangles $RR'T'$ and $T'S'S$ are congruent (and similar to triangle ABC).
- (10) Triangles ATB, BCS, ARC are similar.
- (11) Angle $ATB = \text{angle } BSC = \text{angle } ARC = 135^\circ$.
- (12) $AC = AQ$; $PB = CB$.
- (13) $CU = CV$; CT is the perpendicular bisector of UV .
- (14) PS, RQ, CH are concurrent at K , the orthocenter of triangle CPQ .
- (15) PS is parallel to AT ; RQ is parallel to TB ; triangles PBS and CSB are congruent; triangles AQR and ARC are congruent.
- (16) The midpoint of RS is the nine point center of triangle CPQ .
- (17) The circumcircle of triangle HSR is the nine point circle of triangle CPQ .
- (18) The circumcircles of triangles ARC and CSB are tangent at C , and CT is their common internal tangent.
- (19) $RT = KS = SQ$; $RP = RK = TS$; triangles PKR and KQS are isosceles right triangles. (Also triangle $RT'S$.)
- (20) A, P, T, C are concyclic; B, Q, T, C are concyclic.
- (21) The perimeter of triangle $T'S'S = \text{perimeter of triangle } RR'T' = CH$ (since $SS' = r_2$, $T'S' = r_1$, $T'S = r$).
- (22) Area of triangle $RST = (a + b - c)^3 / (8c) = r^3 / c$.
- (23) Area of pentagon $PQSTR = 2r^3 / c + r^2$.
- (24) Area of triangle $CPQ = abr / c$.

Also solved by W. A. Al-Salam, L. C. Barrett, Robert Bart, G. E. Bills, R. L. Caskey, G. B. Charlesworth, N. A. Childress, T. Y. Chow, Mary Constable, R. J. Cormier, K. W. Crain, A. E. Danese, D. E. D'Atri, G. W. Day, Hazel Evans, Herta Freitag, Michael Goldberg, A. J. Goldman, Peter Gould, Cornelius Groenewoud, D. J. Hansen, Vern Hoggatt, R. T. Hood, Roger Hou, J. P. Hoyt, Raymond Huck, Louise Hutchinson, A. R. Hyde, P. W. M. John, Edgar Karst, M. S. Klamkin, W. G. Koellner, Sam Kravitz, M. A. Laframboise, L. E. Laird, B. R. Leeds, L. I. Lokomowitz, Robert Lynch, D. C. B. Marsh, Beckham Martin, C. N. Mills, C. S. Ogilvy, Margaret Olmsted, M. J. Pascual, Walter Penney, L. L. Pennisi and N. C. Scholomiti (jointly), C. F. Pinzka, P. W. A. Raine, M. A. Rachid, L. A. Ringenberg, Azriel Rosenfeld, Donald Rubin, C. M. Sandwick, Sr., E. D. Schell, G. J. Simmons, Bernard Smilowitz, Sister M. Stephanie, A.

V. Sylwester, W. R. Talbot, Chih-yi Wang, R. M. Warten, Dale Woods, Roscoe Woods, André Yandl, David Zeitlin, and the proposer. Late solutions by Paul Herzberg and Alan Wayne.

It was pointed out that this problem appears in N. A. Court, *College Geometry*, 2nd ed., p. 93, ex. 19b, and in *Scripta Mathematica*, vol. 16 (1950), p. 167.

E1209. A Cross Ratio Associated with Any Triangle

American Mathematical Monthly **63** (1956), 186

Source PDF p. 36.

E1209 [1956, 186]. Proposed by Hüseyin Demir, Zonguldak, Turkey.

Let ABC be any triangle and (I) its incircle. Let (I) touch BC, CA, AB at D, E, F , and intersect the cevians BE, CF at E', F' respectively. Show that the anharmonic ratio $D(E, F, E', F')$ is the same for all triangles ABC .

I. Solution by W. B. Carver, Cornell University. This is obviously a metrically special case of a more general projective theorem. The incircle may be replaced by any conic tangent to the sides at D, E, F , with the conic cutting the lines BE and CF at E' and F' respectively. By one of the limiting cases of Brianchon's theorem the lines AD, BE, CF meet in a point G . We set up a homogeneous coordinate system with A, B, C, G as the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ respectively. It then follows readily that D, E, F are the points $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$; the conic has the equation

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0;$$

E', F' are the points $(1, 4, 1)$, $(1, 1, 4)$; the lines through D have the equations $kx + y - z = 0$ with $k = 1, -1, -3, 3$ for DE, DF, DE', DF' respectively; and the required anharmonic ratio is therefore

$$\frac{(1+3)(-1-3)}{(1-3)(-1+3)} = 4.$$

II. Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. By a central projection, triangle ABC and its incircle (I) can be transformed into an equilateral triangle and its incircle. The anharmonic ratio $D(E, F, E', F')$ is invariant under this transformation and consequently is constant for all triangles. It is easy to show that $D(E, F, E', F') = 4$.

Also solved by N. A. Court, P. W. M. John, D. C. B. Marsh, O. J. Ramler, Roscoe Woods, and the proposer.

E1217. A Property of Euler's Function

American Mathematical Monthly **64** (1957), 45

Source PDF p. 37.

E1217 [1956, 342]. Proposed by Hüseyin Demir, Zonguldak, Turkey.

Evaluate $\prod_{d|n} d^{\phi(n/d)+\phi(d)}$.

Solution by J. B. Johnston, Cornell University. Let f be any function defined on the integers.

Then

$$\begin{aligned}\prod_{d|n} d^{f(d)+f(n/d)} &= \prod_{d|n} d^{f(d)} \prod_{d|n} d^{f(n/d)} \\ &= \prod_{d|n} d^{f(d)} \prod_{d|n} (n/d)^{f(d)} \\ &= \prod_{d|n} n^{f(d)} = n^{\sum_{d|n} f(d)}.\end{aligned}$$

Since

$$\sum_{d|n} \phi(d) = n, \quad (1)$$

the answer to the given problem is n^n .

Also solved by W. J. Buckingham, Leonard Carlitz, A. E. Danese, M. P. Drazin, L. T. Gardner, A. J. Goldman, D. S. Greenstein, Cornelius Groenewoud, Emil Grosswald, Virginia Hanly, A. R. Hyde, Richard Kelisky, Sidney Kravitz, R. G. McDermot, D. C. B. Marsh, Leo Moser, J. B. Muskat, F. R. Olson, Hiram Paly, M. Perisastri, Azriel Rosenfeld, A. V. Sylwester, Chih-yi Wang, David Zeitlin, and the proposer. Late solution by M. S. Klamkin.

Editorial note (original journal). For a proof of (1) see, e.g., Uspensky and Heaslet, *Elementary Number Theory*, p. 113. As another application of the general result established above we have

$$\prod_{d|n} d^{(n/d)+d} = n^{\sigma(n)},$$

where $\sigma(n)$ is the sum of the divisors of n .

E1778. The Radii of a Triangle

American Mathematical Monthly **73** (1966), 667

Source PDF p. 38.

E1778 [1965, 420]. Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

If R, r, r_1, r_2, r_3 are the circumradius, inradius and exradii of a triangle, prove that

$$\frac{1}{r^3} - \frac{1}{r_1^3} - \frac{1}{r_2^3} - \frac{1}{r_3^3} = \frac{12R}{r \cdot r_1 \cdot r_2 \cdot r_3}.$$

Solution by Ralph Schreiber, Warsaw High School, Warsaw, Ind. Denote by Δ the area of triangle ABC , by s the semiperimeter, by r_a the exradius corresponding to side a , and so forth. We recall familiar identities:

$$\Delta = rs = r_a(s - a) = r_b(s - b) = r_c(s - c) = \sqrt{rr_ar_br_c} = abc/(4R).$$

Thus

$$\begin{aligned}\frac{1}{r^3} - \frac{1}{r_a^3} - \frac{1}{r_b^3} - \frac{1}{r_c^3} &= \frac{1}{\Delta^3} [s^3 - (s - a)^3 - (s - b)^3 - (s - c)^3] \\ &= \frac{3abc}{\Delta^3} = \frac{12R}{\Delta^2} = \frac{12R}{rr_ar_br_c}.\end{aligned}$$

Also solved by A. N. Aheart, Leon Bankoff, W. J. Blundon, D. I. A. Cohen, Ragnar Dybvik (Norway), Mrs. A. C. Garstang, Michael Goldberg, Louise S. Grinstein, D. M. Hancasky, E. S. Langford, Ruth S. Lefkowitz, F. Leuenberger (Switzerland), Andrzej Makowski (Poland), D. C. B. Marsh, F. R. Prieto, J. M. Quoniam (France), S. Bhaskara Rao (India), Simeon Reich (Israel), P. A. Scheinok, Klaus Schmitt, R. Sivaramakrishnan (India), Sidney Spital, Sister M. Stephanie, M. V. Tamhankar and M. B. Suryanarayana (India), Simon Vatriquant (Belgium), C. S. Venkataraman (India), William Wernick, and the proposer.

E1779. The Altitudes and Exradii of a Triangle

American Mathematical Monthly **73** (1966), 668

Source PDF p. 39.

E1779 [1965, 420]. Proposed by Hüseyn Demir, Middle East Technical University, Ankara, Turkey.

If h_i and r_i are the altitudes and exradii of a triangle prove that

$$\frac{r_1}{h_1} + \frac{r_2}{h_2} + \frac{r_3}{h_3} \geq 3.$$

I. Solution by D. C. B. Marsh, Colorado School of Mines. Since $2/h_i = 1/r_j + 1/r_k$, for i, j, k a cyclic permutation of $1, 2, 3$ (R. A. Johnson, *Modern Geometry*, p. 189; or use the identities in E1778 above together with $1/r = 1/r_1 + 1/r_2 + 1/r_3$) it follows immediately that

$$\sum_i \frac{r_i}{h_i} = \frac{1}{2} \sum_{i \neq j} \frac{r_i}{r_j} \geq \frac{1}{2}(6) = 3,$$

since $(x/y) + (y/x) \geq 2$ for positive x, y . Moreover, equality obtains only if $r_1 = r_2 = r_3$, i.e., the triangle is equilateral.

II. Solution by H. Guggenheimer, University of Minnesota. We may generalize by securing the inequality $\sum (r_i/h_i)^n \geq 3$, $n \geq 1$. Actually more is true: Let t_i be the lengths of the angle bisectors of the triangle. Since $t_i \geq h_i$, the proposed inequality is weaker than

$$\sum_{i=1}^3 \left(\frac{r_i}{t_i} \right)^m \geq 3, \quad m > 0,$$

which we now prove.

Let s be the semiperimeter of the triangle, a_i the sides. Leuenberger has proved (*Elemente Math.*, 17 (1962), 45–46; see also 16 (1961), p. 129) that $t_i \leq [s(s - a_i)]^{1/2}$. Hence

$$\begin{aligned} \sum_i \left(\frac{r_i}{t_i} \right)^m &\geq \sum_{\text{cyc}} \left[\frac{s(s - a_j)(s - a_k)}{s(s - a_i)^2} \right]^{m/2} \\ &= \frac{1}{[(s - a_1)(s - a_2)(s - a_3)]^m} \sum_{\text{cyc}} [(s - a_j)(s - a_k)]^{3m/2}. \end{aligned}$$

The desired result now follows from the geometric-arithmetic mean inequality, and again equality holds only for the equilateral triangle.

Also solved by A. N. Aheart, Leon Bankoff, W. J. Blundon, D. I. A. Cohen, Mrs. A. C. Garstang, Michael Goldberg, H. Guggenheimer, D. M. Hancasky, E. S. Langford, F. Leuenberger (Switzerland), Andrzej Makowski (Poland), F. R. Prieto, J. M. Quoniam (France), S. Bhaskara Rao

(India), Simeon Reich (Israel), P. A. Scheinok, Ralph Schreiber, R. Sivaramakrishnan (India), Sidney Spital, Sister M. Stephanie, M. V. Tamhankar and M. B. Suryanarayana (India), P. D. Thomas, Simon Vatriquant (Belgium), C. S. Venkataraman (India), and the proposer.

Makowski's student, Tadeusz Figiel observed that the required inequality is equivalent to the fact that the area of an orthic triangle is not greater than one-quarter of the area of a given (acute-angled) triangle. [*Proof:* Let ABC be an orthic triangle of $A_1B_1C_1$. Then A_1, B_1, C_1 are the centers of ex-circles and the ratio of areas of ABC_1 and ABC is equal to the ratio of altitudes on the common side AB , i.e., r_3/h_3 .]

E1877. Convex Pentagon Inscribed in a Semicircle

American Mathematical Monthly **74** (1967), 869

Source PDF p. 40.

E1877 [1966, 410]. Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Let $ABCDE$ be a convex pentagon inscribed in a unit circle with AE as diameter, and let $AB = a$, $BC = b$, $CD = c$, $DE = d$. Then prove that

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4.$$

Solution by Allan Wachs, Student at Far Rockaway (N. Y.) High School. Draw AC and CE and put $\theta = \angle CEA$. Then $\angle CAE = 90^\circ - \theta$, $\angle CBA = 180^\circ - \theta$, $\angle CDE = 180^\circ - (90^\circ - \theta) = 90^\circ + \theta$. By the law of cosines,

$$\begin{aligned} (1) \quad AC^2 &= a^2 + b^2 - 2ab \cos(180^\circ - \theta) = a^2 + b^2 + 2ab \cos \theta, \\ (2) \quad CE^2 &= c^2 + d^2 - 2cd \cos(90^\circ + \theta) = c^2 + d^2 + 2cd \sin \theta. \end{aligned}$$

From the right triangle ACE , $AC^2 + CE^2 = AE^2 = 4$, also $AC = 2 \sin \theta > b$ and $CE = 2 \cos \theta > c$ (because of the obtuse angles). Substitution of these results into the sum of (1) and (2) gives at once

$$4 > a^2 + b^2 + c^2 + d^2 + abc + bcd.$$

Also solved by Leon Bankoff, W. J. Blundon, L. Carlitz, Mannis Charosh, M. A. Ettrick, Michael Goldberg, M. G. Greening (Australia), Ned Harrell, Donald Jeffords, Erwin Just, J. D. E. Konhauser, Dan Marcus, Lieselotte Miller, Norman Miller, C. B. A. Peck, Al Somoyajulu, J. L. Standig, C. S. Venkataraman (India), J. C. Williams, Dale Woods, and the proposer.

E1878. Incenters Associated with a Regular Polygon

Proposal: *American Mathematical Monthly* **73** (1966), 410

Source PDF p. 26, 41.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Let $A_1A_2 \cdots A_n$ be a regular polygon inscribed in circle (O) of radius R . Denote the incenter of the triangle $A_{i-1}A_iA_{i+1}$ (indices mod n) by I_i , and that of the triangle formed by $A_iA_{i+2}, A_{i+2}A_{i+1}, A_{i+1}A_{i+3}$ by J_i . Then show that the points $I_1, \dots, I_n$ and $J_1, \dots, J_n$ all lie on the same circle of radius $R' = R \cos(3\pi/2n) / \cos(\pi/2n)$.

Editorial replacement.^[E005] The supplied collection repeats the E1877 solution under the E1878 heading. The following proof is supplied for this edition; it is not attributed to the original journal.

Proof. Put $\theta = \pi/n$. The symmetry axes through the vertices and the side midpoints all pass through O . For the triangle $A_{i-1}A_iA_{i+1}$, place A_i at $(R, 0)$ and its neighbors at $(R \cos 2\theta, \pm R \sin 2\theta)$. Its altitude is $h = R(1 - \cos 2\theta)$ and its half vertex angle is $\pi/2 - \theta$. If u is its inradius, $u = (h - u) \cos \theta$, so its incenter has abscissa

$$R \cos 2\theta + u = R - \frac{R(1 - \cos 2\theta)}{1 + \cos \theta} = R(2 \cos \theta - 1).$$

For the second triangle, rotate the coordinates so that A_{i+1}, A_{i+2} have arguments $-\theta, \theta$ and A_i, A_{i+3} have arguments $-3\theta, 3\theta$. The two diagonal lines meet at $(R \cos 2\theta / \cos \theta, 0)$, whereas their base lies on $x = R \cos \theta$. The altitude is now $R \sin^2 \theta / \cos \theta$ and the half vertex angle is again $\pi/2 - \theta$. Its inradius is $R(1 - \cos \theta)$; consequently its incenter also has abscissa $R(2 \cos \theta - 1)$. Rotational symmetry gives the asserted common circle, since

$$R(2 \cos \theta - 1) = R \frac{\cos(3\theta/2)}{\cos(\theta/2)}.$$

□

E2100. Six Relations

American Mathematical Monthly **76** (1969), 563

Source PDF p. 42.

E2100 [1968, 670]. Proposed by H. Demir, Middle East Technological University, Ankara, Turkey.

Show that any five of the relations

$$\begin{aligned} (1) \quad \frac{x - a_1}{a_1 - a_2} &= \frac{a - b}{b - c}, & (2) \quad \frac{x - b_1}{b_1 - b_2} &= \frac{b - c}{c - a}, & (3) \quad \frac{x - c_1}{c_1 - c_2} &= \frac{c - a}{a - b}, \\ (4) \quad x + a &= b_2 + c_1, & (5) \quad x + b &= c_2 + a_1, & (6) \quad x + c &= a_2 + b_1 \end{aligned}$$

imply the sixth. Interpret this set of consistent relations geometrically letting a, b, c be the affixes, in the complex plane, of a triangle of reference ABC and other numbers be those of other points.

Solution by Michael Goldberg, Washington, D. C. Take any triangle, represented by the vertices a, b, c . Take any point x in the plane of the triangle. Draw parallels to the sides of the triangle through the point x . Let the intersections of these parallels with the sides be $a_1, a_2, b_1, b_2, c_1, c_2$, as shown in the figure. Then the six given relations hold. If one of the six points, say c_2 , is omitted, then x can be found as the intersection of a_2c_1 with b_2a_1 , and then c_2 is determined as the intersection of b_1x with ab .

Also solved by A. F. Gentzel, Jr., Simeon Reich (Israel), and the proposer.

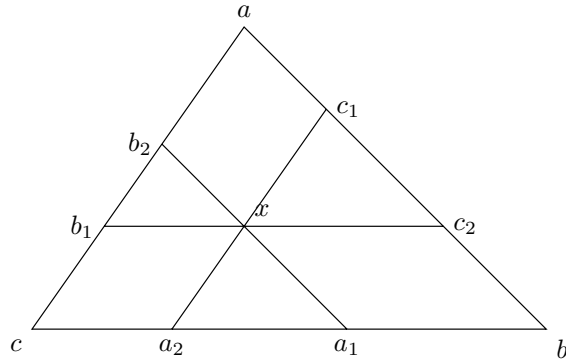
E2101. Three Parabolas and a Triangle

American Mathematical Monthly **76** (1969), 564

Source PDF p. 43.

E2101 [1968, 670]. Proposed by H. Demir, Middle East Technological University, Ankara, Turkey.

ABC is a triangle. Let P_a denote the parabola tangent to the sides AB, AC at B, C respectively. The parabolas P_b and P_c are similarly defined. Let these parabolas intersect

Figure 4.1: Parallels through x in the reference triangle.

to the points A', B', C' inside ABC . Denote the areas of the (curvilinear) triangular regions $ABC, A'B'C', AB'C', BC'A', CA'B', A'BC, B'CA, C'AB$, by $\Delta, \Delta_0, \Delta'_a, \Delta'_b, \Delta'_c, \Delta''_a, \Delta''_b, \Delta''_c$. Then prove

- (1) $\Delta'_a = \Delta'_b = \Delta'_c = (\Delta_1), \quad \Delta''_a = \Delta''_b = \Delta''_c = (\Delta_2),$
- (2) $\Delta_0 : \Delta_1 : \Delta_2 : \Delta = 15 : 17 : 5 : 81.$

Solution by the proposer. Under parallel projections the nature of conics, the tangency and ratios of segments and areas are invariant, and any triangle can be transformed into an equilateral triangle. Hence it will suffice to prove the assertion for an equilateral triangle. So, part (1) is already proved.

To prove (2), let ABC be an equilateral triangle located in the coordinate plane such that $A = (1, \sqrt{3}), B = (0, 0), C = (2, 0)$. The equations of parabolas P_a and P_c are found to be

$$(1) \quad P_a : y = (x - \frac{1}{2}x^2)\sqrt{3},$$

$$(2) \quad P_c : \sqrt{3}x^2 + 6xy + 3\sqrt{3}y^2 - 16y = 0.$$

From (1) and (2) we obtain

$$(3) \quad \Delta_0 + 2\Delta_1 + \Delta_2 = \sqrt{3} \int_0^2 (x - \frac{1}{2}x^2) dx = \frac{2}{3}\sqrt{3} = \frac{2}{3}\Delta,$$

$$(4) \quad y = \frac{8\sqrt{3}}{9} - \frac{\sqrt{3}}{3}x - \frac{4\sqrt{3}}{9}\sqrt{4-3x}, \quad 0 \leq x \leq \frac{4}{3}.$$

We find, therefore,

$$(5) \quad \Delta_2 = 2 \int_0^1 y dx = \frac{5\sqrt{3}}{81} = \frac{5}{81}\Delta,$$

with

$$(6) \quad \Delta_0 + 3\Delta_1 + 3\Delta_2 = \Delta.$$

Solving the system (3), (5), (6) for $\Delta_0, \Delta_1, \Delta_2$, we get the required result.

Also solved by Anders Bager (Denmark), Jordi Dou (Spain), Michael Goldberg, M. G. Greening (Australia), Norman Miller, J. M. Quoniam (France), and A. Zujus.

E2109. Triangle Construction

American Mathematical Monthly **76** (1969), 698

Source PDF p. 44.

E2109 [1968, 780]. Proposed by H. Demir, Middle East Technical University, Ankara, Turkey.

Let ABC be a triangle and A' be any fixed point on the side BC . Construct the inscribed triangle $A'B'C'$ which is directly similar to a given triangle XYZ .

Note by A. W. Walker, Toronto, Canada. The required construction will be found in N. A. Court, *College Geometry*, ed. 1, 1925, p. 47. It is a simple application of the following theorem, established on p. 46: *If one vertex of a triangle of variable size and given shape remains fixed and a second vertex moves on a given straight line, then the locus of the third vertex is also a straight line.*

Also solved by Anders Bager (Denmark), Walter Bluger, C. W. Eliason, Jr., Michael Goldberg, M. G. Greening (Australia), Beckham Martin, D. N. Page, and the proposer.

E2110. Similar Triangles

American Mathematical Monthly **76** (1969), 698

Source PDF p. 45.

E2110 [1968, 780]. Proposed by H. Demir, Middle East Technical University, Ankara, Turkey.

If, in a plane, the triangles AUV, VBU, UVC are directly similar to a given triangle, then so is ABC , provided A, B, C are not all coincident.^[E006]

Solution by M. G. Greening, University of New South Wales, Australia. Represent the points by complex numbers using the appropriate lower case letters and take the given triangle as $Z_1Z_2Z_3$. Let the direct similarities be $z \mapsto \alpha_i z + \beta_i$ ($i = 1, 2, 3$). Then $u = \alpha_i z_{i+1} + \beta_i$; $v = \alpha_i z_{i+2} + \beta_i$ ($i = 1, 2, 3$), taking subscripts modulo 3. Then $\alpha_i z_i(z_{i+1} - z_{i+2}) = z_i(u - v)$ and

$$\beta_i(z_{i+1} - z_{i+2}) = z_{i+1}v - z_{i+2}u$$

so that

$$\sum_{i=1}^3 (\alpha_i z_i + \beta_i)(z_{i+1} - z_{i+2}) = 0.$$

As $\sum_{i=1}^3 (z_{i+1} - z_{i+2}) = 0$ and $\sum_{i=1}^3 z_i(z_{i+1} - z_{i+2}) = 0$ we get

$$0 = \begin{vmatrix} \alpha_1 z_1 + \beta_1 & z_1 & 1 \\ \alpha_2 z_2 + \beta_2 & z_2 & 1 \\ \alpha_3 z_3 + \beta_3 & z_3 & 1 \end{vmatrix} = \begin{vmatrix} a & z_1 & 1 \\ b & z_2 & 1 \\ c & z_3 & 1 \end{vmatrix}$$

which is a sufficient condition for a direct similarity: $z_1 \mapsto a$, $z_2 \mapsto b$, $z_3 \mapsto c$ to exist, provided the resulting map is nonconstant.^[E006]

Also solved by Leon Bankoff, Jordi Dou (Spain), C. W. Eliason, Jr., Michael Goldberg, Norman Miller, Simeon Reich (Israel), A. W. Walker, and the proposer.

Walker points out that the result may be found on p. 289 of R. A. Johnson, *Modern Geometry* (1929).

E2124. All Triangles Generate Right Triangles

American Mathematical Monthly **76** (1969), 938

Source PDF p. 46.

E2124 [1968, 899]. Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Construct on the sides BC, CA, AB of a triangle ABC , exteriorly, the squares $BCDE, ACFG, BAHK$ and build the parallelograms $FCDQ, EBKP$. Show that APQ is an isosceles right triangle.

Solution by W. E. Buker, Pittsburgh Public Schools. Assign coordinates $A(0, 0); B(a, 0); C(b, c)$. Then find by inspection the coordinates

$$\begin{aligned} F(b - c, b + c), \quad D(b + c, c + a - b), \quad Q(b, a + c), \\ K(a, -a), \quad E(a + c, a - b), \quad P(a + c, -b). \end{aligned}$$

Since AQ and AP have equal lengths and are perpendicular, the theorem follows.

Also solved by forty other readers.

Note. It follows at once that if parallelogram $HAGR$ is constructed, then BQR and CRP are also isosceles right triangles.

A. W. Walker points out that an extensive investigation of triangles bordered by squares is found in a paper by Musselman, this *American Mathematical Monthly*, 43 (1936), 539–548.

E2160. Some New Triangle Inequalities

American Mathematical Monthly **76** (1969), 1146

Source PDF p. 47–48.

E2160 [1969, 300]. Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Let p_i, x_i be the distances of an interior or a boundary point P of a triangle $A_1A_2A_3$ from the vertex A_i and the side opposite to A_i , $i = 1, 2, 3$, with r the inradius. Prove the inequalities

$$\begin{aligned} (a) \quad \sum_{i=1}^3 p_i \left(\frac{1}{2} \sin A_i \right) &\leq \sum_{i=1}^3 x_i \leq \sum_{i=1}^3 p_i \sin \left(\frac{1}{2} A_i \right), \\ (b) \quad p_2 p_3 + p_3 p_1 + p_1 p_2 &\geq 8x_1 x_2 x_3 / r. \end{aligned}$$

Solution by M. G. Greening, University of New South Wales, Australia. Let a_i be the side opposite A_i , let P_i be the angle $A_{i-1}PA_{i+1}$, $B_{i,j}$ be the angle between p_i and a_j at A_i , so that $B_{i,i+1} + B_{i,i-1} = A_i$. (All additions of subscripts are modulo 3.) Then $x_i = p_{i+1} \sin B_{i+1,i} = p_{i-1} \sin B_{i-1,i}$ and

$$\begin{aligned} \sum_i x_i &= \frac{1}{2} \sum_j p_j (\sin B_{j,j+1} + \sin B_{j,j-1}) \\ &= \frac{1}{2} \sum_j p_j \cdot 2 \sin \left(\frac{1}{2} A_j \right) \cos \frac{1}{2} (B_{j,j+1} - B_{j,j-1}). \end{aligned}$$

The inequality $0 \leq |B_{j,j+1} - B_{j,j-1}| \leq A_j$ then yields (a).

As $p_1 p_2 \sin P_3 = x_3 a_3$, we obtain, first for an interior point,

$$\begin{aligned} \sum_i p_i p_{i+1} &= \sum_i \frac{x_i a_i}{\sin P_i} \geq \frac{3 (\prod_i a_i \cdot \prod_i x_i)^{1/3}}{(\prod_i \sin P_i)^{1/3}} \\ &\geq 2\sqrt{3} \left(\prod_i a_i \cdot \prod_i x_i \right)^{1/3}. \end{aligned}$$

The last statement follows from the fact that $\prod_i \sin P_i$ with $\sum_i P_i = 2\pi$ has a maximum when $P_1 = P_2 = P_3$.

For (b) we now show

$$(i) \quad 2\sqrt{3}r \left(\prod_i a_i \right)^{1/3} \geq 8 \left(\prod_i x_i^2 \right)^{1/3}.$$

As $\sum_i a_i x_i = 2\Delta$,^[E007] $\prod_i x_i$ has a maximum when $a_1 x_1 = a_2 x_2 = a_3 x_3 = 2\Delta/3$, so that

$$\max 8 \left(\prod_i x_i^2 \right)^{1/3} = 2^5 \Delta^2 3^{-2} \left(\prod_i a_i \right)^{-2/3}.$$

(i) will follow if $3^{5/2}r \cdot \prod_i a_i \geq 2^4 \Delta^2$, or

$$(ii) \quad 3^{5/2}R \geq 4s,$$

as $\prod_i a_i = 4R\Delta$, where R is the circumradius. But the triangle of largest perimeter which can be inscribed in a given circle is equilateral and the inequality (ii) is certainly true then, so that (b) is established. In fact, 8 could be replaced by 12 in (b). Boundary points follow by continuity.^[E007]

Also solved by Simeon Reich (Israel), T. Tamura (Japan), C. S. Venkataraman (India), A. W. Walker and the proposer.

The improved inequality for part (b) was conjectured by Walker and proved by Reich. It is interesting to note that aside from a solution to part (a) by L. Carlitz, all solvers and the proposer reside outside the United States of America.

E2213. Quadrilaterals with the Nagel Property

American Mathematical Monthly **77** (1970), 1109

Source PDF p. 49–51.

E2213 [1970, 79]. Proposed by H. Demir, Middle East Technical University, Turkey.

Let us say that a (planar) polygon has the Nagel property if the lines through the vertices of the polygon and bisecting the perimeter of the polygon are concurrent. It is known that all triangles have the Nagel property and that not all quadrilaterals have the property. Determine the simple nondegenerate quadrilaterals that have the Nagel property.

Solution by the editor based on the proposer's solution. Let $ABCD$ be a quadrilateral having the Nagel property. Let each of the lines AA' , BB' , CC' , DD' bisect the perimeter and pass through the Nagel point N . Let $AB = a$, $BC = b$, $CD = c$, $DA = d$. We may suppose $a + b \geq c + d$ and $b + c \geq a + d$. It then follows that C' lies on segment AB , B' on CD , and A' and D' on BC (Figure 4.2). Set $BD' = u$, $A'C = v$, $CB' = w$, and $a + b + c + d = 2s$. Since

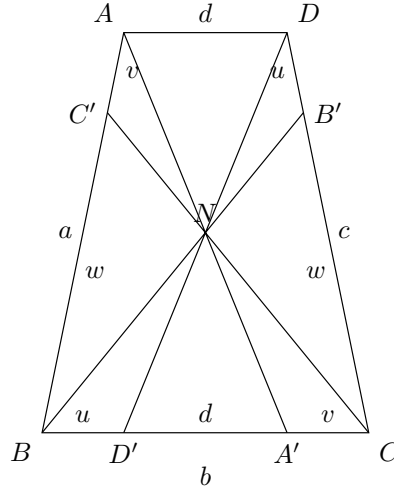


Figure 4.2: The perimeter-bisecting lines in Case 1 (original Figure 1).

$BC + CB' = s = D'C + CD$, then $DB' = u$. Similarly $AC' = v$, $BC' = w$ and $D'A' = d$. We distinguish three cases.

CASE 1. $u \neq 0$ and $v \neq 0$. Thinking of AA' and DD' meeting at N , we see that a necessary and sufficient condition for CC' to pass through N is

$$\frac{v}{w} \cdot \frac{b}{-v} \cdot \frac{A'N}{NA} = -1$$

by Menelaus' theorem applied to triangle ABA' cut by line CNC' . Similarly, BB' passes through N if and only if

$$\frac{-u}{b} \cdot \frac{w}{u} \cdot \frac{DN}{ND'} = -1,$$

using triangle $DD'C$ cut by BNB' . Then

$$\frac{DN}{ND'} = \frac{b}{w} = \frac{AN}{NA'}.$$

Hence AD is parallel to $D'A'$, and since these segments are also equal, it follows that $ADA'D'$ is a parallelogram. The diagonals AA' and DD' then bisect each other, so $AN = NA'$ and $b = w$. But $b + w = s$, so $2w = a - b + c + d = 2b$, from which we obtain

$$b = \frac{a + c + d}{3}.$$

Hence $ABCD$ is a trapezoid such that the longer of its two parallel bases is the arithmetic mean of its other three sides. Reversing the argument of this paragraph shows that every such trapezoid has the Nagel property. For example, the trapezoid with vertices $(0, 0)$, $(19, 0)$, $(38/7, 24)$, $(0, 24)$ has $N = (7, 12)$.^[E008]

CASE 2. $u = v = 0$. Then $a + b = c + d = s$ and $a + d = b + c = s$, so $2a + b + d = 2c + b + d$. Hence $a = c$ and $b = d$, so $ABCD$ is a parallelogram. Clearly every parallelogram has the Nagel property.

CASE 3. $u = 0$ and $v \neq 0$ (or vice versa). Let AC and BD meet at M (Figure 4.3). Now $a + d = b + c = s$, so $d + v + w = C'B + BC = CD + DA + AC' = c + d + v$, whence $c = w$. Applying Menelaus' theorem to triangle ABA' cut by CNC' and to triangle $AA'C$ cut by BNM ,

we obtain

$$\begin{aligned} \frac{b}{-v} \frac{A'N}{NA} \frac{v}{c} &= -1, & \text{so } \frac{A'N}{NA} &= \frac{c}{b}, \\ \frac{AN}{NA'} \frac{-d}{b} \frac{CM}{MA} &= -1, & \text{so } \frac{CM}{MA} &= \frac{b}{d} \frac{A'N}{NA} = \frac{b}{d} \frac{c}{b} = \frac{c}{d}. \end{aligned}$$

Since M divides side CA of triangle DAC in the ratio c/d of the adjacent sides, then DM bisects angle D . Hence $\angle ADM = \angle MDC = \alpha$. Applying the law of cosines to triangles ABD and CBD and letting $BD = f$, we have

$$\begin{aligned} a^2 &= d^2 + f^2 - 2df \cos \alpha, \\ b^2 &= c^2 + f^2 - 2cf \cos \alpha, \end{aligned}$$

so

$$a^2 - b^2 = d^2 - c^2 - 2f(d - c) \cos \alpha.$$

Assuming $d \geq c$, let $d = c + e$, so $b = c + v + e$ and $a = c + v$. Now $a^2 - (a + e)^2 = (c + e)^2 - c^2 - 2ef \cos \alpha$, which simplifies to

$$ef \cos \alpha = ce + ae + e^2.$$

If $e \neq 0$, then $\cos \alpha = (c + a + e)/f = (c + b)/f$, an impossibility since $c + b > f$. Hence $e = 0$, so $c = d$ and $a = b$. The figure is therefore a kite. By symmetry, every kite has the Nagel property.

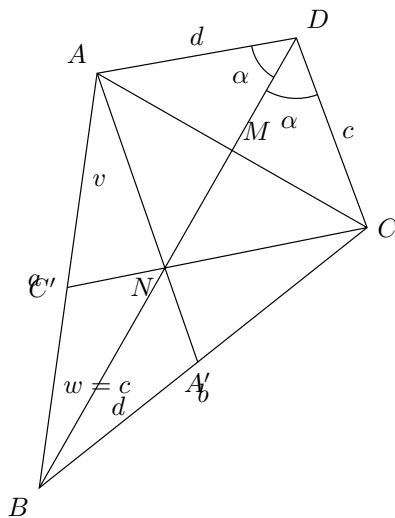


Figure 4.3: The kite configuration in Case 3 (original Figure 2).

Three other correspondents mentioned the kite and the parallelogram.

E2311. The Compleat Cyclic Quadrilateral

American Mathematical Monthly **79** (1972), 777

Source PDF p. 52–53.

E2311 [1971, 793]. Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Prove that, if a quadrilateral $A_1A_2A_3A_4$ can be inscribed in a circle, then the (six) lines drawn from the midpoints of A_pA_q perpendicular to A_rA_s (p, q, r, s are distinct) are concurrent.

Solution by Sister Stephanie Sloyan, Georgian Court College, Lakewood, N. J. Assume that the circle is the unit circle and identify the point A_i with the complex number a_i in the usual manner. Then the line from the midpoint of the segment $A_p A_q$ perpendicular to $A_r A_s$ is given by

$$z - a_r a_s \bar{z} = \frac{1}{2}(a_p + a_q) \frac{a_p a_q - a_r a_s}{a_p a_q},$$

and it is easily calculated that all six lines pass through the point $\frac{1}{2}(a_1 + a_2 + a_3 + a_4)$. J. W. Clawson, *The complete quadrilateral*, *Annals of Math.* 20 (1918–1919), 232–261, calls this point the *orthic center* of the quadrilateral.

In a similar fashion one can show that the three lines joining the midpoint of $A_p A_q$ to that of $A_r A_s$ (p, q, r, s distinct) are each bisected by a point identified by Clawson as the *mean center* of the quadrilateral. Since the mean center is given by $\frac{1}{4}(a_1 + a_2 + a_3 + a_4)$, it follows that it lies halfway between the orthic center and the circumcenter.

Also solved by Michael Goldberg, Leonard Goldstone, M. G. Greening (Australia), N. G. Gunderson, V. F. Ivanoff, Lew Kowarski, Harry Lass, O. P. Lossers (Netherlands), Rick Troxel, and the proposer.

Editorial note (original journal). This theorem and its solution appear on page 59 of Yaglom, *Complex Numbers in Geometry*, Academic Press, 1968, along with many other interesting properties of cyclic quadrilaterals, cyclic pentagons, etc. (see pages 54–68). The point of concurrence of this problem is called the *anticenter* by Lucien Droussent (*On a theorem of J. Griffiths*, this *American Mathematical Monthly*, 54 (1947), 538–540). The anticenter N is the midpoint of the quadrilateral's *Euler segment* which joins its circumcenter O to the center H of the circle through the four orthocenters H_m of the triangles $A_i A_j A_k$ ($\{i, j, k, m\} = \{1, 2, 3, 4\}$); these orthocenters form a quadrilateral congruent to the given one and symmetric to it in point N . Furthermore, the eight points A_i and H_i lie by fours on four distinct pairs of circles, each pair having N as center of symmetry.

The eight congruent nine-point circles for the four triangles $A_i A_j A_k$ and four triangles $H_i H_j H_k$ all pass through N , and their centers lie on another congruent circle centered at N . Thus N can be called the *eight circle point* and this last circle the *eight point circle* for the quadrilateral.

There are four distinct Simson lines for the eight points A_m with triangles $A_i A_j A_k$ and H_m with triangles $H_i H_j H_k$, and these Simson lines all pass through N . In fact, one can form 280 (180 of which are distinct) pedal circles (and lines) by taking any one of these eight points with the triangle formed by any three others, and all of them pass through N .

The nine point centers N_m for the four triangles $A_i A_j A_k$ form a quadrilateral homothetic to $H_1 H_2 H_3 H_4$ in center O with ratio $\frac{1}{2}$, hence homothetic to $A_1 A_2 A_3 A_4$ in center G , $1/3$ of the way from O to H , with ratio $-\frac{1}{2}$. Similarly, the nine-point centers N'_m for the triangles $H_i H_j H_k$ are homothetic to $H_1 H_2 H_3 H_4$ in center G' , $2/3$ of the way from O to H , with ratio $-\frac{1}{2}$. Their common circumcircle has center N and radius half the given quadrilateral's circumradius. In a similar manner (see E1740 [1965, 1026]) the centroids G_m for the triangles $A_i A_j A_k$ form a quadrilateral homothetic to $H_1 H_2 H_3 H_4$ in center O with ratio $1/3$, hence homothetic to $A_1 A_2 A_3 A_4$ in center S (the mean center) $1/4$ of the way from O to H , with ratio $-\frac{1}{3}$. Its circumcenter is G . Similarly, the centroids G'_m (whose circumcenter is G') for the triangles $H_i H_j H_k$ determine the other quadrisecion point S' of OH . Furthermore, N is the center of symmetry for the two quadrilaterals $N_1 N_2 N_3 N_4$ and $N'_1 N'_2 N'_3 N'_4$ and also for $G_1 G_2 G_3 G_4$ and $G'_1 G'_2 G'_3 G'_4$.

There are eight *orthocentroidal circles* (see Droussent) on the segments $G_i H_i$ and on $G'_i H'_i$ as diameters, pairs of which determine 16 distinct radical axes all passing through N , so N is the

center of a circle orthogonal to all these eight circles.

We see that the Euler segment could well be renamed the *seven point line* (points O, S, G, N, G', S', H). With this notation, since points G and N trisect and bisect OH , the resemblance to the Euler line of a triangle is striking.

See also H. G. Forder, *Higher Course Geometry*, Cambridge University Press, 1949, pp. 232–235, and R. A. Johnson, *Modern Geometry*, Houghton-Mifflin, 1929, pp. 169, 207, 243, and 251–253.

E2312. An Application of Ceva's Theorem

American Mathematical Monthly **79** (1972), 778

Source PDF p. 53–54.

E2312 [1971, 793]. Proposed by Hüseyn Demir, Middle East Technical University, Ankara, Turkey.

Let D be a point in the plane of a positively oriented triangle ABC and let AD, BD, CD intersect the respective opposite sides in A_1, B_1, C_1 . If the oriented segments BA_1, CB_1, AC_1 are equal ($= \delta$), then D is uniquely determined and lies in the interior of ABC . (Notice the analogy between D and the Brocard point Ω .)

Solution by Michael Goldberg, Washington, D. C. Let the lengths of the sides of the triangle be a, b, c , where $a \leq b \leq c$. Then by Ceva's Theorem, we have the equation

$$(*) \quad (a - \delta)(b - \delta)(c - \delta) = \delta^3.$$

The left member of $(*)$ is a function which decreases monotonically from abc at $\delta = 0$ to zero at $\delta = a$, and the right member is a function which increases monotonically from zero at $\delta = 0$. Hence the two functions are equal for exactly one real value of δ which lies in the interval $(0, a)$; it is easy to see also that there are no other real solutions to $(*)$.

Note that if, instead, the segments CA_1, BC_1 , and AB_1 are equal, then the value of δ is the same, but the transversals cross at another point E . The points D and E coincide only for the equilateral triangle.

Also solved by Bernhard Andersen (Denmark), Harold Donnelly, Jordi Dou (Spain), M. G. Greening (Australia), V. F. Ivanoff, and the proposer.

Editorial note (original journal). L. Goldstone located a complete discussion of this point, its isotomic conjugate, and their properties in Peter Yff, *An analogue of the Brocard Points*, this *American Mathematical Monthly* 70 (1963), 495–501.

E2363. On Spherical Triangles

American Mathematical Monthly **80** (1973), 694

Source PDF p. 55.

E2363 [1972, 662].^[E009] Proposed by Hüseyn Demir, Middle East Technical University, Ankara, Turkey.

Characterize pairs of spherical triangles ABC and $A'B'C'$ for which $A' = a, B' = b, C' = c, A = a', B = b', C = c'$.

Solution by M. G. Greening, University of New South Wales, Australia. For any spherical triangle $A'B'C'$ we have:

$$(1) \quad \cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'$$

and the two others following from the cyclic permutations of (a, b, c) .^[E009] So

$$(2) \quad \cos A = \cos B \cos C + \sin B \sin C \cos a,$$

and so on. But from consideration of the polar triangle of ABC ,

$$(3) \quad \cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

Then $\cos B \cos C = \cos C \cos A = \cos A \cos B = 0$ and we have, say, $A = B = \pi/2$, yielding $a = b = \pi/2$ from (2). Also $\cos C = \cos c$, which must give $C = c$ as $c > 0$, $C < \pi$. Consequently

$$A' = B' = a' = b' = \pi/2 \quad \text{and} \quad c' = C' = c = C,$$

so that the two triangles are necessarily congruent.

Also solved by Michael Goldberg, Lew Kowarski, Clellie Oursler and Eric Sturley, and the proposer.

E2462. The Extended Erdős–Mordell Inequality

American Mathematical Monthly **92** (1985), 360

Source PDF p. 55–63.

E2462 [1974, 281]. Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Let P be a point interior to the triangle $A_1A_2A_3$. Denote by R_i the distance from P to the vertex A_i , and denote by r_i the distance from P to the side a_i opposite to A_i . The Erdős–Mordell inequality asserts that

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3).$$

Prove that the above inequality holds for every point P in the plane of $A_1A_2A_3$ when we make the interpretation $R_i \geq 0$ always and r_i is positive or negative depending on whether P and A_i are on the same side of a_i or on opposite sides.

Editorial note (original journal). Professor Clayton W. Dodge, Department of Mathematics, University of Maine refereed the “solutions” submitted to this problem in 1974 and found that there were no solutions. Since that time Professor Dodge himself has solved the problem. His solution appears in *Cruz Mathematicorum*, vol. 10, no. 9, November 1984, pages 274–281.

The Extended Erdős–Mordell Inequality

Clayton W. Dodge

Cruz Mathematicorum **10** (1984), 274–281.

Source PDF p. 56–63.

Ten years ago the *American Mathematical Monthly* published the following Problem E2462 [81 (1974), 281], which is an extension of the earlier Problem 3740 proposed by Paul Erdős [42 (1935), 396] and first solved by L. J. Mordell [44 (1937), 252–254]:

E2462. Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Let P be a point interior to the triangle $A_1A_2A_3$. Denote by R_i the distance from P to the vertex A_i , and denote by r_i the distance from P to the side a_i opposite to A_i . The Erdős–Mordell inequality asserts that

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3).$$

Prove that the above inequality holds for every point P in the plane of $A_1A_2A_3$ when we make the interpretation $R_i \geq 0$ always and r_i is positive or negative depending on whether P and A_i are on the same side of a_i or on opposite sides.

It was my pleasure in 1974 to referee the solutions to this problem. Curiously, each of the solvers started with the solution to the original Erdős inequality given by Kazarinoff [1] and modified it for the case where r_1, r_2 , or r_3 is negative. Each made the same error, invalidating the proof. Curiously, Kazarinoff stated that his proof “holds even if P lies outside the triangle, provided it remains inside the circumcircle”, but the Elementary Problem Department editors could not see that such an extension of the proof was possible without committing the same error the other solvers had made. We outline Kazarinoff’s proof and describe the error. Since we shall rely heavily on this proof, our outline is quite complete. It is interesting to note that, if Kazarinoff’s statement could have been verified then, a proof would have been published in 1975.

In Demir’s notation, Kazarinoff let P lie within angle A_1 and then he reflected triangle $A_1A_2A_3$ in the bisector A_1T of angle A_1 into triangle $A_1A'_2A'_3$, as shown in Figure 4.4. Noting that the bisector of angle A_1 also bisects the angle between the altitude A_1D and the circumradius OA_1 , he used a theorem of Pappus which states that the area of the parallelogram whose adjacent sides are $A_1A'_2$ and A_1P plus the area of the parallelogram whose adjacent sides are A_1P and $A_1A'_3$ is equal to the area of the parallelogram erected on $A'_2A'_3$ whose sides emanating from A'_2 and A'_3 are equal, as vectors, to $\overrightarrow{A_1P}$. Since $A_1P = R_1$, Kazarinoff obtained the first of equalities (1), and the other two are obtained in the same way when P lies within angles A_2 and A_3 :

$$\begin{cases} a_1 R_1 \cos(OA_1P) = a_2 r_3 + a_3 r_2, \\ a_2 R_2 \cos(OA_2P) = a_3 r_1 + a_1 r_3, \\ a_3 R_3 \cos(OA_3P) = a_1 r_2 + a_2 r_1. \end{cases} \quad (1)$$

From this follows, when P is an interior point of the triangle,

$$\begin{aligned} R_1 + R_2 + R_3 &\geq R_1 \cos(OA_1P) + R_2 \cos(OA_2P) + R_3 \cos(OA_3P) \\ &= \left(\frac{a_2}{a_3} + \frac{a_3}{a_2}\right) r_1 + \left(\frac{a_3}{a_1} + \frac{a_1}{a_3}\right) r_2 + \left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right) r_3 \\ &\geq 2(r_1 + r_2 + r_3) \end{aligned} \quad (2)$$

because $x + 1/x \geq 2$ when $x > 0$.

Equations (1) hold for all locations of P , provided Demir’s sign convention is observed. Then also the first two lines of (2) hold. We shall make use of this in our proofs later in this paper, so a proof is presented.

Let P lie outside angle A_1 and outside angle A_2 but inside angle A_3 and inside the circumcircle of triangle $A_1A_2A_3$, as shown in Figure 4.5. Using the notation of Figure 4.4, we see that the parallelogram on side $A'_2A'_3$ now is the difference between those on sides $A_1A'_2$ and $A_1A'_3$. Accordingly,

$$a_1 R_1 \cos(OA_1P) = -a_2 r_3 + a_3 r_2,$$

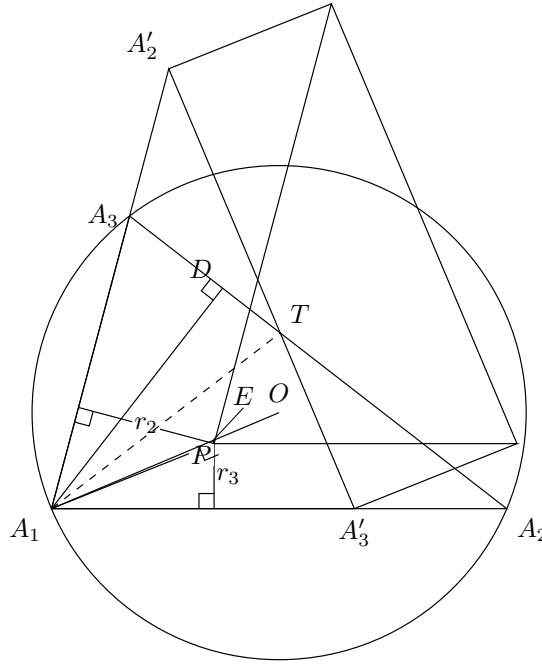


Figure 4.4: Reflection in the angle bisector (Dodge's Figure 1).

where we take the r_i all nonnegative; similarly,

$$a_2 R_2 \cos(OA_2P) = a_3 r_1 - a_1 r_3,$$

and we have as before

$$a_3 R_3 \cos(OA_3P) = a_1 r_2 + a_2 r_1,$$

since P lies within angle A_3 . Thus equations (1) are true for this case if we observe Demir's sign convention. That they also hold in other cases is not needed here. Since the cosines of the angles OA_iP are all still positive because P lies inside the circumcircle, the first two lines of (2) both still hold. Only the third line of (2) is in doubt. In fact, Kazarinoff's argument fails at this point, as explained in the next paragraph. The error in the submitted solutions to Problem E2462 occurred when one of the r_i , say r_3 , is negative. Then we still have

$$\frac{a_1}{a_2} + \frac{a_2}{a_1} \geq 2,$$

but, since $r_3 < 0$, the inequality reverses to give

$$\left(\frac{a_1}{a_2} + \frac{a_2}{a_1} \right) r_3 \leq 2r_3,$$

rendering the argument inconclusive. The editors could find no simple remedy for this flaw since the extended theorem requires that either one or two of the distances r_i be negative. We wrote to those who had submitted solutions, and Leon Bankoff and I corresponded for perhaps a year in attempting to put together a satisfactory proof. Over the next nine years I returned to the problem from time to time, fascinated by its challenge.

Two cases were disposed of almost immediately.

Case 1. Point P lies inside the angle vertical to a vertex angle.

For example, let P lie inside the angle vertical to A_1 , as shown in Figure 4.6. Then r_2 and r_3 are to be taken negative and we must prove that

$$R_1 + R_2 + R_3 \geq 2(r_1 - r_2 - r_3),$$

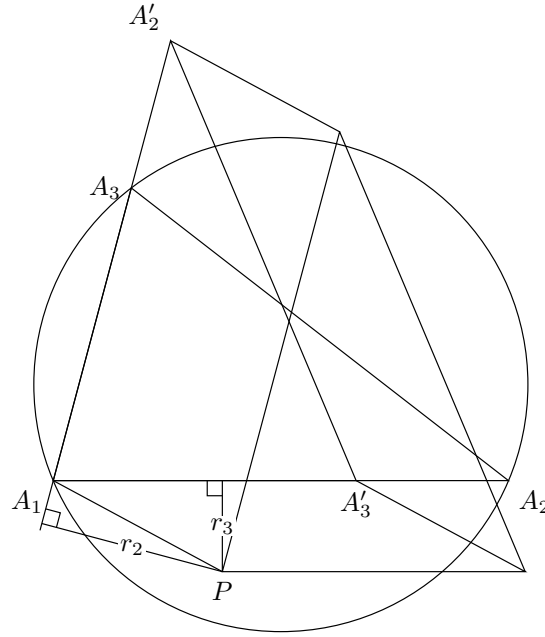


Figure 4.5: The reflected configuration with P outside the triangle (Dodge's Figure 2).

where the r_i have all been taken nonnegative and we have inserted the appropriate negative signs. Because R_2 and r_1 are hypotenuse and leg of a right triangle, we have

$$R_2 \geq r_1, \quad \text{and similarly} \quad R_3 \geq r_1.$$

Thus

$$R_1 + R_2 + R_3 \geq R_2 + R_3 \geq r_1 + r_1 \geq 2(r_1 - r_2 - r_3). \quad \square$$

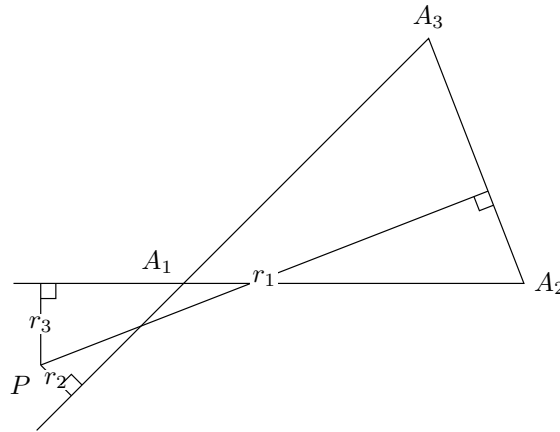


Figure 4.6: The region vertically opposite a vertex angle (Dodge's Figure 3).

Case 2. Point P is interior to an angle of the triangle, but far enough outside the triangle so that a foot F_i of a distance r_i lies outside the triangle.

As shown in Figure 4.7, we take P lying within angle A_1 and far enough outside the triangle so that, say, the foot F_3 of distance r_3 lies outside the triangle. Then r_1 is taken negative. Choose point A'_2 so that F_3 is the midpoint of segment $A_2A'_2$. Then $PA_2 = PA'_2$ and the three distances R_i for triangle $A_1A_2A_3$ are the same as those for triangle $A_1A'_2A_3$. Also r_2 and r_3

remain unchanged, and only r_1 changes to r'_1 . If, as pictured, P lies outside triangle $A_1A'_2A_3$, then $|r_1| > |r'_1|$ and $-r_1 < -r'_1$ since they both must be taken negative. If P lies inside triangle $A_1A'_2A_3$, we get $-r_1 < 0 < r'_1$. So in either case, using the appropriate sign, we have

$$\pm r'_1 + r_2 + r_3 \geq -r_1 + r_2 + r_3.$$

It therefore suffices to prove the extended theorem in the case where all three feet F_i of the distances r_i lie inside the triangle's sides or at its vertices, and when this occurs P lies inside the circumcircle. $\square$

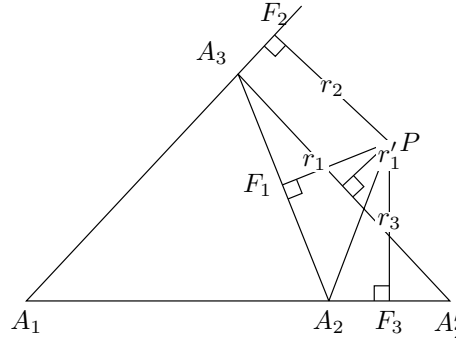


Figure 4.7: Reflection of a vertex across the perpendicular foot (Dodge's Figure 4).

A comprehensive computer run showed the theorem apparently true for all points inside the circumcircle, so all that remained was to prove the theorem when the point P lies outside the triangle and inside the circumcircle. Moreover, Case 2 eliminates a portion of even that region (when, say, P lies inside triangle $A_1A'_2A_3$, for the original Erdős inequality applies to that triangle). Let D be diametrically opposite vertex A_1 on the circumcircle of triangle $A_1A_2A_3$, as shown in Figure 4.8. Without loss of generality, we must prove the theorem whenever P lies within or on triangle A_2A_3D . We may assume that $A_2 < 90^\circ$ and $A_3 < 90^\circ$ since otherwise the indicated region is empty. In this region, since r_1 is to be given a negative sign, we must prove

$$R_1 + R_2 + R_3 + 2r_1 \geq 2r_2 + 2r_3. \quad [E010]$$

If Kazarinoff's statement that his proof holds whenever P lies inside the circumcircle had been substantiated, then the proof of Problem E2462 would have been complete at this point. The following cases, all developed in the past year, do complete the desired proof.

Case 3. Point P lies in triangle A_2A_3D and at least one of angles A_2 and A_3 does not exceed 30° .

Referring to Figure 4.9, let $A_2 \leq 30^\circ$, so that $\sin A_2 \leq 1/2$. If $\angle A_3A_2P = \varepsilon$, then

$$r_3 = R_2 \sin(A_2 + \varepsilon), \quad r_1 = R_2 \sin \varepsilon,$$

and

$$\begin{aligned} \sin(A_2 + \varepsilon) - \sin \varepsilon &= \sin A_2 \cos \varepsilon + \cos A_2 \sin \varepsilon - \sin \varepsilon \\ &= \sin A_2 \cos \varepsilon + \sin \varepsilon (\cos A_2 - 1) \\ &\leq \sin A_2 \leq \frac{1}{2}. \end{aligned}$$

Now

$$R_2 = PA_2 \geq PA_2 \{2 \sin(A_2 + \varepsilon) - 2 \sin \varepsilon\} = 2r_3 - 2r_1.$$

Hence

$$R_1 + R_2 + R_3 + 2r_1 \geq r_2 + (2r_3 - 2r_1) + r_2 + 2r_1 = 2r_2 + 2r_3. \quad \square$$

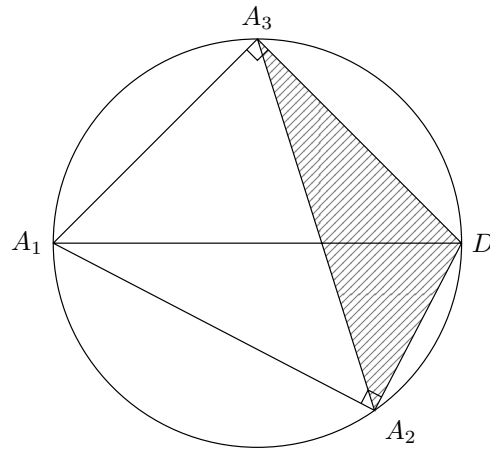


Figure 4.8: The remaining triangular region (Dodge's Figure 5).

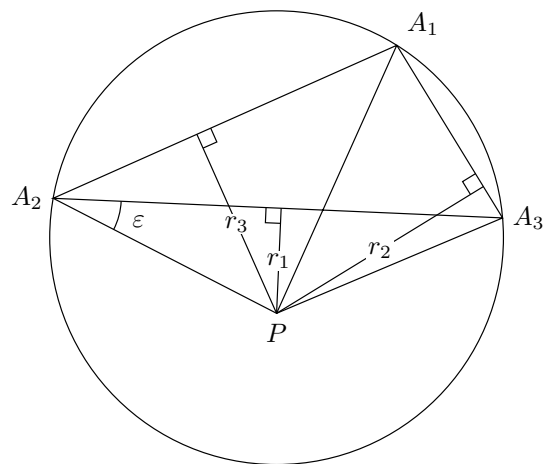


Figure 4.9: The case of an angle at most 30° (Dodge's Figure 6).

Case 4. Point P lies inside the largest angle of the triangle.

Let P lie in triangle A_2A_3D and suppose $A_1 \geq A_2 \geq A_3$. Then we have $a_1 \geq a_2 \geq a_3$ and $r_1 \leq r_2$, and also

$$2 \leq U \equiv \frac{a_2}{a_3} + \frac{a_3}{a_2} \leq V \equiv \frac{a_3}{a_1} + \frac{a_1}{a_3}.$$

Hence, if for some number N we have $Ur_1 = Vr_2 + N$, then

$$Ur_1 \geq Ur_2 + N$$

and, since also

$$(U - 2)r_1 \leq (U - 2)r_2,$$

we may subtract to get

$$2r_1 \geq 2r_2 + N.$$

Therefore, since we have, by the first two lines of (2),

$$R_1 + R_2 + R_3 + \left(\frac{a_2}{a_3} + \frac{a_3}{a_2}\right)r_1 \geq \left(\frac{a_3}{a_1} + \frac{a_1}{a_3}\right)r_2 + \left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right)r_3,$$

it follows that

$$R_1 + R_2 + R_3 + 2r_1 \geq 2r_2 + \left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right)r_3 \geq 2r_2 + 2r_3. \quad \square$$

Now only one case remains to be settled, but first we prove a pair of lemmas.

Lemma 1. The function

$$f(x) = 1 - \cos x - \frac{1}{2} \sin(x - 15^\circ)$$

has a minimum at approximately $(29^\circ, 0.004419)$,^[E011] so $f(x) > 0$ for all x in the interval $[15^\circ, 90^\circ]$. (See Figure 4.10.)

We have

$$f'(x) = \sin x - \frac{1}{2} \cos(x - 15^\circ),$$

which vanishes when

$$\tan x = \frac{\cos 15^\circ}{2 - \sin 15^\circ},$$

that is, when $x \approx 29.019466^\circ$, at which point $f(x) \approx 0.004419 > 0$. Since

$$f''(x) = \cos x + \frac{1}{2} \sin(x - 15^\circ) > 0,$$

the critical point is a minimum. $\square$

Lemma 2. If $1 \leq x \leq 2$, then $g(x) = x + 1/x \leq 2.5$.

Clearly $g'(x) \geq 0$ in the given interval, so $g(x) \leq g(2) = 2.5$. $\square$

Our last case takes P inside the triangle A_2A_3D of Figure 4.8, where $A_2 > 30^\circ$ and $A_3 > 30^\circ$ (by Case 3), and A_1 is not the largest angle of the triangle (by Case 4). We may without loss of generality assume that A_2 is the largest angle. Since we need not consider $A_2 \geq 90^\circ$ (by Figure 4.8), we have the following case:

Case 5. Point P lies inside triangle A_2A_3D , and $30^\circ < A_3 \leq A_2$ and $A_1 \leq A_2 < 90^\circ$. (See Figure 4.11.)

Let $\delta = \angle OA_3P$. Since $A_3 > 30^\circ$, we have $A_1 + A_2 < 150^\circ$ and $A_1 < 75^\circ$. So $\angle A_3OA_2 < 150^\circ$ and $\angle OA_3A_2 > 15^\circ$. From

$$\frac{a_2}{a_3} = \frac{\sin A_2}{\sin A_3} \quad \text{and} \quad 30^\circ < A_3 \leq A_2 < 90^\circ,$$

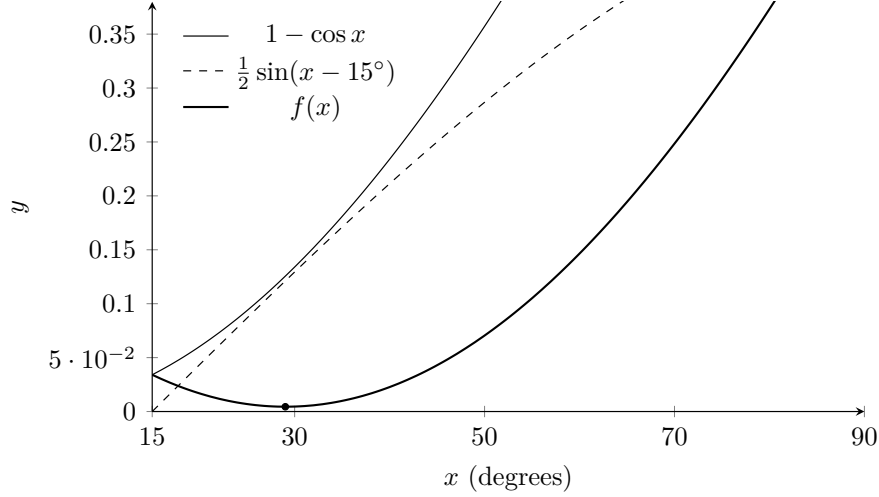


Figure 4.10: The three functions in Lemma 1 (Dodge's Figure 7).

we get

$$1 \leq \frac{a_2}{a_3} < \frac{\sin 90^\circ}{\sin 30^\circ} = 2,$$

and so, by Lemma 2,

$$\frac{a_2}{a_3} + \frac{a_3}{a_2} < 2.5.$$

Now $r_1 \leq R_3 \sin(\delta - 15^\circ)$, so, by Lemma 1,

$$R_3(1 - \cos \delta) \geq \frac{1}{2} R_3 \sin(\delta - 15^\circ) \geq \frac{1}{2} r_1.$$

Then

$$\begin{aligned} R_1 + R_2 + R_3 + 2r_1 &= R_1 + R_2 + R_3 \cos \delta + R_3(1 - \cos \delta) + 2r_1 \\ &\geq R_1 + R_2 + R_3 \cos \delta + 2.5r_1 \\ &\geq R_1 \cos(OA_1P) + R_2 \cos(OA_2P) + R_3 \cos \delta + \left(\frac{a_2}{a_3} + \frac{a_3}{a_2}\right) r_1 \\ &= \left(\frac{a_3}{a_1} + \frac{a_1}{a_3}\right) r_2 + \left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right) r_3 \quad \text{by (2)} \\ &\geq 2(r_2 + r_3), \end{aligned}$$

and we are at last finished. The proof of Problem E2462 is complete.

Finally, we use the extended Erdős–Mordell inequality for triangles to get, as a corollary, a corresponding result for convex quadrilaterals.

Let $A_1A_2A_3A_4$ be a convex quadrilateral, and let P be any point in its plane. We set $PA_i = R_i \geq 0^{[E012]}$ and denote the signed distance between P and line A_iA_j by r_{ij} , the sign being determined by Demir's convention for any triangle of which A_iA_j is a side. Thus (see Figure 4.12), r_{12} is associated with triangles $A_1A_2A_3$ and $A_1A_2A_4$ and has the same sign for both triangles regardless of the location of point P ; and similar statements can be made about r_{23}, r_{34} , and r_{41} . The distance $|r_{13}|$, on the other hand, is associated with triangles $A_1A_2A_3$ and $A_1A_3A_4$; and if r_{13} is the signed distance associated with triangle $A_1A_2A_3$, then $-r_{13}$ is the signed distance associated with triangle $A_1A_3A_4$. Similarly, if r_{24} corresponds to triangle $A_1A_2A_4$, then $-r_{24}$ corresponds to triangle $A_2A_3A_4$. Our inequality extended to quadrilaterals reads as follows:

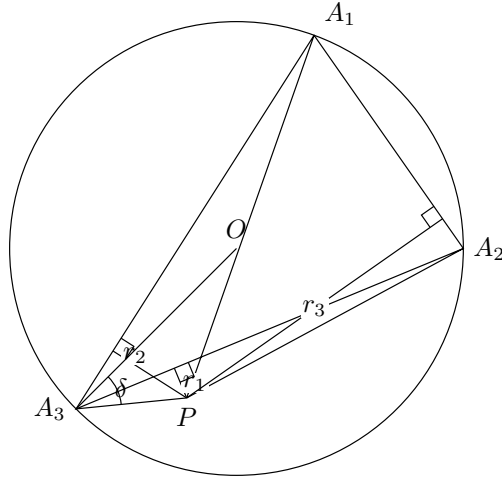


Figure 4.11: The final angular configuration (Dodge's Figure 8).

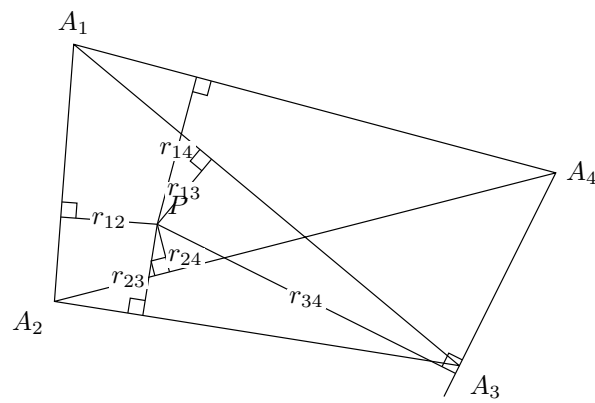


Figure 4.12: Signed distances for a convex quadrilateral (Dodge's Figure 9).

Corollary. *If $A_1A_2A_3A_4$ is a convex quadrilateral, P is any point in its plane, and the distances R_i and r_{ij} are as defined above, then*

$$3(R_1 + R_2 + R_3 + R_4) \geq 4(r_{12} + r_{23} + r_{34} + r_{41}). \quad (3)$$

Proof. We apply the extended Erdős–Mordell inequality successively to triangles $A_1A_2A_3$, $A_1A_2A_4$, $A_1A_3A_4$, and $A_2A_3A_4$:

$$R_1 + R_2 + R_3 \geq 2(r_{12} + r_{23} + r_{13}),$$

$$R_1 + R_2 + R_4 \geq 2(r_{12} + r_{24} + r_{41}),$$

$$R_1 + R_3 + R_4 \geq 2(r_{34} + r_{41} - r_{13}),$$

$$R_2 + R_3 + R_4 \geq 2(r_{23} + r_{34} - r_{24}),$$

and adding these four inequalities yields (3). $\square$

Reference

- [1] D. K. Kazarinoff, “A Simple Proof of the Erdős–Mordell Inequality for Triangles”, *Michigan Math. J.* **4** (1957), 97–98.

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E2625. A Property of Conics

American Mathematical Monthly **85** (1978), 121

Source PDF p. 64.

E2625 [1976, 812]. Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Let A_i ($i = 0, 1, 2, 3 \pmod{4}$) be four points on a circle Γ . Let t_i be the tangent to Γ at A_i and let p_i and q_i be the lines parallel to t_i passing through the points A_{i-1} and A_{i+1} , respectively. If $B_i = t_i \cap t_{i+1}$, $C_i = p_i \cap q_{i+1}$, show that the four lines B_iC_i have a common point.

Solution by Jordi Dou, Barcelona, Spain. We shall prove a more general result.

Theorem. *Let K be a non-degenerate conic in a real projective plane, A_i ($0 \leq i \leq 3$) be four distinct points on K and r be a line such that $A_i \notin r$. Let t_i be the tangent of K at A_i , $B_i = t_i \cap t_{i+1}$, $T_i = t_i \cap r$, $p_i = T_iA_{i-1}$, $q_i = T_iA_{i+1}$ and $C_i = p_i \cap q_{i+1}$. Then the four lines B_iC_i are concurrent.*

Proof. Let π be the polarity with respect to K and $S = A_0A_2 \cap A_1A_3$. Put $s = \pi(S)$ and $R = \pi(r)$. Let σ be the harmonic homology with center S and axis s . Thus we have $\sigma^2 = 1$ and $\sigma(A_i) = A_{i+2}$. We claim that the point $Q = \sigma(R)$ lies on each of the lines B_iC_i .

Note that π interchanges S and s and consequently σ and π commute. Therefore, $\tau = \sigma\pi = \pi\sigma$ is also a polarity. We have

$$\tau(t_i) = \sigma\pi(t_i) = \sigma(A_i) = A_{i+2},$$

$$\tau(t_{i+1}) = A_{i+3} = A_{i-1},$$

$$\tau(r) = \sigma\pi(r) = \sigma(R) = Q$$

and consequently the two triangles $T_{i+1}T_iB_i$ and $A_{i+2}A_{i-1}Q$ are polar to each other with respect to τ .^[E014] By Chasles’ theorem (see H. S. M. Coxeter, *The Real Projective Plane*, Cambridge University Press, 1961, p. 71) this is a pair of Desargues’ triangles. Hence the lines $T_{i+1}A_{i+2} = q_{i+1}$, $T_iA_{i-1} = p_i$ and B_iQ are concurrent at $C_i = p_i \cap q_{i+1}$. Therefore we see that Q lies on the lines B_iC_i as claimed. $\square$

The statement of the problem is obtained by choosing $K = \Gamma$ and $r = \text{line at infinity}$.

Also solved by L. Kuipers (Switzerland), and the proposer.

E3135. Matching Distances to Vertices

American Mathematical Monthly **95** (1988), 764

Source PDF p. 65.

E3135 [1986, 215]. Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

For a scalene triangle ABC inscribed in a circle, prove that there is a point D on the arc of the circle opposite to some vertex whose distance from this vertex is the sum of its distances from the other two vertices. Show how D may be constructed with straightedge and compass.

Solution I by J. Leech, University of Stirling, Scotland. Suppose $BC > AC > AB$. Perturb BAC into an isosceles triangle BXY by determining X on the ray BA and Y on BC such that $BX = BY = AC$. The circumcircle of BXY meets the circumcircle of BAC at the required point D . To prove D has the required property, extend AD to a point Z such that $DZ = DC$. Then triangle DZC is similar to triangle BXY , because the angles at D and B are both supplementary to angle ADC . Consequently, triangle DBY is congruent to triangle ZAC , since $AC = BY$ by construction, the angles at A and B are equal and angle $BDY = \text{angle } BXY = \text{angle } DZC$. Hence $DB = ZA = DA + DC$.

Solution II by P. Tzermias (student), University of Patras, Greece. Let a, b, c be the lengths of the sides opposite A, B, C , and let x, y, z be the lengths of DA, DB, DC . We seek point D such that $y = x + z$. By Ptolemy's Theorem, $ax + cz = by$. Substituting for y yields $x/z = (c - b)/(b - a)$. Thus, D lies on the "Circle of Apollonius" determined by A and C using the ratio $(c - b)/(b - a)$. This circle has a standard construction (see N. Altshiller-Court, *College Geometry*, 1952, p. 15).

Editorial note (original journal). Since $c - b$ and $b - a$ must have the same sign, D must lie on the arc cut by the side of intermediate length. Consequently, if ABC is isosceles, then D can only be the vertex common to the two equal sides. On the other hand, if ABC is equilateral, then D can be any point on the circumference; Leech's two circles then coincide.

Several solvers noted that the existence of D follows from the Intermediate Value Theorem. If D on the arc opposite B is close to the shortest side of ABC , then its distance to B is less than $DA + DC$, but if D is close to the longest side, then $DB > DA + DC$.

Other solutions independent of Ptolemy's Theorem were submitted by J. Dou (Spain), L. Kuipers (Switzerland), and by P. L. Hon (Hong Kong).

E. Morgantini (Italy) submitted a paper entitled "Una Quartica Bicircolare Della Geometria Del Triangolo" making reference to this problem.

In addition to the solvers mentioned above, correct solutions were received from S. Arslanagić (Yugoslavia), A. Bager, H. Eves, J. Fukuta (Japan), H. Kappus (W. Germany), O. P. Lossers (Netherlands), J. P. Robertson, J. S. Robertson, V. Schindler (E. Germany), R. A. Simon (Chile), B. A. Troesch, M. Vowe (Switzerland), and the proposer.

E3164. Elliptical Tangents

American Mathematical Monthly **95** (1988), 660

Source PDF p. 66.

E3164 [1986, 566]. Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Let s, t be the lengths of the tangent line segments to an ellipse from an exterior point. Find the extreme values of the ratio s/t .

Solution by Gene Arnold and Vaclav Konecny, Ferris State College, Big Rapids, MI. Consider an ellipse as the normal projection of a circle, from one plane to another in $\mathbb{R}^3$. Clearly tangents project to tangents and the minor axis of the ellipse is perpendicular to the intersection of the two planes. Since the ratio of any two intersecting tangents to the circle is 1, the extreme ratios of two such tangents to the ellipse will be attained when the ratio of one tangent to the circle to its projection is minimum while the other ratio is maximum. This happens when the projected tangents are respectively parallel to and perpendicular to the intersection of the planes. Thus the extreme ratios are those of the major axis to the minor axis, and its reciprocal.

Editorial note (original journal). M. S. Klamkin suggested study of the more difficult problem of the extreme values of $|s - t|$.

Also solved by M. Barr (Canada), J. C. Binz (Switzerland), J. M. Cohen, J. Dou (Spain), J. Fukuta (Japan), P. L. Hon (Hong Kong), L. R. King, M. S. Klamkin (Canada), K.-W. Lau (Hong Kong), O. P. Lossers (The Netherlands), M. Pachter (South Africa), K. Schilling, J. H. Steelman, P. Tracy, D. B. Tyler, C. Vandermee (The Netherlands), and the proposer. One incorrect solution was received.

E3422. Tangents Intersect on the Axis of Involution

American Mathematical Monthly **99** (1992), 679

Source PDF p. 66–68.

E3422 [1991, 158]. Proposed by H. Demir and C. Tezer, Middle East Technical University, Ankara, Turkey.

Suppose F and F' are points situated symmetrically with respect to the center of a given circle, and suppose S is a point on the circle not on the line FF' . Let P and P' be the second points of intersection of SF and SF' respectively with the circle. If the tangents to the circle at P and P' intersect at T , prove that the perpendicular bisector of FF' passes through the midpoint of the line segment ST .

Solution I by Jean-Pierre Grivaux, Paris, France. We work in the complex plane, with lower-case letters denoting the complex representations of points designated by the corresponding upper-case letters. We may assume that the circle is $\mathbf{U} = \{Z : |z| = 1\}$ and that the points F and F' are on the real axis.

If $A, B \in \mathbf{U}$, then Z is on the line through A and B if and only if $z + ab\bar{z} = a + b$, which we shall refer to as equation $\mathcal{E}_{ab}$. To derive this equation, note that the line is the set of Z whose numerical representation satisfies $z = a + r(b - a)$, where r is real. Conjugating this and using $\bar{a} = 1/a$ and $\bar{b} = 1/b$ yields $\bar{z} = \bar{a} + r(\bar{b} - \bar{a})$, which when multiplied by ab and added to the first equation yields $\mathcal{E}_{ab}$. This form of $\mathcal{E}_{ab}$ remains valid when $a = b$.

Since $\mathcal{E}_{pp}$ and $\mathcal{E}_{p'p'}$ are the equations of the tangents to $\mathbf{U}$ at P and P' , we have $t + p^2\bar{t} = 2p$ and $t + (p')^2\bar{t} = 2p'$. Solving for t by eliminating $\bar{t}$ (when $p \neq p'$) yields $t = 2/(\bar{p} + \bar{p}')$. Note

that $p + p' \neq 0$ because the tangents have a finite intersection T .^[E013] The midpoint of ST is Z , where

$$z = \frac{1}{2}(s + t) = \frac{1}{2} \left(s + \frac{2}{\bar{p} + \bar{p}'} \right),$$

and the result we want to prove is $z + \bar{z} = 0$, which by the above is

$$\left(s + \frac{2}{1/p + 1/p'} \right) + \left(\frac{1}{s} + \frac{2}{p + p'} \right) = 0.$$

This is equivalent by algebraic manipulation to

$$\left(\frac{-2s}{1 + s^2} \right) (1 + pp') = p + p'. \quad (*)$$

Since F and F' belong to the lines PS and $P'S$ respectively, f and $f' (= -f)$ satisfy the equations $\mathcal{E}_{ps}$ and $\mathcal{E}_{p's}$, respectively, namely $f(1 + ps) = p + s$ and $(-f)(1 + p's) = p' + s$,^[E013] where we use the fact that $\bar{f} = f$. Elimination of f from these two equations produces the desired equality (*), understood after clearing denominators when $1 + s^2 = 0$.^[E013]

Solution II by the proposers. We exclude the case in which F and F' coincide. Let K be the point diametrically opposite S . Let S' be the additional point where the line through S parallel to FF' intersects the circle (S' may coincide with S). The lines SS', SP, SK, SP' form a harmonic pencil, as the center of the circle bisects FF' . Consequently, for any point X on the circle, the lines XS', XP, XK, XP' form a harmonic pencil. Choosing $X = P$ or $X = P'$ in particular, we find that the pencils PS', PT, PK, PP' and $P'S', P'P, P'K, P'T$ are harmonic. Since the line PP' is common to both pencils, the points S', K, T lie on a line which is clearly perpendicular to FF' . Hence the perpendicular bisector of FF' bisects ST .

Editorial note (original journal). Most solvers used straightforward analytic geometry and brute force calculation to prove the result. Several used synthetic Euclidean geometry. H. Kappus gave another proof using complex numbers. O. P. Lossers gave another proof using projective geometry. A nice approach by J. Dou uses a classical property of projective involutions of a conic (involutions sending a conic to itself and preserving cross ratios). We briefly describe this and its relationship to Grivaux's solution, using the notational conventions of that solution.

The mapping $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ given by $\sigma(x) = -x$ yields a projective involution of the real line which extends to a projective involution of the real projective line $\mathbf{P}$ by defining $\sigma(\infty) = \infty$. With point S given on the unit circle $\mathbf{U}$, we define $\pi : \mathbf{P} \rightarrow \mathbf{U}$ by letting $\pi(X)$ be the point where the line joining S to $X \in \mathbf{P}$ again intersects $\mathbf{U}$. In particular, π applied to the point at infinity is the other point of intersection with $\mathbf{U}$ of the line through S parallel to the real axis. It then follows that the mapping $g : \mathbf{U} \rightarrow \mathbf{U}$ given by $g = \pi \circ \sigma \circ \pi^{-1}$ is a projective involution of $\mathbf{U}$.

The numbers corresponding to the fixed points of g are $-s$ and $-\bar{s}$, since σ fixes 0 and ∞ . Now a classical result of projective geometry implies that for each $P \in \mathbf{U}$, the tangent lines of $\mathbf{U}$ at P and $g(P)$ intersect on the line $\mathbf{l}$ through the fixed points of g . Since T is the intersection of the tangent lines at P and $P' = g(P)$, we see that T is on the line $\mathbf{l}$, and it is then obvious that the midpoint of ST is on the pure-imaginary axis, as was to be proved.

It is easy to calculate that $\pi(X)$ is represented by $(x - s)/(1 - xs)$ for $x \in \mathbb{R}$, and $g(P)$ is represented by $(\lambda - p)/(1 - \lambda p)$ for $p \in \mathbf{U}$, where $\lambda = -2s/(1 + s^2)$. In fact, λ is real (or ∞) and it represents the intersection of the tangent lines at the fixed points of g . Indeed, the relation (*) in Grivaux's solution expresses the fact that λ satisfies the equation $\mathcal{E}_{pp'}$; thus the line through P and P' always passes through Λ . This shows that the involution g is obtained by sending each point $P \in \mathbf{U}$ to the other point where $\mathbf{U}$ intersects the line through Λ and P . The line $\mathbf{l}$ (the "axis" of the involution g) is the polar of Λ with respect to $\mathbf{U}$.

For a detailed discussion of involutions of conics, see H. F. Baker, *An Introduction to Plane Geometry* (Cambridge University Press, 1943), Chapter IX, or M. Berger, *Geometry II* (Springer, 1987), Section 16.3. In Berger's book the above point Λ is called the "Frégier point" of the involution g .

Solved by 26 readers (including those cited) and the proposers.

E3469. Six Barycenters in Search of a Conic

American Mathematical Monthly **100** (1993), 875

Source PDF p. 69–70.

E3469 [1991, 955]. Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.

Suppose P is a point in the interior of triangle ABC and suppose AP, BP, CP meet the lines BC, CA, AB respectively at the points D, E, F . Prove that the centroids of the six triangles $PBD, PDC, PCE, PEA, PAF, PFB$ lie on a conic if and only if P lies on at least one of the three medians of the triangle.

Restatement of problem and fixing of notation. Applying the homothety with center P and ratio $3 : 2$ we see that the centroids of triangles are on a conic if and only if the midpoints of AF, FB, BD, DC, CE and EA are on one conic. Let x, y, z, u, v, w denote half the lengths of AF, FB, BD, DC, CE, EA , respectively. Let the midpoints of AF, FB, BD, DC, CE, EA be denoted by $1, 2, 3, 4, 5, 6$ respectively.^[E015]

Solution I by Victor Prasolov, Independent University of Moscow, Moscow, Russia. By Carnot's Theorem (see Howard W. Eves, *A survey of geometry* (Revised Edition), Allyn and Bacon, 1972, pages 256 and 262) the six centroids lie on a conic if and only if

$$x(2x + y)z(2z + u)v(2v + w) = w(2w + v)u(2u + z)y(2y + x). \quad (1)$$

By Ceva's Theorem, $xzv = wuy$, so (1) simplifies to $xzw + zvy + vxu - (wux + uyv + ywz) = 0$, or $(x - y)(z - u)(w - v) = 0$. This condition corresponds to P lying on a median.

Solution II by Albert Nijenhuis, Seattle, WA. By Pascal's Theorem, the points $1, 2, 3, 4, 5$, and 6 lie on a conic if and only if the three points $Q = AB \cap 45$, $R = BC \cap 61$ and $S = CA \cap 23$ are collinear. (There is no real difficulty if any of these points are at infinity. The ratio AQ/QB , for example, is replaced by -1 if $AB \parallel 45$.)

By Menelaus' Theorem, we have

$$\begin{aligned} \frac{AQ}{QB} \cdot \frac{2z + u}{u} \cdot \frac{v}{2w + v} &= -1, & \frac{BR}{RC} \cdot \frac{2v + w}{w} \cdot \frac{x}{2y + x} &= -1, \\ \frac{CS}{SA} \cdot \frac{2x + y}{y} \cdot \frac{z}{2u + z} &= -1. \end{aligned}$$

Multiplying these together and using Ceva's theorem, as in Solution I, we see that

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CS}{SA} = -1$$

if and only if $(x - y)(z - u)(w - v) = 0$. Thus Q, R, S are collinear and hence the points $1, 2, 3, 4, 5, 6$ lie on a conic if and only if P is on a median.

Comments by Neela Lakshmanan, University of Scranton, Scranton, PA. The restriction that P is interior to the triangle may be relaxed: we need only that P does not lie on any side of the triangle.

We can prove that the result is true not only for the midpoints but also for the points that divide each of those six segments in a constant ratio: If $1, 2, 3, 4, 5, 6$ are points on the sides of the triangle defined by

$$A1 : 1F = F2 : 2B = B3 : 3D = D4 : 4C = C5 : 5E = E6 : 6A,$$

then the six points lie on a conic if and only if P is on a median. Also, if P is an interior point, the hexagon $1, 2, 3, 4, 5, 6$ is convex and attains its maximum area when P is the centroid of $\triangle ABC$.

Editorial note (original journal). Many of the solvers supplemented the use of Carnot's Theorem or Pascal's Theorem with homogeneous coordinates and analytic methods. Some others worked directly with conditions on the six coefficients of a general conic.

Also solved by F. Bellot and M. A. López (Spain), R. J. Chapman (U.K.), J. Fukuta (Japan), H. Kappus (Switzerland), O. P. Lossers (The Netherlands), I. A. Sakmar (Turkey), Anchorage Math Solutions Group, and the proposer. One incorrect solution was received.

Contributed Solutions to Monthly Problems

4057. Euler Line

American Mathematical Monthly **51** (1944), 168

Source PDF p. 72.

4057 [1942, 616]. Proposed by J. R. Musselman, Western Reserve University.

Let B_1, B_2, B_3 be the points symmetric to the vertices of triangle $A_1A_2A_3$ in its circumcenter O , and let C_1, C_2, C_3 be the reflections of A_i in the perpendicular bisector of the sides of $A_1A_2A_3$. It is known that the circles $OB_1C_1, OB_2C_2, OB_3C_3$ meet at a point P . Show that P lies on the Euler line of $A_1A_2A_3$ and that O is the midpoint of PD , where D is the inverse in the circumcircle of the orthocenter H of $A_1A_2A_3$.

Solution by Hüseyin Demir, Columbia University. Let $G_1G_2G_3$ be the triangle formed by the straight lines A_iC_i so that $A_1A_2A_3$ is its medial triangle, the circumcircle (O) of the latter is its ninepoint circle, G_iC_i are its altitudes, its orthocenter H' is the symmetric of H with respect to O , and the straight lines C_iB_i are concurrent in H' . Let P be the point where the circle (OB_iC_i) cuts OH' , i.e., OH . We have $H'O \cdot H'P = H'C_i \cdot H'B_i = H'C_j \cdot H'B_j$; hence the circles (OB_iC_i) intersect again in P . The inverse of (OB_iC_i) with respect to (O) is B_iC_i , and hence $OH' \cdot OP = OC_i^2 = R^2$. Since $OH \cdot OD = R^2$ and $OH = H'O$, we must have $OD = PO$.

Also solved by H. Eves using inversion with respect to O and power $-R^2$ which gives a concise proof.

E1107. A Pencil of Planes Associated with a Tetrahedron

American Mathematical Monthly **61** (1954), 643

Source PDF p. 73.

E1107 [1954, 194]. Proposed by Victor Thébault, Tennesse, Sarthe, France.

On the edges AB, AC, AD of a tetrahedron $ABCD$ are marked points M, N, P such that $AB = nAM$, $AC = (n+1)AN$, $AD = (n+2)AP$. Show that the plane MNP contains a fixed line as n varies.

I. Solution by Hüseyin Demir, Zonguldak, Turkey. From the relations it is evident that the ranges of points $[M]$ and $[P]$ are projective. But since A is a self-corresponding element, the projectivity is a perspectivity. Hence MP is on a fixed point P' . Similarly MN is on a fixed point N' . Hence the plane MNP is on the fixed line $P'N'$.

E1142. Semi-vertical Angle of a Right Circular Cone

American Mathematical Monthly **62** (1955), 444

Source PDF p. 73.

E1142 [1954, 711]. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.

Find the semi-vertical angle of a right circular cone if three generating lines make angles of $2\alpha, 2\beta, 2\gamma$ with each other.

Demir gave the equivalent answer, with the numerical factor corrected here,^[E017]

$$\sin^2 \phi = - \frac{4 \sin^2 \alpha \sin^2 \beta \sin^2 \gamma}{\begin{vmatrix} 0 & \sin \alpha & \sin \beta & \sin \gamma \\ \sin \alpha & 0 & \sin \gamma & \sin \beta \\ \sin \beta & \sin \gamma & 0 & \sin \alpha \\ \sin \gamma & \sin \beta & \sin \alpha & 0 \end{vmatrix}}.$$

E1148. Two Equiareal Triangles

American Mathematical Monthly **62** (1955), 495

Source PDF p. 74.

E1148 [1955, 40]. Proposed by Victor Thébault, Tennie, Sarthe, France.

Let a, b, c be arbitrary points on the sides BC, CA, AB of triangle ABC , and let A', B', C' be the reflections of A, B, C in the midpoints of the segments bc, ca, ab . Show that triangles abc and $A'B'C'$ have equal areas.

Solution by Hüseyin Demir, Zonguldak, Turkey. Let a', b', c' be the reflections of a, b, c in the midpoints of BC, CA, AB . Since, by a well known property, abc and $a'b'c'$ have equal areas, we shall prove that $a'b'c'$ and $A'B'C'$ have equal areas. From

$$\overrightarrow{aB'} = \overrightarrow{Bc} = \overrightarrow{c'A}, \quad \overrightarrow{aC'} = \overrightarrow{Cb} = \overrightarrow{b'A}$$

we get $\overrightarrow{b'c'} = \overrightarrow{B'C'}$. Similarly $\overrightarrow{c'a'} = \overrightarrow{C'A'}$, $\overrightarrow{a'b'} = \overrightarrow{A'B'}$, and triangles $a'b'c'$ and $A'B'C'$ are actually congruent.

Also solved by W. B. Carver, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, C. S. Ogilvy, C. F. Pinzka, Roscoe Woods, and the proposer.

Pinzka called attention to two similar results in R. A. Johnson, *Modern Geometry* (1929), p. 80. Carver, Hyde, Ogilvy, and Woods gave simple solutions using oblique coordinates.

Editorial note (original journal). The above solution shows that triangles $a'b'c'$, $A'B'C'$ are not only congruent, but also homothetic. It follows that if a, b, c are collinear on a line L , then A', B', C' are also collinear on a line L' parallel to the reciprocal transversal of L . Consequently, if L is a Simson line of triangle ABC , then L and L' are perpendicular.

E1166. Chain of Circles in a Segment

American Mathematical Monthly **63** (1956), 42

Source PDF p. 75.

E1166 [1955, 364]. Proposed by Leon Bankoff, Los Angeles, Calif.

Let DE be a variable chord perpendicular to diameter AB of a given circle (O) . The maximum circle (ω_0) inscribed in the smaller segment, DEB , touches chord DE in C . The circle (ω_1) is tangent to (ω_0) , (O) , and DC and another circle (ω_2) is tangent to (ω_1) , (O) , and DC . Find the ratio BC/CA for which the radius of circle (ω_2) is a maximum.

Solution by Hüseyin Demir, Zonguldak, Turkey. Denote the radii of (O) and (ω_i) by R and r_i respectively. Let $(\omega_1), (\omega_2)$ touch CD in C_1, C_2 . Then we easily get

$$CC_1 = 2\sqrt{r_0 r_1}, \quad C_1 C_2 = 2\sqrt{r_1 r_2}.$$

From right triangles having hypotenuses $O\omega_1 = R - r_1$, $O\omega_2 = R - r_2$ we get

$$\begin{aligned} (1) \quad & (R - 2r_0 + r_1)^2 + 4r_0 r_1 = (R - r_1)^2, \\ (2) \quad & (R - 2r_0 + r_2)^2 + 4(\sqrt{r_0 r_1} + \sqrt{r_1 r_2})^2 = (R - r_2)^2. \end{aligned}$$

The value

$$r_1 = r_0(R - r_0)/R,$$

obtained from (1), when substituted in (2) yields

$$r_2 = (R - r_0)^2 r_0 / (R + r_0)^2.$$

Now, introducing $k = BC/CA = r_0/(R - r_0)$ and applying the derivative test for a maximum, we get

$$k = (\sqrt{5} - 1)/4.$$

Also solved by G. B. Charlesworth, Walter Guber, A. R. Hyde, R. B. Plymale, and the proposer. Some of these solutions were based upon a misinterpretation of the figure of the problem.

The proposer remarked that the problem was suggested by an attempt to display circle (ω_2) to best advantage in a diagram. The following interesting allied facts were pointed out by the proposer:

1. Circles (ω_0) and (ω_1) are maximum when C coincides with O , but (ω_2) is a maximum when $BC/CA = (\sqrt{5} - 1)/4$, with the unexpected consequence that CB is the side of a regular decagon inscribed in the circle on AC as diameter.
2. $r_{2(\max)} = r_1/2$.
3. $r_n = 2AB \cos^2 u / [\tan^n(u/2) + \cot^n(u/2)]^2$, u being the angle ABD (communicated to the proposer by Victor Thébault).
4. r_n is rational if AC and CB are rational.

E1687. An Application of Menelaus' Theorem*American Mathematical Monthly* **72** (1965), 425

Source PDF p. 76–77.

E1687 [1964, 430]. Proposed by Daniel Pedoe, Purdue University.

UVW is an equilateral triangle; A, B, C are the respective midpoints of the sides VW, WU, UV; A' is any point on line VW, B' any point on line WU, and C' any point on line UV. If P is the intersection of BC and B'C', Q of CA and C'A', R of AB and A'B', prove that (1) the lines A'P, B'Q, C'R are concurrent, (2) the areal coordinates of the point of concurrency with respect to triangle ABC are, with a suitable sign convention, $(AA')^{-1} : (BB')^{-1} : (CC')^{-1}$.

Generalize both (1) and (2) by means of an affine projection, and generalize (1) by a general projection.

I. Solution by Hüseyin Demir, Middle East Technical University, Ankara, Turkey. (1) We first set $BC = CA = AB = 1$ and consider VW, WU, UV ; $AA' = a'$, $BB' = b'$, $CC' = c'$ as directed segments. Let λ, μ, ν be the ratios in which P, Q, R divide the sides of $A'B'C'$. Applying the Menelaus theorem to the pair $UC'B', CB$ we get $(PB'/PC') \cdot (CC'/CU) \cdot (BU/BB') = 1$ or $\lambda(-c')(1/b') = 1$; i.e., $\lambda = -b'/c'$. Considering also two other pairs we get $\mu = -c'/a'$ and $\nu = -a'/b'$ which give $\lambda\mu\nu = -1$ proving the concurrency at a point T .

(2) We denote the areal coordinates of T by the matrices $(l' m' n')$ and $(l m n)$ in the triangles $A'B'C'$ and ABC respectively, and from $l' : n' = -\mu$, $m' : l' = -\nu$ we obtain

$$(\alpha) \quad l' : m' : n' = l' : \frac{a'}{b'}l' : \frac{a'}{c'}l' = \frac{1}{a'} : \frac{1}{b'} : \frac{1}{c'}.$$

Now to find $l : m : n = (l m n)$, let us first introduce the following symbol

$$LMN/XYZ = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

where the rows are the areal coordinates of L, M, N in the triangle XYZ .^[E016] We have

$$(\beta) \quad A'B'C'/UVW = \begin{bmatrix} 0 & 1-a' & 1+a' \\ 1+b' & 0 & 1-b' \\ 1-c' & 1+c' & 0 \end{bmatrix}, \quad UVW/ABC = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix},$$

and $T/A'B'C' = (1/a' \ 1/b' \ 1/c')$. With consistent normalization of the coordinates, it is not difficult to see the general identity

$$(\gamma) \quad T/ABC = (T/A'B'C') \cdot (A'B'C'/UVW) \cdot (UVW/ABC).$$

Substituting (α) and (β) in (γ) ,

$$\begin{aligned} (l \ m \ n) &= (l' \ m' \ n') \begin{bmatrix} 0 & 1-a' & 1+a' \\ 1+b' & 0 & 1-b' \\ 1-c' & 1+c' & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \\ &= (1/a' \ 1/b' \ 1/c') \begin{bmatrix} 2 & 2a' & -2a' \\ -2b' & 2 & 2b' \\ 2c' & -2c' & 2 \end{bmatrix} = (2/a' \ 2/b' \ 2/c'), \end{aligned}$$

$$(\delta) \quad T/ABC = l : m : n = 1/a' : 1/b' : 1/c' = (AA')^{-1} : (BB')^{-1} : (CC')^{-1}.$$

(3) An affine projection transforms UVW into an arbitrary triangle and ABC into its medial triangle in which the concurrency holds.

In an affine transformation the ratio of segments and the ratio of areas being preserved, replacing $AA'/1, BB'/1, CC'/1$ by $AA'/VW, BB'/WU, CC'/UV$, (δ) becomes

$$T/ABC = \left(\frac{AA'}{VW}\right)^{-1} : \left(\frac{BB'}{WU}\right)^{-1} : \left(\frac{CC'}{UV}\right)^{-1}.$$

By a general projection the perspective triangles UVW and ABC are transformed into such triangles. So the generalization is obtained for any UVW and for any inscribable triangles ABC and $A'B'C'$, such that AU, BV, CW are concurrent. Furthermore, in the affine setting above, the point T has the same areal coordinates in ABC and $A'B'C'$.^[E016]

Also solved by the proposer who points out that the generalization of (1) to general projections occurs as a problem in a 1909 Mathematical Tripos, Part I.

E2122. An Extension of Napoleon's Theorem

American Mathematical Monthly **76** (1969), 833

Source PDF p. 78.

E2122 [1968, 898]. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Let D, E, F be points in the plane of a nonequilateral triangle ABC so that triangles BDC, CEA, AFB are directly similar. Prove that triangle DEF is equilateral if and only if the three triangles are isosceles (with a side of triangle ABC as base) with base angles 30° . (The "if" part, Napoleon's theorem, is known. See the *Mathematics Magazine*, 1966, p. 166.)

Solution by Hüseyin Demir, Middle East Technical University, Ankara, Turkey. The following lemma is easily proved:

Lemma. A triangle in the complex plane with vertices a, b, c is equilateral if and only if $a^2 + b^2 + c^2 - bc - ca - ab = 0$.

Let $a, b, c; d, e, f$ be the affixes of the vertices of the triangles ABC, DEF . Since the triangles DBC, ECA, FAB are directly similar, then for some t

$$d = b + (c - b)t, \quad e = c + (a - c)t, \quad f = a + (b - a)t.$$

Forming the expression $U = d^2 + e^2 + f^2 - ef - fd - de$, we find

$$U = (a^2 + b^2 + c^2 - bc - ca - ab)(3t^2 - 3t + 1).$$

If ABC is equilateral, then by the lemma, $U = 0$, and again by lemma, DEF is equilateral. Now suppose that DEF is equilateral, that is $U = 0$ by lemma. Since ABC is supposed to be non-equilateral, we must have $3t^2 - 3t + 1 = 0$. Solving for t , we find $t = \frac{1}{3}\sqrt{3} \operatorname{cis}(\pm\pi/6)$ which proves the assertion.

Also solved by Walter Bluger, Slobodan Ćuk (Yugoslavia), M. G. Greening (Australia), L. Kuipers, C. F. Merrill, and the proposer. Jordi Dou (Spain) shows the uniqueness of the solution. A. W. Walker mentions a weaker result given in a paper by Wong, this *American Mathematical Monthly*, 48 (1941), p. 530.

E2398. A Result Known to Johnson

American Mathematical Monthly **81** (1974), 89

Source PDF p. 79.

E2398 [1973, 202]. Proposed by C. W. Dodge, University of Maine at Orono.

Prove that the point of intersection of the diagonals of a parallelogram lies on the pedal circle for any vertex with respect to the triangle formed by the other three vertices.

I. Solution by Hüseyin Demir, Middle East Technical University, Ankara, Turkey. Let $ABCD$ be a given parallelogram with I as center. Let the projections of D on sides BC, CA, AB of triangle ABC be A', B', C' , respectively. If $\angle D = \pi/2$, the pedal triangle degenerates into the Simson line AC containing the point I .

We give the proof in the case where $\angle D > \pi/2$ and A' is on the segment BC and C' is on the segment AB . Similar proofs may be given in other cases. We need only show that $\angle C'A'I = \angle AB'C'$. In obtaining this equality we use the properties that $AC'B'D$ and $DC'BA'$ are cyclic and triangle DIA' is isosceles. We have

$$\begin{aligned}\angle C'A'I &= \angle C'A'D - \angle IA'D = \angle C'BD - \angle IDA' = \angle A'DC \\ &= \pi/2 - \angle C = \pi/2 - \angle A = \angle ADC' = \angle AB'C'.\end{aligned}$$

II. Solution by A. W. Walker, Toronto, Canada. Let D be the reflection of the vertex A of triangle ABC in the midpoint M of the side BC . If BAC is a right triangle, the pedal “circle” of D for triangle ABC is the line BC ; if not, let E be the meet of the lines tangent to circle ABC at B and C . Then BD and BE are isogonal conjugate lines in the angle ABC , and likewise for CD and CE in angle BCA , so D and E are isogonal conjugate points in triangle ABC and therefore (R. A. Johnson, *Modern Geometry*, p. 155) have a common pedal circle passing through the projection M of E on BC .

Remark. E2398 is a special case of the theorem: For a plane non-orthocentric quadrangle $ABCD$ there are four pedal circles (and four nine-point circles) passing through the center of the rectangular hyperbola $ABCD$ (Johnson, p. 242).

Also solved by Günter Bach (Germany), Leon Bankoff, Howard Eves, Michael Goldberg, M. G. Greening (Australia), Lew Kowarski, L. Kuipers, and the proposer.

Proposals for Mathematics Magazine

The sixty-three proposals composed by Hüseyin Demir are reproduced below. The chronological reference list appears in Chapter 13. Corrections already made in the historical solutions, as well as new editorial corrections, are identified in Appendix A.

208

Mathematics Magazine **28** (1954–1955), 27

Source PDF p. 81.

Proposed by Hüseyin Demir, Zonguldak, Turkey

Evaluate the following trigonometric expressions without using numerical tables:

$$A = \cos 5^\circ \cos 10^\circ \cos 15^\circ \cdots \cos 75^\circ \cos 80^\circ \cos 85^\circ,$$

$$B = \cos 1^\circ \cos 3^\circ \cos 5^\circ \cdots \cos 85^\circ \cos 87^\circ \cos 89^\circ,$$

$$C = \cos 4^\circ \cos 8^\circ \cos 12^\circ \cdots \cos 80^\circ \cos 84^\circ \cos 88^\circ.$$

217

Mathematics Magazine **28** (1954–1955), 103

Source PDF p. 82.

Proposed by Hüseyin Demir, Zonguldak, Turkey

Prove that a necessary and sufficient condition for the convex polygon $A_1A_2A_3A_4$ to be inscriptable is that

$$D = \begin{vmatrix} A_1A_1 & A_1A_2 & A_1A_3 & A_1A_4 \\ A_2A_1 & A_2A_2 & A_2A_3 & A_2A_4 \\ A_3A_1 & A_3A_2 & A_3A_3 & A_3A_4 \\ A_4A_1 & A_4A_2 & A_4A_3 & A_4A_4 \end{vmatrix} = 0,$$

where A_iA_j denotes the distance between the vertices A_i and A_j if $j > i$, and $A_jA_i = -A_iA_j$.

227

Mathematics Magazine **28** (1954–1955), 160

Source PDF p. 82.

Proposed by Hüseyin Demir, Zonguldak, Turkey

Let A_1B_1 , A_2B_2 and A_3B_3 be three bars of lengths l_1, l_2, l_3 with weights W_1, W_2, W_3 , respectively. The ends B_1, B_2, B_3 rest on a horizontal surface while the other ends A_1, A_2, A_3 are supported by the bars A_3B_3 , A_1B_1 and A_2B_2 , respectively. Find the reactions R_1, R_2, R_3 at B_1, B_2, B_3 .

234

Mathematics Magazine **28** (1954–1955), 234

Source PDF p. 82.

Proposed by Hüseyin Demir, Zonguldak, Turkey

Given an m by n rectangular lattice containing mn points, find the total number of (a) squares, (b) rectangles having vertices at the points of the lattice. Consider $m \geq n$.

242

Mathematics Magazine **28** (1954–1955), 284

Source PDF p. 83.

Proposed by Hüseyin Demir, Zonguldak, Turkey

Let A', B', C' be the points dividing the sides of triangle ABC in the ratio k , and let A'', B'', C'' be the points dividing the sides of triangle $A'B'C'$ in the ratio $1/k$. Prove that the triangle $A''B''C''$ is homothetic with the original triangle ABC .

248

Mathematics Magazine **29** (1955–1956), 46

Source PDF p. 83.

Proposed by Hüseyin Demir, Zonguldak, Turkey

Let Γ_1 and Γ_2 be two plane curves. Let t be a variable line intersecting these curves at the points M_1, M_2 , where the tangents t_1 and t_2 to the curves are parallel to each other. Prove that the centers of curvature C_1 and C_2 of Γ_1 and Γ_2 at M_1 and M_2 are collinear with the characteristic point C of the straight line t .

258

Mathematics Magazine **29** (1955–1956), 163

Source PDF p. 83.

Proposed by Hüseyin Demir, Zonguldak, Turkey

A triangle ABC inscribed in a circle varies such that AB and AC keep fixed directions. Find the locus of the orthocenter H .

266

Mathematics Magazine **29** (1955–1956), 222

Source PDF p. 83.

Proposed by Hüseyin Demir, Zonguldak, Turkey

If M and M' are points inverse to each other with respect to the circumcircle of a triangle ABC ,

then prove the following for oriented angles, modulo 2π , wherever the angles are defined:^[E038]

$$\begin{aligned}\angle BMC + \angle BM'C &= 2\angle A, \\ \angle CMA + \angle CM'A &= 2\angle B, \\ \angle AMB + \angle AM'B &= 2\angle C.\end{aligned}$$

298

Mathematics Magazine **30** (1956–1957), 164

Source PDF p. 84.

Proposed by Hüseyin Demir, Kandilli, Bölgesi, Turkey

Let $y = f(x)$ be a curve with the following properties:

- (a) $f(x) = f(-x)$;
- (b) $f'(x) > 0$ for $x > 0$;
- (c) $f''(x) > 0$.

Determine the weight per unit length $w(x)$ at the point (x, y) such that when the curve is suspended under gravity by any two points on it, the curve will keep its original shape.

304

Mathematics Magazine **30** (1956–1957), 223

Source PDF p. 84.

Proposed by Hüseyin Demir, Kandilli Bölgesi, Turkey

Let ABC be a triangle, $AB \neq AC$, inscribed in a circle O , and let K be the point where the exterior angle bisector of A meets O . A variable circle with center at K meets AB, AC at E and F , respectively, such that A is not an interior point of KEF . Find the limiting position m of the common point M of EF, BC as EF approaches BC .

334

Mathematics Magazine **31** (1957–1958), 228

Source PDF p. 84.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Find the simplest expression for the area S enclosed by the arc AM of a cycloid, the arc TM of the rolling circle $\Omega(a)$ and the base line segment AT .

349

Mathematics Magazine **32** (1958–1959), 47

Source PDF p. 84.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

If $ABCD$, $AEBK$ and $CEFG$ are squares of the same orientations, prove that B bisects DF .

372

Mathematics Magazine **32** (1958–1959), 220

Source PDF p. 85.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Prove the identity^[E027]

$$\sin^2(\theta_1 + \theta_2 + \cdots + \theta_n) = \sin^2 \theta_1 + \cdots + \sin^2 \theta_n \\ + 2 \sum_{1 \leq i < j \leq n} \sin \theta_i \sin \theta_j \cos(\theta_i + 2\theta_{i+1} + \cdots + 2\theta_{j-1} + \theta_j).$$

380

Mathematics Magazine **32** (1958–1959), 278

Source PDF p. 85.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Solve the system of equations

$$\begin{aligned} x(z - a) + u(x + u) &= 0, \\ y(x - b) + u(y + u) &= 0, \\ z(y - c) + u(z + u) &= 0, \end{aligned}$$

where $abc \neq 0$ and $a^{-1} + b^{-1} + c^{-1} = u^{-1}$.

384

Mathematics Magazine **33** (1959–1960), 51

Source PDF p. 85.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Let (a_{ij}) be a nonsingular real matrix of n th order, $n \geq 2$, the sum of the elements of each of whose rows equals 1. Prove that the totality of such matrices forms a group of infinite order.^[E024]

398

Mathematics Magazine **33** (1959–1960), 165

Source PDF p. 85.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Determine the roots of the equations

$$x^2 + y_1x + y_2 = 0, \quad y^2 + x_1y + x_2 = 0,$$

where the coefficients (real numbers) in one equation are the roots of the other.

407

Mathematics Magazine **33** (1959–1960), 225

Source PDF p. 86.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

The twelve edges of a cube are made of wires of one ohm resistance each. The cube is inserted into an electrical circuit by:

- (a) two adjacent vertices;*
- (b) two opposite vertices of a face;*
- (c) two opposite vertices of the cube.*

Which offers the least resistance?

415

Mathematics Magazine **33** (1959–1960), 296

Source PDF p. 86.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Prove

$$\sum_{p=0}^n \binom{n}{p} \cos(px) \sin((n-p)x) = 2^{n-1} \sin nx.$$

419

Mathematics Magazine **34** (1960–1961), 49

Source PDF p. 86.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Determine the path in a vertical plane such that when a particle moved, under gravity, with an initial velocity v_0 from a point of the path, the particle maintained a constant speed along the path. Assume no friction.

425

Mathematics Magazine **34** (1960–1961), 109

Source PDF p. 86.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

If $n > 4$ and $n - 1$ and $n + 1$ are twin prime numbers, prove that $3\varphi(n) \leq n$, where φ denotes Euler's φ -function.^[E025]

437

Mathematics Magazine **34** (1961), 174

Source PDF p. 86.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Prove or disprove the statement: the number of odd coefficients in the binomial expansion of $(a + b)^n$ is $2^{s_2(n)}$, where $s_2(n)$ is the number of 1's appearing in the expression of n in the binary number system.^[E028]

440

Mathematics Magazine **34** (1961), 237

Source PDF p. 87.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Consider a packing of circles of radius r such that each is tangent to its six surrounding circles. Let a larger circle of radius R be drawn concentric with one of the small circles. If N is the number of small circles contained in the larger circle, prove that^[E026]

$$N = 1 + 6n + 6 \sum_{p=1}^n \left\lfloor \frac{\sqrt{4\rho^2 - 3p^2} - p}{2} \right\rfloor, \quad \rho = \frac{1}{2} \left(\frac{R}{r} - 1 \right), \quad n = \lfloor \rho \rfloor,$$

the floor brackets designating the greatest integer function.

458

Mathematics Magazine **34** (1961), 364

Source PDF p. 87.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

A student used De Moivre's theorem incorrectly as

$$(\sin \alpha + i \cos \alpha)^n = \sin n\alpha + i \cos n\alpha.$$

For what values of α does the equation hold for every integer n ?

472

Mathematics Magazine **35** (1962), 55

Source PDF p. 87.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Let (C) be a noncircular ellipse, or a hyperbola with semiaxes $a, b > 0$ such that $a^2 \neq 3b^2$ and $b^2 \neq 3a^2$. Let M be a variable point off the axes; take axis values by continuity.^[E049] Let T be the point symmetric to M with respect to the main axis, and t the tangent line at T . Denote the intersection of the perpendicular from M to t with the line joining T to the center of the conic by I . If M' is symmetric to M with respect to I , prove that:

1. The locus of M' is another conic (C') of the same kind as (C) .
2. The conics (C) and (C') are confocal.

487

Mathematics Magazine **35** (1962), 186

Source PDF p. 87.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Find the square root of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

498

Mathematics Magazine **35** (1962), 309

Source PDF p. 88.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

If m and n are positive integers and δ, D are their g.c.d. and l.c.m., respectively, and $d(n)$ denotes the number of divisors of n , $\varphi(n)$ being the Euler function, prove that

$$d(m)d(n) = d(\delta)d(D), \quad (1)$$

$$\varphi(m)\varphi(n) = \varphi(\delta)\varphi(D). \quad (2)$$

509

Mathematics Magazine **36** (1963), 133

Source PDF p. 88.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Solve the cryptarithm

$$\begin{array}{rcccccc} U & N & I & T & E & D \\ S & T & A & T & E & S \\ \hline A & M & E & R & I & C & A \end{array}$$

in the base 11, introducing the digit α .

517

Mathematics Magazine **36** (1963), 197

Source PDF p. 88.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Here the regions share the arc MN of the parabola, rather than the straight chord.^[E030] Let F and d be the focus and directrix of a parabola. If M and N are any two points on the parabola and M', N' are their respective projections on d , show that

$$\frac{\text{Area } FMN}{\text{Area } N'M'MN} = \text{Constant}.$$

529

Mathematics Magazine **36** (1963), 264

Source PDF p. 89.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

A cycloid (cardioid) rolls on a straight line without sliding. Prove that the locus of the center of curvature at the point of tangency lies on a circle (ellipse), and identify the portion traced by one arch (one circuit).^[E054]

537

Mathematics Magazine **37** (1964), 55

Source PDF p. 89.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Determine the relative positions of an equilateral triangle and a square inscribed in the same circle so that their common area shall be an extremum.

544

Mathematics Magazine **37** (1964), 119

Source PDF p. 89.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Solve the cryptarithm (alphametic)

$$ONE + TWO + SIX = NINE$$

in the base 10, with the following conditions:

(a) $ONE < TWO < SIX$;

(b) $2 \mid TWO$, $6 \mid SIX$, $9 \mid NINE$, where $a \mid b$ means “ a divides b ”.

563

Mathematics Magazine **37** (1964), 276

Source PDF p. 89.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let A, B', A', B be four consecutive vertices of a regular hexagon. If M is in the open minor arc $A'B'$ of the circumcircle^[E057] and MA, MB intersect BB' and AA' in the points E and F , respectively, then prove that:

$$\angle MEF = 3\angle MAF, \quad (\text{a})$$

$$\angle MFE = 3\angle MBE. \quad (\text{b})$$

572

Mathematics Magazine **38** (1965), 52

Source PDF p. 90.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

To the memory of President Kennedy. Mr. J. F. Kennedy was killed on November 22, 1963. That is, on the day 11-22-1963. Solve the cryptarithm

$$JF \cdot (KEN + NEDY) = (11 + 22) \cdot 1963$$

in the decimal system.

587

Mathematics Magazine **38** (1965), 179

Source PDF p. 90.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Prove the following inequality:^[E029]

$$\left(\frac{\theta + \sin \theta}{\pi} \right)^2 + \cos^4 \frac{\theta}{2} < 1, \quad 0 < |\theta| < \pi.$$

599

Mathematics Magazine **38** (1965), 241

Source PDF p. 90.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

If a, b, c are any three vectors in 3-space, then show that the vectors

$$a \times (b \times c), \quad b \times (c \times a), \quad c \times (a \times b)$$

are linearly dependent.

600

Mathematics Magazine **38** (1965), 317

Source PDF p. 90.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

If the area of a triangle ABC is S and the areas of the in- and ex-contact triangles are T, T_a, T_b, T_c , then show that

$$T_a + T_b + T_c - T = 2S, \quad (1)$$

$$T_a^{-1} + T_b^{-1} + T_c^{-1} - T^{-1} = 0. \quad (2)$$

609

Mathematics Magazine **39** (1966), 69

Source PDF p. 91.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Solve the following cryptarithm in the decimal system:

$$4 \cdot NINE = 9 \cdot FOUR.$$

628

Mathematics Magazine **39** (1966), 246

Source PDF p. 91.

Proposed by B. Suer and Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Solve the alphametic

$$\text{COS}^2 + \text{SIN}^2 = \text{UNO}^2$$

in the decimal system.

639

Mathematics Magazine **39** (1966), 306

Source PDF p. 91.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let $ABCD$ be a convex quadrangle and P be the intersection of diagonals AC and BD . Let the distance of P from the sides $AB = a$, $BC = b$, $CD = c$, $DA = d$ be x, y, z, t , respectively. Prove that

$$x + y + z + t < \frac{3}{4}(a + b + c + d).$$

649

Mathematics Magazine **40** (1967), 100

Source PDF p. 91–92.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Solve the cryptarithm

$$\begin{array}{rcccccc} & T & H & R & E & E & \\ + & & F & O & U & R & \\ \hline & S & E & V & E & N & \end{array}$$

in the decimal system such that:

- 3 does not divide *THREE*, in which the digit 3 is missing;
- 4 does not divide *FOUR*, in which the digit 4 is missing;
- 7 does not divide *SEVEN*, in which the digit 7 is missing.

680

Mathematics Magazine **41** (1968), 42

Source PDF p. 92.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let E be an ellipse and t', t'' be two variable parallel tangents to it. Consider a circle C , tangent to t', t'' and to E externally. Show that the locus of the center of C is a circle.

724

Mathematics Magazine **42** (1969), 96

Source PDF p. 92.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Find the probability that for a point P taken at random in the interior of a triangle ABC ($a \geq b \geq c$), the distances of P from the sides of ABC form the lengths of sides of a triangle.

738

Mathematics Magazine **42** (1969), 214

Source PDF p. 92.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Historical data (inconsistent for a positive-width river).^[E061] There is a river with parallel and straight shores. A is located on one shore and B on the other, with $AB = 72$ miles. A ferry boat travels the straight path AB from A to B in four hours and from B to A in nine hours. If the boat's speed on still water is $v = 13$ mph, what is the velocity of the flow?

743

Mathematics Magazine **42** (1969), 267

Source PDF p. 93.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Historical proposal (false as printed).^[E062] Let P be an interior point of a regular tetrahedron, $T \equiv A_1A_2A_3A_4$, with $p_i = PA_i$, and let x_{ij} denote the distance of P from the edge A_iA_j . Then prove

$$\sum_{i=1}^4 p_i \geq \frac{2\sqrt{3}}{3} \sum_{i < j} x_{ij},$$

equality holding if and only if P is at the center O of T .

756

Mathematics Magazine **43** (1970), 103

Source PDF p. 93.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Determine closed and centrally symmetric curves C , other than circles, such that the product of two perpendicular radius vectors (issued from the center) be a constant.

763

Mathematics Magazine **43** (1970), 166

Source PDF p. 93.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Prove:

$$\left(1 + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \cdots\right) = \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) \left(1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \cdots\right).$$

775

Mathematics Magazine **43** (1970), 278

Source PDF p. 93.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Prove

$$\int_0^1 \sqrt[p]{1-x^p} dx = \int_0^1 \sqrt[q]{1-x^q} dx, \quad p, q > 0.$$

806*Mathematics Magazine* **44** (1971), 228

Source PDF p. 94.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let H be the orthocenter of a non-right isosceles triangle ABC , and let AH, BH, CH intersect the opposite sides in D, E, F , respectively. Prove that the incenters of the right triangles $HBD, HDC, HCE, HEA, HAF, HFB$ lie on a conic, possibly degenerate.^[E065]

839*Mathematics Magazine* **45** (1972), 228

Source PDF p. 94.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Given three boxes each containing w white balls and r red balls identical in shape. Take a ball from the first box and put it in the second box, then take a ball from the second box and put it in the third, and finally take a ball from the third box and put it in the first. Find the probability that the boxes have their original contents as to color.

859*Mathematics Magazine* **46** (1973), 103

Source PDF p. 94.

*Proposed by B. Suer and H. Demir, Middle East Technical University, Ankara, Turkey**Solve the cryptarithm*

$$THREE + NINE = EIGHT + FOUR.$$

916*Mathematics Magazine* **47** (1974), 286

Source PDF p. 94.

Proposed by H. Demir, M.E.T.U., Ankara, Turkey

Let XYZ be the pedal triangle of a point P with regard to the triangle ABC . Then find the trilinear coordinates x, y, z of P such that

$$YA + AZ = ZB + BX = XC + CY.$$

963*Mathematics Magazine* **49** (1976), 43

Source PDF p. 95.

*Proposed by Hüseyin Demir, Ankara, Turkey**Characterize convex quadrilaterals with sides a, b, c, d such that*

$$\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} = 0.$$

998*Mathematics Magazine* **49** (1976), 252

Source PDF p. 95.

*Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey**Characterize all triangles in which the triangle whose vertices are the feet of the internal angle bisectors is a right triangle.***1197***Mathematics Magazine* **57** (1984), 238

Source PDF p. 95.

*Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey**Characterize the triangles of which the midpoints of the altitudes are collinear.***1206***Mathematics Magazine* **58** (1985), 46

Source PDF p. 95.

*Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey**Let ABC be a triangle with sides a, b, c and semiperimeter s . Let the side BC be subdivided using the points $B = P_0, P_1, \dots, P_{n-1}, P_n = C$ in order. If r_i is the inradius of triangle $AP_{i-1}P_i$ for $i = 1, \dots, n$, prove that*

$$r_1 + \dots + r_n < \frac{1}{2}h_a \ln \frac{s}{s-a},$$

where h_a is the length of the altitude from vertex A .

1211

Mathematics Magazine **58** (1985), 111

Source PDF p. 95.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Find the locus of points under which an ellipse is seen under a constant angle.

1298

Mathematics Magazine **61** (1988), 195

Source PDF p. 96.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

A quadrilateral $ABCD$ is circumscribed about a circle, and P, Q, R, S are the points of tangency of sides AB, BC, CD, DA , respectively. Let $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$, and $p = |QS|$, $q = |PR|$. Show that

$$\frac{ac}{p^2} = \frac{bd}{q^2}.$$

1305

Mathematics Magazine **61** (1988), 261

Source PDF p. 96.

Proposed by H. Demir and C. Tezer, Middle East Technical University, Ankara, Turkey

Let $P_0 = B, P_1, P_2, \dots, P_n = C$ be points, taken in that order, on the side BC of the triangle ABC . If r, r_i, h denote, respectively, the inradii of the triangles $ABC, AP_{i-1}P_i$ and the common altitude, prove that

$$\prod_{i=1}^n \left(1 - \frac{2r_i}{h}\right) = 1 - \frac{2r}{h}.$$

1327

Mathematics Magazine **62** (1989), 274

Source PDF p. 96.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let the sides PQ, QR, RS, SP of a convex quadrangle $PQRS$ touch an inscribed circle at A, B, C, D and let the midpoints of the sides AB, BC, CD, DA be E, F, G, H . Show that the angle between the diagonals PR, QS is equal to the angle between the bimedians EG, FH .

1356*Mathematics Magazine* **63** (1990), 274

Source PDF p. 96.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let P, Q be points taken on the side BC of a triangle ABC , in the order B, P, Q, C . Let the circumcircles of PAB, QAC intersect at M ($\neq A$) and those of PAC, QAB at $N \neq A$, assuming both distinct second intersections exist.^[E070] Show that A, M, N are collinear if and only if P and Q are symmetric in the midpoint A' of BC .

1371*Mathematics Magazine* **64** (1991), 132

Source PDF p. 97.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let A, B, C be vertices of a triangle and let D, E, F be points on the sides BC, AC, AB , respectively. Let U, X, V, Y, W, Z be the midpoints of, respectively, BD, DC, CE, EA, AF, FB . Prove that

$$\text{Area}(\triangle UVW) + \text{Area}(\triangle XYZ) - \frac{1}{2} \text{Area}(\triangle DEF)$$

is a constant independent of D, E, F .

1377*Mathematics Magazine* **64** (1991), 197

Source PDF p. 97.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let D, E, F lie on the side segments BC, CA, AB , respectively, of triangle ABC , and let U, X, V, Y, W, Z be the midpoints of the line segments BD, DC, CE, EA, AF, FB , respectively. Show that the expression

$$|UVW| + |XYZ| - \frac{1}{2}|DEF|$$

for areas is constant.^[E071]

1405*Mathematics Magazine* **65** (1992), 265

Source PDF p. 97.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Two circles inscribed in distinct angles of a triangle are isogonally related if the tangents from the third vertex not coinciding with the sides are symmetric with respect to the bisector of the third angle. Given three circles inscribed in distinct angles of a triangle, prove that if any two of the three pairs of circles are isogonally related then so is the third pair.

Solutions of Proposals

208

Mathematics Magazine **28** (1954–1955), 27

Source PDF p. 98–99.

I. Solution by E. P. Starke, Rutgers University.

$$\begin{aligned}\sqrt{2}A &= \cos 5^\circ \cos 10^\circ \cdots \cos 40^\circ \sin 40^\circ \cdots \sin 10^\circ \sin 5^\circ \\ &= \frac{\sin 10^\circ \sin 20^\circ \cdots \sin 80^\circ}{2^8},\end{aligned}$$

by use of the double-angle formula. A repetition of the same device gives

$$2^{12}\sqrt{2}A = \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{\sqrt{3}}{2}k,$$

say, where

$$\begin{aligned}k &= \sin 20^\circ (\sin 40^\circ \sin 80^\circ) = \sin 20^\circ (\sin^2 60^\circ - \sin^2 20^\circ) \\ &= \frac{1}{4}(3 \sin 20^\circ - 4 \sin^3 20^\circ) = \frac{1}{4} \sin 60^\circ = \frac{\sqrt{3}}{8}.\end{aligned}$$

So $A = 3 \cdot 2^{-33/2}$. Similarly,

$$B = \cos 1^\circ \sin 1^\circ \cos 3^\circ \sin 3^\circ \cdots \cos 43^\circ \sin 43^\circ \cos 45^\circ,$$

and $2^{22}\sqrt{2}B = \sin 2^\circ \sin 6^\circ \sin 10^\circ \cdots \sin 86^\circ = C$. Let $x = \cos 2^\circ \cos 6^\circ \cos 10^\circ \cdots \cos 86^\circ$. Then

$$\begin{aligned}2^{22} \cdot 2^{22}\sqrt{2}B x &= \sin 4^\circ \sin 12^\circ \sin 20^\circ \cdots \sin 172^\circ \\ &= \sin 4^\circ \sin 8^\circ \sin 12^\circ \sin 16^\circ \cdots \sin 88^\circ = x,\end{aligned}$$

whence $B = 2^{-89/2}$ and $C = 2^{-22}$.

II. Solution by H. M. Feldman, Washington University, St. Louis, Missouri. From the identities^[E031]

$$\begin{aligned}x^{2n} - 1 &= (x^2 - 1)(x^{2n-2} + x^{2n-4} + \cdots + 1) \\ &= (x^2 - 1) \prod_{k=1}^{n-1} \left(x^2 - 2x \cos \frac{k\pi}{n} + 1 \right), \\ x^{2n+1} + 1 &= (x + 1)(x^{2n} - x^{2n-1} + \cdots + 1) \\ &= (x + 1) \prod_{k=1}^n \left(x^2 - 2x \cos \frac{(2k-1)\pi}{2n+1} + 1 \right),\end{aligned}$$

we get, by letting $x = \pm 1$, the following relations:

$$\begin{aligned}\prod_{k=1}^{n-1} \sin \frac{k\pi}{2n} &= \prod_{k=1}^{n-1} \cos \frac{k\pi}{2n} = 2^{-n+1} \sqrt{n}, \\ \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} &= 2^{-n+1} n, \\ \prod_{k=1}^n \sin \frac{(2k-1)\pi}{2(2n+1)} &= 2^{-n}, \\ \prod_{k=1}^n \cos \frac{(2k-1)\pi}{2(2n+1)} &= \prod_{k=1}^n \sin \frac{(2k-1)\pi}{2n+1} = 2^{-n} \sqrt{2n+1}.\end{aligned}$$

By means of these relations, we find

$$A = 2^{-17}(3\sqrt{2}), \quad B = 2^{-45}\sqrt{2}, \quad C = 2^{-22}.$$

Also solved by Leon Bankoff, Los Angeles, California; Kwan Moon (partially), Mississippi State College; George Mott, Republic Aviation Corp., New York; T. F. Mulcrone, St. Charles College, Louisiana; L. A. Ringenberg, Eastern Illinois State College; Chih-yi Wang, University of Minnesota; Hazel S. Wilson, Jacksonville State College, Alabama; and the proposer.

217

Mathematics Magazine **28** (1954–1955), 103

Source PDF p. 99.

Solution by H. M. Feldman, St. Louis, Missouri. Since $A_i A_i$ must clearly be zero, the determinant is skew-symmetric and its value is ^[E035]

$$D = [(A_1 A_2)(A_3 A_4) + (A_1 A_4)(A_2 A_3) - (A_1 A_3)(A_2 A_4)]^2.$$

The vanishing of the expression within the brackets is a necessary and sufficient condition for the quadrilateral to be inscriptable in a circle (Ptolemy's theorem).

Also solved by Ben K. Gold, Los Angeles City College; M. S. Klamkin, Polytechnic Institute of Brooklyn; E. P. Starke, Rutgers University; Chih-yi Wang, University of Minnesota; and the proposer.

227: Forces in Equilibrium

Mathematics Magazine **28** (1954–1955), 160; proposal January 1955

Source PDF p. 100–101.

Proposed by Hüseyin Demir, Zonguldak, Turkey

Let $A_1 B_1$, $A_2 B_2$ and $A_3 B_3$ be three bars of lengths l_1, l_2, l_3 with weights W_1, W_2, W_3 , respectively. The ends B_1, B_2, B_3 rest on a horizontal surface while the other ends A_1, A_2, A_3 are supported by the bars $A_3 B_3$, $A_1 B_1$ and $A_2 B_2$, respectively. Find the reactions R_1, R_2, R_3 at B_1, B_2, B_3 .

Solution by the proposer. Let the reactions of the bars at the ends A_1, A_2, A_3 be denoted by r_1, r_2, r_3 and the lengths $A_1A_2, A_2B_1; A_2A_3, A_3B_2; A_3A_1, A_1B_3$ by $a_1, b_1; a_2, b_2; a_3, b_3$, respectively. Then considering the equilibrium of one of the bars, say A_1B_1 , we have by taking moments of the forces r_1, r_2, W_1, R_1 at the point B_1 :

$$\frac{1}{2}l_1W_1 - l_1r_1 + b_1r_2 = 0.$$

Setting $b_i = k_i l_i$ ($i = 1, 2, 3$) and considering the other equations corresponding to the two other bars, we get the system of equations with unknowns r_1, r_2, r_3 :

$$r_1 - k_1 r_2 = \frac{1}{2}W_1, \quad r_2 - k_2 r_3 = \frac{1}{2}W_2, \quad r_3 - k_3 r_1 = \frac{1}{2}W_3.$$

The determinant of this system being

$$D = \begin{vmatrix} 1 & -k_1 & 0 \\ 0 & 1 & -k_2 \\ -k_3 & 0 & 1 \end{vmatrix} = 1 - k_1 k_2 k_3,$$

we have

$$\begin{aligned} r_1 &= (W_1 + k_1 W_2 + k_1 k_2 W_3)/(2D), \\ r_2 &= (W_2 + k_2 W_3 + k_2 k_3 W_1)/(2D), \\ r_3 &= (W_3 + k_3 W_1 + k_3 k_1 W_2)/(2D), \end{aligned}$$

and

$$\begin{aligned} R_1 &= W_1 - r_1 + r_2 \\ &= W_1 - \frac{W_1 + k_1 W_2 + k_1 k_2 W_3}{2D} + \frac{W_2 + k_2 W_3 + k_2 k_3 W_1}{2D}. \end{aligned}$$

Thus^[E032]

$$\begin{aligned} R_1 &= \frac{(1 + k_2 k_3 - 2k_1 k_2 k_3)W_1 + (1 - k_1)W_2 + k_2(1 - k_1)W_3}{2(1 - k_1 k_2 k_3)}, \\ R_2 &= \frac{k_3(1 - k_2)W_1 + (1 + k_3 k_1 - 2k_1 k_2 k_3)W_2 + (1 - k_2)W_3}{2(1 - k_1 k_2 k_3)}, \\ R_3 &= \frac{(1 - k_3)W_1 + k_1(1 - k_3)W_2 + (1 + k_1 k_2 - 2k_1 k_2 k_3)W_3}{2(1 - k_1 k_2 k_3)}. \end{aligned}$$

Also solved by George R. Mott, Republic Aviation Company.

234

Mathematics Magazine **28** (1954–1955), 234

Source PDF p. 102.

Solution by the proposer. We distinguish two kinds of squares. A square is an N - or L -square according as its sides are or are not parallel to the sides of the lattice.

Every L -square is inscribed in a unique N -square. By a $p \times p$ N -square we mean one having p points on each of its sides. In such a square are inscribed evidently $p - 2$ L -squares. Including the N -square itself the number is $p - 1$.

The number of $p \times p$ N -squares is easily seen to be $(m - p + 1)(n - p + 1)$. Hence the number of $p \times p$ N -squares together with L -squares inscribed in them is $(p - 1)(m - p + 1)(n - p + 1)$. Hence the required total number of squares is given by^[E033]

$$\begin{aligned}
 N &= \sum_{p=2}^n (p-1)(m-p+1)(n-p+1) \\
 &= mn \sum_{p=2}^n (p-1) - (m+n) \sum_{p=2}^n (p-1)^2 + \sum_{p=2}^n (p-1)^3 \\
 &= mn \frac{n(n-1)}{2} - (m+n) \frac{n(n-1)(2n-1)}{6} + \frac{n^2(n-1)^2}{4} \\
 &= \frac{n(n-1)}{12} [6mn - 2(m+n)(2n-1) + 3n(n-1)] \\
 &= \frac{n(n^2-1)(2m-n)}{12}.
 \end{aligned}$$

Editorial note (original journal). No solution of the rectangular case has been received. Solutions restricting the squares and rectangles to those with sides parallel to the lines of lattice points were received from Julian H. Braun, White Sands Proving Ground, and E. P. Starke, Rutgers University. Braun noted that the restricted case was a variation of Problem E1127 of the *American Mathematical Monthly*.

Editorial completion of part (b). The total number of rectangles, including squares, is

$$\binom{m}{2} \binom{n}{2} + \sum_{\substack{p,q \geq 1 \\ \gcd(p,q)=1}} \sum_{\substack{a,b \geq 1 \\ ap+bq < m \\ aq+bp < n}} (m - ap - bq)(n - aq - bp).$$

The first term counts rectangles with horizontal and vertical sides. Every other rectangle has a unique positive-slope primitive direction (p, q) and perpendicular direction $(-q, p)$. Its adjacent edge vectors have lengths represented by $a(p, q)$ and $b(-q, p)$, with positive integers a, b . The bounding box has width $ap + bq$ and height $aq + bp$, so it has precisely the displayed number of translations in the lattice. This counts each tilted rectangle once.

242

Mathematics Magazine **28** (1954–1955), 284

Source PDF p. 102–103.

Solution by P. W. Allen Raine, Newport News High School, Newport News, Virginia. Let $A, B, C, A', B', C', A'', B'', C''$ represent the vector coordinates of the respective points and k a scalar quantity. Thus

$$A' = \frac{kB + C}{k+1}, \quad B' = \frac{kC + A}{k+1}, \quad C' = \frac{kA + B}{k+1},$$

and^[E034]

$$\begin{aligned}
 A'' &= \frac{B' + kC'}{1+k} = \frac{kC + A + k^2A + kB}{(1+k)^2}, \\
 B'' &= \frac{C' + kA'}{1+k} = \frac{kA + B + k^2B + kC}{(1+k)^2}, \\
 C'' &= \frac{A' + kB'}{1+k} = \frac{kB + C + k^2C + kA}{(1+k)^2}.
 \end{aligned}$$

Now we can easily show that

$$\begin{aligned} A'' - B'' &= \frac{1 - k + k^2}{(1 + k)^2} (A - B), \\ B'' - C'' &= \frac{1 - k + k^2}{(1 + k)^2} (B - C), \\ C'' - A'' &= \frac{1 - k + k^2}{(1 + k)^2} (C - A), \end{aligned}$$

which tells us that the sides of the two triangles are parallel and hence the triangles are homothetic, the homothetic ratio being $(1 - k + k^2)/(1 + k)^2$.

Also solved by Maimouna Edy, Hull, P. Q., Canada; M. S. Klamkin, Polytechnic Institute of Brooklyn; Chih-yi Wang, University of Minnesota; and the proposer.

248

Mathematics Magazine **29** (1955–1956), 46

Source PDF p. 104.

Solution by the proposer. Considering the new position t' of t very close to t , we have

$$\frac{M_1 M'_1}{\sin \Delta\theta} = \frac{C' M'_1}{\sin \mu_1},$$

where M'_1 is close to M_1 on Γ_1 , and $\Delta\theta = (t, t')$, the angle between t and t' . Infinitesimally,

$$\frac{ds_1}{d\theta} = \frac{CM_1}{\sin \mu_1}, \quad \mu_1 = (t, t_1) \pm \pi,$$

and similarly

$$\frac{ds_2}{d\theta} = \frac{CM_2}{\sin \mu_2}, \quad \mu_2 = (t, t_2) \pm \pi.$$

Having $\sin \mu_1 = \sin \mu_2$, as t_1 is parallel to t_2 , we get

$$\frac{ds_1}{CM_1} = \frac{ds_2}{CM_2},$$

which in turn yields^[E036]

$$\frac{ds_1/d\alpha}{CM_1} = \frac{ds_2/d\alpha}{CM_2}, \quad \text{i.e.} \quad \frac{R_1}{CM_1} = \frac{R_2}{CM_2},$$

where $d\alpha$ is the infinitesimal angle relative to the parallel normals at M_1, M_2 , and R_1, R_2 the corresponding radii of curvature. The last equality proves the statement.

Also solved by Richard K. Guy, University of Malaya, Singapore; and Chih-yi Wang, University of Minnesota.

258: An Orthocentric Locus

Mathematics Magazine **30** (1956–1957), 47; proposal January 1956

Source PDF p. 104–106.

Proposed by Hüseyin Demir, Zonguldak, Turkey

A triangle ABC inscribed in a circle varies such that AB and AC keep fixed directions. Find the locus of the orthocenter H .

I. Solution by Major H. S. Subba Rao, Defense Science Organization, New Delhi, India. The vertical angle A and the base BC are fixed in magnitude. Let $A_1B_1C_1$ be the isosceles triangle satisfying the conditions imposed on ABC . Let P be the mid-point of the smaller of the two arcs AC of the circumcircle and similarly Q the mid-point of the arc AB . Let O be the centre of the circle. The points P, Q are fixed.

Take the diameter through A_1 as the y -axis and the perpendicular diameter as the x -axis. With reference to these axes we can represent any point on the circle ABC by the parametric representation $a \cos t, a \sin t$.

Let $B_1 = (t_2)$, $C_1 = (t_3)$, $P = (t_4)$ and $Q = (t_5)$. Noting that $\angle A_1B_1C_1 = \angle A_1C_1B_1 = 90^\circ - A/2$, it can be easily shown that

$$t_2 = \frac{3\pi}{2} - A, \quad t_3 = \frac{3\pi}{2} + A, \quad t_4 = \frac{A}{2}, \quad t_5 = 2\pi - \frac{A}{2}.$$

In any position of the triangle ABC let $t = \angle B_1OB = \angle C_1OC$. Then $B = (t_2 + t)$ and $C = (t_3 + t)$. Further, BH being perpendicular to AC is parallel to OP , and similarly CH is parallel to OQ .

The equations to BH and CH are easily found to be^[E037]

$$\begin{aligned} x \sin \frac{A}{2} - y \cos \frac{A}{2} &= a \cos \left(t - \frac{3A}{2} \right), \\ x \sin \frac{A}{2} + y \cos \frac{A}{2} &= -a \cos \left(t + \frac{3A}{2} \right). \end{aligned}$$

Eliminating t between the two equations, the locus of H is found to be

$$\frac{x^2 \sin^2(A/2)}{a^2 \sin^2(3A/2)} + \frac{y^2 \cos^2(A/2)}{a^2 \cos^2(3A/2)} = 1.$$

This is an ellipse with its centre at O and semiaxes

$$\left| \frac{a \sin(3A/2)}{\sin(A/2)} \right| \quad \text{and} \quad \left| \frac{a \cos(3A/2)}{\cos(A/2)} \right|.$$

At $A = 60^\circ$ or 120° , the parametrization instead gives a degenerate ellipse, a line segment; the displayed quotient equation is used only when its denominators are nonzero. An interesting corollary to this is that the loci of the nine-point centre and centroid of the triangle ABC are also ellipses, with the same possible degeneracy.

II. Solution by the proposer. Let OX, OY be the lines parallel to external and internal bisectors of A , respectively. Let the altitude AH intersect these fixed lines at X, Y . Since AO, AH are equally inclined to the bisectors of A , we have $AX = AO = AY$. Hence $XY = 2R = \text{const.}$

We may think then of XY as a rod of constant length having the ends moving on OX, OY . Now the angle A being constant, BC will envelop, or the mid-point D of BC will describe, a

circle with center O . Hence the directed altitude is $AH = 2OD = 2R \cos A = \text{const}$. This proves that H is a fixed point of the moving bar XAY . Hence H describes an ellipse.

The semidiameters of the ellipse are easily determined:^[E037]

$$a = R|1 + 2 \cos A|, \quad b = R|1 - 2 \cos A|.$$

Also solved by J. W. Clawson, Collegeville, Pennsylvania; R. K. Guy, University of Malaya, Singapore; Sister M. Stephanie, Georgian Court College, New Jersey; Harry D. Ruderman, The Bronx, New York; and Chih-yi Wang, University of Minnesota.

266: Points Inverse in a Circumcircle

Mathematics Magazine **30** (1956–1957), 105; proposal March 1956

Source PDF p. 107–108.

Proposed by Hüseyn Demir, Zonguldak, Turkey

If M and M' are points inverse to each other with respect to the circumcircle of a triangle ABC , then prove the following for oriented angles, modulo 2π , wherever the angles are defined:^[E038]

$$\begin{aligned} \angle BMC + \angle BM'C &= 2\angle A, \\ \angle CMA + \angle CM'A &= 2\angle B, \\ \angle AMB + \angle AM'B &= 2\angle C. \end{aligned}$$

I. Solution by Richard K. Guy, University of Malaya, Singapore. In triangles COM and $M'OC$ angle O is common and, as $OM \cdot OM' = OC^2$, we have $OC/OM = OM'/OC$. Hence the triangles are similar and $\angle OM'C = \angle OCM$. In the same way $\angle OM'B = \angle OBM$. Adding these to $\angle OMC$ and $\angle OMB$, we have

$$\angle BMC + \angle BM'C = \pi - \angle COM + \pi - \angle BOM = \angle BOC = 2\angle A.$$

Similarly we have $\angle CMA + \angle CM'A = 2\angle B$ and $\angle AMB + \angle AM'B = 2\angle C$.^[E038]

II. Solution by Maimouna Edy, Hull, P. Q., Canada. Represent points A, B, C, M, M' by complex numbers z_1, z_2, z_3, z, z' , respectively. Let parentheses represent cross ratios and the bars the complex conjugate. We then have

$$(z_1, z_2, z_3, z) = \overline{(z_1, z_2, z_3, z')}. \quad (1)$$

This says that the homographic transformation which sends z_1, z_2, z_3 into $1, 0, \infty$, respectively, that is, the transformation of the given circle into the axis of reals, sends z and z' into two conjugate complex points.

Now equation (1) reads explicitly

$$\frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} = \overline{\frac{z' - z_2}{z' - z_3}} \cdot \overline{\frac{z_1 - z_3}{z_1 - z_2}}.$$

Evidently,

$$\frac{z_2 - z}{z_3 - z} \div \frac{\overline{z_2 - z'}}{\overline{z_3 - z'}} = \frac{\overline{z_3 - z_1}}{\overline{z_2 - z_1}} \div \frac{z_3 - z_1}{z_2 - z_1}.$$

Therefore,

$$\begin{aligned}\arg \frac{z_2 - z}{z_3 - z} + \arg \frac{z_2 - z'}{z_3 - z'} &= -2 \arg \frac{z_3 - z_1}{z_2 - z_1} \\ &= 2 \arg \frac{z_2 - z_1}{z_3 - z_1}.\end{aligned}$$

This means that for oriented angles,

$$\angle(\overrightarrow{MC}, \overrightarrow{MB}) + \angle(\overrightarrow{M'C}, \overrightarrow{M'B}) = 2\angle(\overrightarrow{AC}, \overrightarrow{AB}).$$

The oriented angles form an additive group isomorphic with the multiplicative group of the unit circle. In other words, we may take arbitrary measures of our angles and add them (mod 2π). The other two relations are proven similarly.

Bankoff's solution also noted the necessity for proper orientation of the angles.

Also solved by Leon Bankoff, Los Angeles, California; J. W. Clawson, Collegeville, Pennsylvania; and the proposer.

298: An Invariant Curve

Mathematics Magazine **31** (1957–1958), 56; proposal January 1957

Source PDF p. 109.

Proposed by Hüseyin Demir, Kandilli, Bölgesi, Turkey

Let $y = f(x)$ *be a curve with the following properties:*

- (a) $f(x) = f(-x)$;
- (b) $f'(x) > 0$ for $x > 0$;
- (c) $f''(x) > 0$.

Determine the weight per unit length $w(x)$ *at the point* (x, y) *such that when the curve is suspended under gravity by any two points on it, the curve will keep its original shape.*^[E039]

Solution by K. L. Cappel, Philadelphia, Pennsylvania. Assume the curve to be suspended at two arbitrary points A, B . Let the weight between A and the y intercept of the curve be W . Then at A , the tension in the curve can be resolved into vertical and horizontal components so that $W/H = \tan \theta$, or $W = H \, dy/dx$.

Now assume the right point of support to be moved from A to A' . If the curve is to retain its shape, there must be no change in the forces at A . This can only be the case if H is a constant. If ds is the length of the segment AA' , and dW is its weight, then the weight per unit length will be

$$W_x = \frac{dW}{ds} = \frac{dW}{\sqrt{1 + (dy/dx)^2} \, dx}, \quad \text{or} \quad W_x = H \frac{d^2y/dx^2}{\sqrt{1 + (dy/dx)^2}},$$

which can be satisfied by any curve obeying the given conditions. This problem is analogous to the problem of finding the optimum shape of a masonry arch, when the material of the arch is the only load to be supported, and it is desired to have the thrust load act along the neutral axis in order to eliminate bending moments.

Also solved by the proposer.

304: Circles Connected with a Triangle

Mathematics Magazine **31** (1957–1958), 117; proposal March 1957

Source PDF p. 110.

Proposed by Hüseyin Demir, Kandilli, Bölgesi, Turkey

Let ABC be a triangle, $AB \neq AC$, inscribed in a circle O , and let K be the point where the exterior angle bisector of A meets O . A variable circle with center at K meets AB, AC at E and F , respectively, such that A is not an interior point of KEF . Find the limiting position m of the common point M of EF, BC as EF approaches BC .

Solution by the proposer. Let E', F' be the points where (K) meets AB, AC other than E, F . Let M' be the common point of BC with $E'F'$. Applying the Menelaus theorem to ABC , considering $EFM, E'F'M'$ as transversals, we have

$$\frac{MB}{MC} \frac{FC}{FA} \frac{EA}{EB} = 1, \quad \frac{M'B}{M'C} \frac{F'C}{F'A} \frac{E'A}{E'B} = 1.$$

Multiplying these equalities member to member and observing that $EA = F'A$, $E'A = FA$, we get

$$\frac{MB}{MC} \frac{M'B}{M'C} = \frac{EB \cdot E'B}{FC \cdot F'C}.$$

Since in the last ratio the numerator and denominator are the powers of B, C with respect to the circle (K) , and since these powers are equal (K is equidistant from B and C), $MB : MC = M'C : M'B$ follows. Hence the points M and M' are symmetric points on BC . The limiting position m of M will also be the symmetric point of m' , the limiting position of M' . It is easy to see that m' is the foot of KA , the exterior angle bisector of A . Hence the construction of m follows immediately.

349: A Bisector

Mathematics Magazine **32** (1958–1959), 223; proposal September 1958

Source PDF p. 111.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

If $ABCD$, $AEBK$ and $CEFG$ are squares of the same orientations, prove that B bisects DF .

Solution by Leon Bankoff, Los Angeles, California. Removing angle CEB from the right angles AEB and CEF , we find that angles BEF, AEC, DEB are equal. But $DE = EC = EF$. Hence the triangles BDE and FBE are congruent and $FB = BD$. The collinearity of F, B, D is established by the fact that angle $EBD = 90^\circ$.

Also solved by Norman Anning, Alhambra, California; D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; J. W. Clawson, Collegeville, Pennsylvania; Norbert Jay, New York, New York; Joseph D. E. Konhauser, Haller, Raymond and Brown, Inc., State College, Pennsylvania; Arne Pleijel, Trollhättan, Sweden; William Sanders, Mississippi Southern College; C. W. Trigg, Los Angeles City College; Dale Woods, Idaho State College; and the proposer.

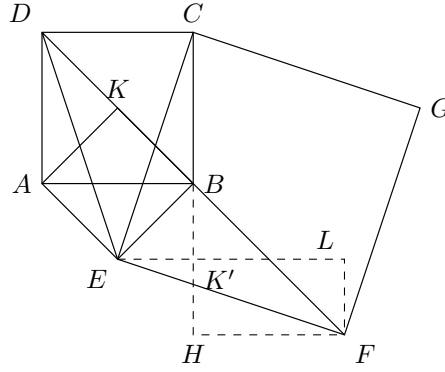


Figure 7.1: Three equally oriented squares in Proposal 349. The auxiliary point on BH is denoted K' to distinguish it from the square vertex K .

334: An Irregular Area

Mathematics Magazine **32** (1958–1959), 106; proposal March 1958

Source PDF p. 112.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Find the simplest expression for the area S enclosed by the arc AM of a cycloid, the arc TM of the rolling circle $\Omega(a)$ and the base line segment AT .

Solution by J. W. Clawson, Collegeville, Pennsylvania. Draw MN and CT perpendicular to AT . Let angle $MCT = \theta$ and $CT = a$. The area required is area AMN plus area of trapezoid $NMCT$ minus area of sector MCT . Now, for M , $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$. Hence

$$\begin{aligned} S &= a^2 \int_0^\theta (1 - \cos t)^2 dt + \frac{a^2 \sin \theta}{2} (2 - \cos \theta) - \frac{a^2 \theta}{2} \\ &= \frac{3}{2} a^2 \theta - 2a^2 \sin \theta + \frac{a^2}{2} \sin \theta \cos \theta + a^2 \sin \theta - \frac{a^2}{2} \sin \theta \cos \theta - \frac{a^2 \theta}{2} \\ &= a^2(\theta - \sin \theta) = ax. \quad [E041] \end{aligned}$$

Also solved by Stanley P. Franklin, Memphis State University; Joseph D. E. Konhauser, State College, Pennsylvania; Arne Pleijel, Trollhättan, Sweden; and the proposer.

372: A Trigonometric Identity

Mathematics Magazine **33** (1959–1960), 112; proposal March 1959

Source PDF p. 112–113.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Prove the identity ^[E027]

$$\begin{aligned} \sin^2(\theta_1 + \theta_2 + \cdots + \theta_n) &= \sin^2 \theta_1 + \cdots + \sin^2 \theta_n \\ &\quad + 2 \sum_{1 \leq i < j \leq n} \sin \theta_i \sin \theta_j \cos(\theta_i + 2\theta_{i+1} + \cdots + 2\theta_{j-1} + \theta_j). \end{aligned}$$

Solution by the proposer. We proceed by induction. The equality holds for $n = 1$ and $n = 2$. Let the property be true for $n = p$. Then setting $\theta = \theta_1 + \cdots + \theta_p$, it will suffice to prove the

equality obtained by subtraction,

$$\begin{aligned} \sin^2(\theta + \theta_{p+1}) - \sin^2 \theta &= \sin^2 \theta_{p+1} \\ &+ 2 \sum_{i=1}^p \sin \theta_i \sin \theta_{p+1} \cos(\theta_i + 2\theta_{i+1} + \cdots + 2\theta_p + \theta_{p+1}). \end{aligned}$$

The left-hand side, A , is seen to be equal to

$$A = \sin \theta_{p+1} \sin(2\theta + \theta_{p+1}).$$

The right-hand side, B , is equal to

$$\begin{aligned} B &= \sin^2 \theta_{p+1} + \sin \theta_{p+1} \sum_{i=1}^p 2 \sin \theta_i \cos(\theta_i + 2\theta_{i+1} + \cdots + 2\theta_p + \theta_{p+1}) \\ &= \sin^2 \theta_{p+1} + \sin \theta_{p+1} \sum_{i=1}^p [\sin(2\theta_i + 2\theta_{i+1} + \cdots + 2\theta_p + \theta_{p+1}) \\ &\quad - \sin(2\theta_{i+1} + \cdots + 2\theta_p + \theta_{p+1})] \\ &= \sin^2 \theta_{p+1} + \sin \theta_{p+1} [\sin(2\theta_1 + 2\theta_2 + \cdots + 2\theta_p + \theta_{p+1}) - \sin \theta_{p+1}] \\ &= \sin \theta_{p+1} \sin(2\theta_1 + \cdots + 2\theta_p + \theta_{p+1}) = A. \end{aligned}$$

The equality $A = B$ proves that the equality holds for $n = p + 1$. The result follows by induction.^[E027]

380: A System of Equations

Mathematics Magazine **33** (1959–1960), 172; proposal May 1959

Source PDF p. 114–115.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Solve the system of equations

$$\begin{aligned} x(z - a) + u(x + u) &= 0, \\ y(x - b) + u(y + u) &= 0, \\ z(y - c) + u(z + u) &= 0, \end{aligned}$$

where $abc \neq 0$ and $a^{-1} + b^{-1} + c^{-1} = u^{-1}$.

Solution by Chih-yi Wang, University of Minnesota. Number the three equations (1), (2), (3). By performing the operations multiply (1) by y , multiply (2) by z , multiply (3) by x and applying (2), (3), (1), respectively, we get

$$xyz - aby + au^2 + buy - u^3 = 0, \tag{4}$$

$$xyz - bcx + bu^2 + cuz - u^3 = 0, \tag{5}$$

$$xyz - cax + cu^2 + aux - u^3 = 0. \tag{6}$$

By performing the operations (4)–(5), (5)–(6), (6)–(4), we get

$$(au + bu - ab)y + (bc - bu - cu)z = (b - a)u^2, \tag{7}$$

$$(ca - cu - au)x + (cu + bu - bc)z = (c - b)u^2, \tag{8}$$

$$(au + cu - ca)x + (ab - au - bu)y = (a - c)u^2. \tag{9}$$

Since the augmented matrix of (7), (8), (9) is of rank 2, we can calculate two variables in terms of the third, so we get

$$y = \frac{c^2}{a^2}z - \frac{c(b-a)}{ab}u, \quad (10)$$

$$x = \frac{b^2}{a^2}z + \frac{b(c-b)}{ca}u. \quad (11)$$

The sign in (11) has been corrected.^[E040] By substituting (10) into (3) we get, after simplification,

$$\left(\frac{c}{a}z - u\right)^2 = 0,$$

whence by aid of (10) and (11),

$$x = \frac{b}{a}u, \quad y = \frac{c}{b}u, \quad z = \frac{a}{c}u.$$

Note that we have used the relations $abc \neq 0$, $a^{-1} + b^{-1} + c^{-1} = u^{-1}$ whenever necessary.

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Victor Ch'in, Kent State University, Kent, Ohio; Melvin Hochster, Stuyvesant High School, New York; and the proposer.

384: An Infinite Group

Mathematics Magazine **33** (1959–1960), 230; proposal September 1959

Source PDF p. 115.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Let (a_{ij}) *be a nonsingular real matrix of* n *th order,* $n \geq 2$ *, the sum of the elements of each of whose rows equals 1. Prove that the totality of such matrices forms a group of infinite order.*^[E024]

Solution by D. A. Breault, Sylvania Electric Products, Inc. We assume that the proposed group operation is multiplication, and that the sum condition means that

$$\sum_{j=1}^n a_{ij} = 1 \quad \text{for } i = 1, 2, \dots, n. \quad (1)$$

The system has:

(1) *Closure*: for if $A = [a_{ij}]$ and $B = [b_{ij}]$, we have

$$c_{ij} = (AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj};$$

whence

$$\sum_{j=1}^n c_{ij} = \sum_{j=1}^n \sum_{k=1}^n a_{ik}b_{kj} = \sum_{k=1}^n a_{ik} \left(\sum_{j=1}^n b_{kj} \right) = \sum_{k=1}^n a_{ik} = 1$$

for each i .

(2) *Associativity*: which can be demonstrated by the use of summations similar to the above.

(3) *Identity*: the usual identity matrix $I = (\delta_{ij})$ serves here also.

- (4) *Inverses*: given a matrix A which satisfies (1), it can be shown that A^{-1} satisfies (1) also, *whenever it exists!* Hence the totality of nonsingular matrices satisfying (1) form a group, but not the unrestricted set.

Editorial completion. For $n \geq 2$, the matrices

$$\begin{pmatrix} 1-t & t \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}, \quad t \in \mathbb{R} \setminus \{1\},$$

are distinct nonsingular matrices with row sums 1. Thus the group is infinite. For $n = 2$ the second block is omitted.^[E024]

Also solved by the proposer.

398: Simultaneous Quadratics

Mathematics Magazine **34** (1960–1961), 51; proposal January 1960

Source PDF p. 116.

Proposed by Hüseyn Demir, Kandilli, Ereğli, Kdz., Turkey

Determine the roots of the equations

$$x^2 + y_1x + y_2 = 0, \quad y^2 + x_1y + x_2 = 0,$$

where the coefficients (real numbers) in one equation are the roots of the other.

Solution by Harry M. Gehman, University of Buffalo. The relations between roots and coefficients give these four equations:

$$x_1 + x_2 = -y_1, \quad x_1x_2 = y_2, \quad y_1 + y_2 = -x_1, \quad y_1y_2 = x_2.$$

From the first and third equations, $x_2 = y_2$.

Case I. If $x_2 = y_2 = 0$, then $x_1 = -y_1 = a$, where a is arbitrary, the equations are

$$x^2 - ax = 0 \quad \text{and} \quad x^2 + ax = 0,$$

whose roots are $a, 0$ and $-a, 0$, respectively.

Case II. If $x_2 = y_2 \neq 0$, then $x_1 = y_1 = 1$ and $x_2 = y_2 = -2$. Both equations become

$$x^2 + x - 2 = 0,$$

whose roots are $1, -2$. Note that there is no need for the condition that the coefficients be real.

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Sidney Kravitz, Dover, New Jersey; A. J. Kokar, School of Mines, Adelaide, Australia; Rostyslaw Lewyckij, University of Toronto; Ernest E. Moyers, University of Mississippi; F. D. Parker, University of Alaska; Charles F. Pinzka, University of Cincinnati; Arne Pleijel, Trollhättan, Sweden; Robert E. Shafer, University of California Radiation Laboratory; C. M. Sidlo, Framingham, Massachusetts; William Squire, Southwestern Research Institute, San Antonio, Texas; Harvey Walden, Rensselaer Polytechnic Institute; Chih-yi Wang, University of Minnesota; Dale Woods, Northeastern Missouri State College; and the proposer.

407: Resistance in a Cube

Mathematics Magazine **34** (1960–1961), 115; proposal March 1960

Source PDF p. 117–118.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

The twelve edges of a cube are made of wires of one ohm resistance each. The cube is inserted into an electrical circuit by:

- (a) *two adjacent vertices;*
- (b) *two opposite vertices of a face;*
- (c) *two opposite vertices of the cube.*

Which offers the least resistance?

Solution by C. W. Trigg, Los Angeles City College. It may be inferred that the least resistance occurs in (a) since there is a single-edge connector between the terminals. For confirmation:

In the figures, the direction of current flow is shown in each case. Below each cube a schematic diagram is shown wherein corners at the same potential, as determined by symmetry, are represented by the same point. Each situation is thus reduced to the simple case of repeated application of the laws of parallel circuits. So:

$$(a) \quad \frac{1}{R} = \frac{1}{r} + \frac{1}{\frac{r}{2} + \frac{1}{2/r + 1/(r/2 + r + r/2)} + \frac{r}{2}},$$

$$\text{whence } R = \frac{7}{12}r,$$

where R is the resistance of the cube and r is 1 ohm.

(b) E, B, H, C are at the same potential, so

$$R = \frac{2}{1/r + 1/r + 1/(r + r/2)} = \frac{3r}{4}.$$

(c)

$$R = \frac{r}{3} + \frac{r}{6} + \frac{r}{3} = \frac{5r}{6}.$$

Cases (a) and (c) are solved on pages 277–279 of *Magnetism and Electricity* by E. E. Brooks and A. W. Poyser, Longmans, Green and Co. (1920). Case (c) is Quickie 32, *Mathematics Magazine*, March 1951, November 1959.

Also solved by Charles F. Pinzka, University of Cincinnati; and the proposer (partially). One incorrect solution was received.

415: A Trigonometric Sum

Mathematics Magazine **34** (1961), 178; proposal May 1960

Source PDF p. 119.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Prove

$$\sum_{p=0}^n \binom{n}{p} \cos(px) \sin((n-p)x) = 2^{n-1} \sin nx.$$

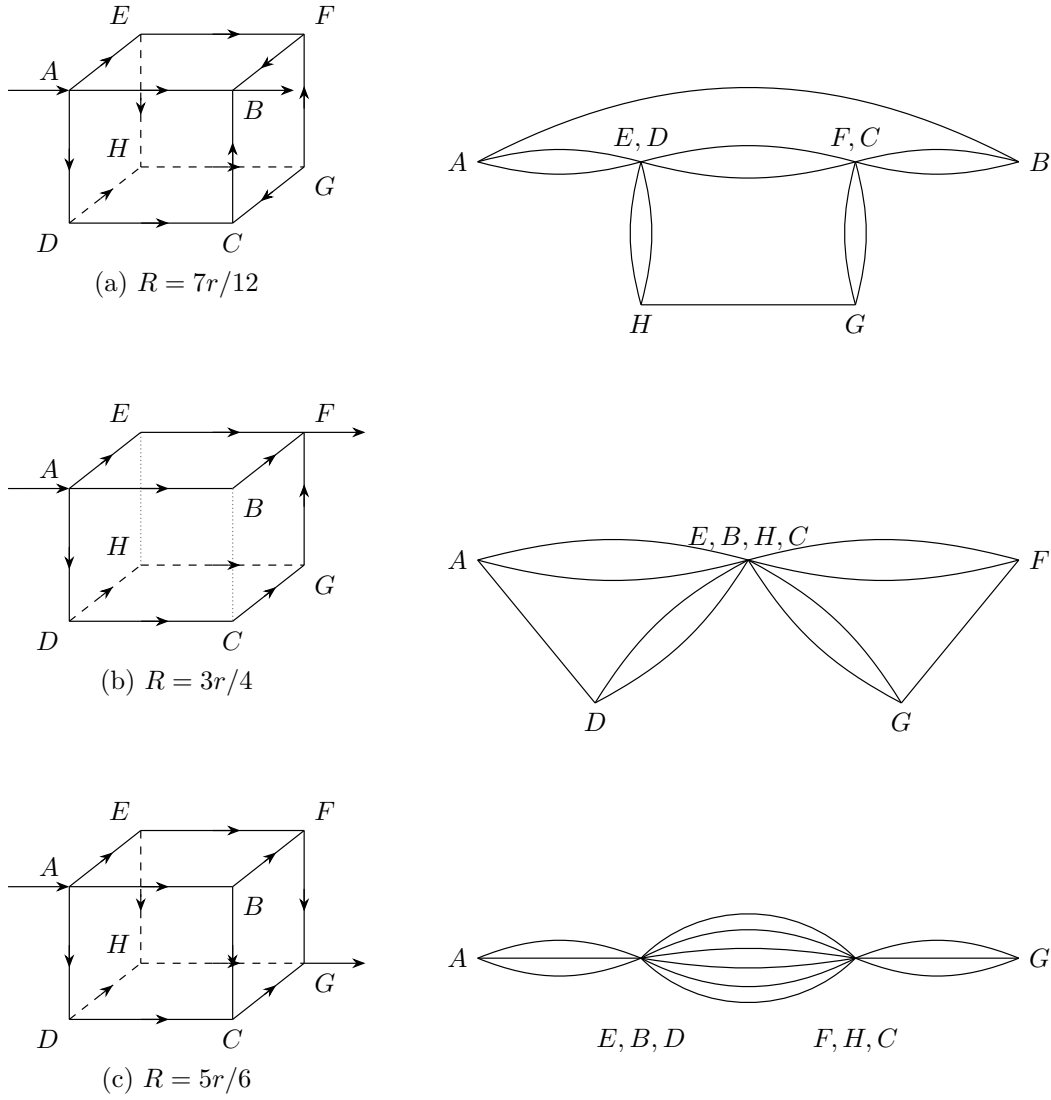


Figure 7.2: Cube currents and the corresponding networks after identifying equipotential vertices. Each drawn branch represents one edge resistance r ; a dashed cube edge is hidden, and a gray edge in (b) carries no current.

Solution by Josef Andersson, Vaxholm, Sweden. (Translated and paraphrased by the original journal editor.) Making use of the formulas

$$\sum_{p=0}^n \binom{n}{p} = 2^n, \quad \binom{n}{n-p} = \binom{n}{p},$$

the original sum can be written

$$\frac{1}{2} \sum_{p=0}^n \binom{n}{p} \sin nx + \frac{1}{2} \sum_{p=0}^n \binom{n}{p} \sin((n-2p)x) = 2^{n-1} \sin nx + \frac{s}{2}.$$

It remains to be proven that $s = 0$. Now from the substitution $p = n - p'$ it follows that

$$s = \sum_{p'=0}^n \binom{n}{n-p'} \sin((2p' - n)x) = -s. \text{[E042]}$$

Therefore $s = 0$.

Also solved by J. L. Brown, Ordnance Research Laboratory, Pennsylvania State University; L. Carlitz, Duke University; James C. Ferguson, Lynnwood, Washington; A. F. Horadam, University of New England, Armidale, NSW, Australia; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; William Squire, Southwest Research Institute, San Antonio, Texas; Chih-Yi Wang, University of Minnesota; and the proposer.

Comment on Problem 415

Mathematics Magazine **34** (1961), 308.

Source PDF p. 120.

Comment by Louis Brand, University of Houston. In the problem of a trigonometric sum a much simpler solution is as follows: Call the sum S and make the index change $p = n - q$; adding the two sums now gives

$$2S = \sum_{p=0}^n \binom{n}{p} \sin nx = 2^n \sin nx.$$

419: Constant Speed Curve

Mathematics Magazine **34** (1961), 239; proposal September 1960

Source PDF p. 120–121.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Determine the path in a vertical plane such that when a particle moved, under gravity, with an initial velocity v_0 from a point of the path, the particle maintained a constant speed along the path. Assume no friction.

Corrected answer (editorial). A horizontal straight path has the required property. For a particle constrained to a fixed smooth path without friction, the constraint force does no work. Conservation of energy gives $\frac{1}{2}mv^2 + mgh = \text{constant}$, so constant nonzero speed forces constant height h . In a vertical plane this is a horizontal line. The published parabolic answer is incompatible with the stated mechanical assumptions. Its argument is retained below as a clearly identified historical error.^[E043]

Historical proposed solution by the proposer (invalid). Let Ox, Oy be the axes of coordinates taken in the vertical plane such that Oy points downward and Ox to the left. Let the particle be dropped from O . It reaches the velocity v_0 at a point A of Oy with $y_0 = OA = v_0^2/(2g)$.

Since there is no friction, the velocity along the path is the projection of the velocity $v = \sqrt{2gy}$, and we write $v_0 = v \cos \alpha$ where

$$v_0 = \sqrt{2gy_0}, \quad v = \sqrt{2gy}, \quad \cos^2 \alpha = \frac{1}{1 + \tan^2 \alpha} = \frac{1}{1 + y'^2},$$

and get $y_0 = y/(1 + y'^2)$.

The variables separate and give

$$x = \int_{y_0}^y \frac{dy}{\sqrt{(y - y_0)/y_0}} = \frac{1}{2} \sqrt{y_0} \sqrt{y - y_0},$$

$$y = \frac{4x^2}{y_0} + y_0 = \frac{8g}{v_0^2} x^2 + \frac{v_0^2}{2g}.$$

The path is a parabola tangent to Oy at A , Oy being the tangent at the vertex.

Editorial explanation. The projection premise above does not describe frictionless constrained motion. Independently, the displayed integral equals $2\sqrt{y_0}\sqrt{y - y_0}$, not one half of that product, and the written parabola has a horizontal tangent at its vertex, not the claimed vertical tangent. Correcting only the integration cannot repair the physical argument.^[E043]

425: Euler's Phi-function

Mathematics Magazine **34** (1961), 300; proposal November 1960

Source PDF p. 121–123.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

If $n > 4$ and $n - 1$ and $n + 1$ are twin prime numbers, prove that $3\varphi(n) \leq n$, where φ denotes Euler's φ -function.^[E025]

I. Solution by Dermott A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts (corrected). If $n + 1$ and $n - 1$ are prime and $n > 4$, then n is both even and a multiple of 3, so that for some m , $n = 6m$. The required computation, with the multiplicativity error in the original solution corrected, is

$$\varphi(n) = \varphi(6m) = 2m \prod_{\substack{p|m \\ p \neq 2,3}} \left(1 - \frac{1}{p}\right).$$

Thus

$$3\varphi(n) = 6m \prod_{\substack{p|m \\ p \neq 2,3}} \left(1 - \frac{1}{p}\right) = n \prod_{\substack{p|m \\ p \neq 2,3}} \left(1 - \frac{1}{p}\right) \leq n,$$

as required. The original argument incorrectly used $\varphi(6m) = \varphi(6)\varphi(m)$ without requiring $(6, m) = 1$; the following historical comment records this error.^[E044]

II. Solution by L. Carlitz, Duke University. It is evidently necessary to assume $n > 4$. Since $n - 1$ and $n + 1$ are primes and $n > 4$ it follows that n is divisible by 3. Also n must be even so that n is divisible by 6. We shall now show that if

$$n = 2^\alpha 3^\beta m \quad (\alpha \geq 1, \beta \geq 1, (m, 6) = 1), \quad (1)$$

then $\varphi(n) \leq n/3$. Indeed from (1),

$$\varphi(n) = 2^\alpha 3^{\beta-1} \varphi(m) \leq 2^\alpha 3^{\beta-1} m = \frac{n}{3}. \quad (2)$$

Remark: It is not difficult to show that

$$\varphi(n) = \frac{n}{3} \quad (3)$$

if and only if

$$n = 2^\alpha 3^\beta \quad (\alpha \geq 1, \beta \geq 1). \quad (4)$$

We have seen above that (4) implies (3). Now if (3) holds it is clear that n is divisible by 3. Put $n = 3^\alpha k$, where $\alpha \geq 1$ and $(k, 3) = 1$; then (3) becomes

$$2 \cdot 3^\alpha \varphi(k) = n,$$

so that n is even. Now put

$$n = 2^\alpha 3^\beta m \quad (\alpha \geq 1, \beta \geq 1, (m, 6) = 1).$$

Then if $m > 1$ it follows from (2) that $\varphi(n) < n/3$. This completes the proof of the equivalence of (3) and (4).^[E045]

Also solved by Brother Alfred, St. Mary's College, California; Leon Bankoff, Los Angeles, California; Maxey Brooke, Sweeny, Texas; B. A. Hausman, S. J., West Baden College, Indiana; Vern Hoggatt, San Jose State College; Sidney Kravitz, Dover, New Jersey; D. L. Silverman, Fort Meade, Maryland; Dale Woods, Northeast Missouri State Teachers College; and the proposer.

Comment on Problem 425

Mathematics Magazine **34** (1961), 433.

Source PDF p. 123.

Comment by David A. Klarner, Napa, California. The solution given by Dermott A. Breault contains an error. In the proof we find the statement, "If $n + 1$ and $n - 1$ are prime, n is even and a multiple of 3, so that, for some m , $n = 6m$, and we have $\varphi(n) = \varphi(6)\varphi(m) = 2\varphi(m)$." This is only true when $(6, m) = 1$. In fact, the twin primes 11, 13 yield the erroneous value

$$\varphi(12) = \varphi(6)\varphi(2) = 2,$$

but $\varphi(12) = 4$. Therefore the method of proof given would have to be altered to make it valid.

437: A Well Known Problem

Mathematics Magazine **34** (1961), 371; proposal January 1961

Source PDF p. 123.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Prove or disprove the statement: the number of odd coefficients in the binomial expansion of $(a + b)^n$ is $2^{s_2(n)}$, where $s_2(n)$ is the number of 1's appearing in the expression of n in the binary number system.^[E028]

Editorial note (original journal). Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania, pointed out that a simpler version of this problem appeared as Problem 7, Part II, in the Putnam Competition of 1956. The given problem appeared as Problem E1288 in the *American Mathematical Monthly* in November 1957 with solution and references given in the May 1958 issue.

E1288: Odd Binomial Coefficients

American Mathematical Monthly **65** (1958), 368; proposal *American Mathematical Monthly* **64** (1957), 671

Source PDF p. 123–125.

Proposed by S. H. Kimball, University of Maine

The number of odd binomial coefficients in any finite binomial expansion is a power of 2 (Putnam Mathematical Competition, American Mathematical Monthly [1957, p. 24]). Prove that the power of 2 is the number of 1's in the binary scale expression for n in $(x + y)^n$.

I. Solution by T. R. Hatcher and J. A. Riley, Parke Mathematical Laboratories, Carlisle, Mass. Let h and n be positive integers with $h < n$. We define the *binary length* of n , $L(n)$, to be the number of ones in the binary representation of n , and the *binary capacity* of n , $C(n)$, to be the exponent of the highest power of two which divides n . We say “ h is contained in n ,” written $h \subset n$, if when h has a one in a certain binary place, n also has a one in the corresponding binary place; that is, the binary representation of h can be obtained from that of n by changing ones to zeros.

The following properties are easily proved:

- (1) $C(n)$ is the number of terminating zeros in the binary representation of n .
- (2) $C(n) = 0$ if and only if n is odd.
- (3) $C(ab) = C(a) + C(b)$, $C(a/b) = C(a) - C(b)$ when a/b is a positive integer.
- (4) $L(n) = L(h) + L(n - h)$ if and only if $h \subset n$.
- (5) $C(n) = 1 + L(n - 1) - L(n)$.
- (6) $C(n!) = n - L(n)$.
- (7) $C\left(\binom{n}{h}\right) = L(h) + L(n - h) - L(n)$.

The corollary of the following theorem gives the solution. In the endpoint cases take $L(0) = 0$ and permit $h = 0, n$ in the containment notation.^[E046]

Theorem. $\binom{n}{h}$ is odd if and only if $h \subset n$.

Proof. If $\binom{n}{h}$ is odd, $C\left(\binom{n}{h}\right) = 0$ and by (7), $L(n) = L(h) + L(n - h)$. Thus, by (4), $h \subset n$. Conversely, if $h \subset n$, then $L(n) = L(h) + L(n - h)$ and $C\left(\binom{n}{h}\right) = 0$. $\square$

Corollary. The number of integers h such that $\binom{n}{h}$ is odd is $2^{L(n)}$.

Proof. The number of integers h with $h \subset n$ and $L(h) = j$ is $\binom{L(n)}{j}$. Thus the number of integers h for which $\binom{n}{h}$ is odd is simply

$$\sum_{j=0}^{L(n)} \binom{L(n)}{j} = 2^{L(n)}.$$

$\square$

II. Remarks by Leo Moser, University of Alberta. Problem E1288 is a special case of 4723 [1957, 116]. The solution of that problem is the following:

If p is prime and $n = a_0 + a_1p + a_2p^2 + \cdots + a_kp^k$, $0 \leq a_i < p$, $i = 0, 1, 2, \dots, k$, then the number of solutions of $\gcd\left(\binom{n}{r}, p\right) = 1$, $r = 0, 1, \dots, n$, is $\prod_{i=0}^k (a_i + 1)$.^[E046]

The result in E1288 is contained in J. W. L. Glaisher, “On the residue of a binomial coefficient with respect to a prime modulus,” *Quarterly Journal of Mathematics* **30** (1899), 150–156. More recently a proof was given by J. B. Roberts, “On binomial coefficient residues,” *Canadian Journal of Mathematics* **9** (1957), 363–370.

Also solved by D. R. Brillinger, Leonard Carlitz, Joe Lipman, D. C. B. Marsh, Paul Schillo, and the proposer.

440: Circle Packing

Mathematics Magazine **34** (1961), 427; proposal March 1961

Source PDF p. 125–127.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Consider a packing of circles of radius r such that each is tangent to its six surrounding circles. Let a larger circle of radius R be drawn concentric with one of the small circles. If N is the number of small circles contained in the larger circle, prove that^[E026]

$$N = 1 + 6n + 6 \sum_{p=1}^n \left\lfloor \frac{\sqrt{4\rho^2 - 3p^2} - p}{2} \right\rfloor, \quad \rho = \frac{1}{2} \left(\frac{R}{r} - 1 \right), \quad n = \lfloor \rho \rfloor,$$

the floor brackets designating the greatest integer function.

Solution by Alan Sutcliffe, Knottingley, Yorkshire, England. The expression [in the original proposal] is not quite correct. For example, when $R/r = 2\sqrt{3} + 1$ we have $\lfloor \rho \rfloor = 1$ and hence the original expression gives $N = 7$, while the correct value is $N = 13$. The correct expression is

$$\begin{aligned} N &= 1 + 6\lfloor \rho \rfloor + 6 \sum_{p=1}^{\lfloor \rho \rfloor} \left\lfloor \frac{1}{2} (\sqrt{4\rho^2 - 3p^2} - p) \right\rfloor \\ &= 1 + 6 \sum_{p=0}^{\lfloor \rho \rfloor} \left\lfloor \sqrt{\rho^2 - \frac{3}{4}p^2} - \frac{p}{2} \right\rfloor, \quad \rho = \frac{1}{2}(R/r - 1). \end{aligned}$$

Here ρ denotes the unrounded radius parameter called n in the original solution. The misprinted coefficient $3/5$ in its first rewritten expression is corrected to $3/4$.^[E047]

To prove this we shall first assume unit distance between adjacent centers, and find the number of centers within a circle of radius ρ . Because of the triangular nature of the array of centers, we need consider only one of the six similar sectors of the circle as shown in the diagram, where the centers marked $\circ$ are in the adjoining sector and the common center C is in no sector. Clearly the number of centers contained within the sector is the sum of the integral part of the lengths, such as AB , from CE to FG . Let $CB = p$, which will be an integer. Then, since $\angle BCD = 30^\circ$, $CD = (\sqrt{3}/2)p$ and $BD = p/2$. As $AC^2 = AD^2 + CD^2$, we have

$$\rho^2 = \left(AB + \frac{p}{2} \right)^2 + \frac{3}{4}p^2, \quad AB = \sqrt{\rho^2 - \frac{3}{4}p^2} - \frac{p}{2}.$$

The number of centers within the sector is the sum of the integral part of this from $p = 0$ to $\lfloor \rho \rfloor$. Since there are six sectors and the common center C , we have

$$N = 1 + 6 \sum_{p=0}^{\lfloor \rho \rfloor} \left\lfloor \sqrt{\rho^2 - \frac{3}{4}p^2} - \frac{p}{2} \right\rfloor.$$

Now in fact the centers are not unit distance, but $2r$ apart. So a radius $2r\rho$ will contain N centers. Thus a radius $R = 2r\rho + r$ will contain N circles, giving $\rho = \frac{1}{2}(R/r - 1)$, which completes the proof.

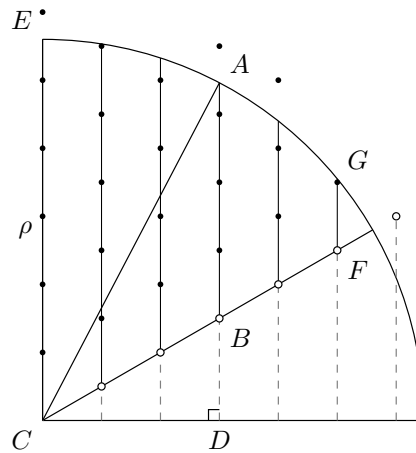


Figure 7.3: A half-open 60° sector of the triangular lattice for Proposal 440. Filled centers inside the arc are counted in this sector; open centers on the lower ray belong to the adjoining sector. The arc has radius ρ in units of the center spacing.

Comment on Proposal 440

Mathematics Magazine **35** (1962), 316.

Source PDF p. 127.

Comment by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Editorial note (original journal). The number N given in the statement is correct. N denotes the number of small circles contained entirely by the larger circle (tangency being included). The number N offered by A. Sutcliffe includes also the partly contained circles and therefore both numbers are correct.

Editorial correction to the historical comment. The assertion above is false: Sutcliffe's radius is $(R - r)/(2r)$, so it already counts only circles contained entirely in the larger circle. At $R/r = 1 + 2\sqrt{3}$, six centers lie at distance $2\sqrt{3}r$ from the central center; their circles are internally tangent to the larger boundary and must be included. Together with the central circle and its six nearest neighbors, they give 13, not 7. The corrected formula and the comment are retained with this explicit distinction.^[E048]

458: De Moivre's Theorem

Mathematics Magazine **35** (1962), 126; proposal September 1961

Source PDF p. 127.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

A student used De Moivre's theorem incorrectly as

$$(\sin \alpha + i \cos \alpha)^n = \sin n\alpha + i \cos n\alpha.$$

For what values of α does the equation hold for every integer n ?

Solution by Dermott A. Breault, Sylvania Applied Research Laboratory, Waltham, Massachusetts. Let $z = \cos \theta + i \sin \theta$. Then using De Moivre's theorem correctly we have $z^n = \cos n\theta + i \sin n\theta$. The proposed relation is that $(i/z)^n = i/z^n$, which implies that

$$\frac{1}{z^n}(i^n - i) = 0.$$

But $z^{-n} \neq 0$, so there are no values of θ for which the proposal holds for every integer n , but it is an identity for all n of the form $n = 4k + 1$.

Also solved by Brother U. Alfred, St. Mary's College, California; Leonard Carlitz, Duke University; Alan B. Delfino, St. Mary's College, California; P. D. Goodstein, University of Leicester, England; Harvey H. Green, R. C. A. Ascension Island (partially); Richard Levitt, Boston Latin School; David L. Silverman, Beverly Hills, California; Paul Stygar, Yale University; W. C. Waterhouse, Harvard University; and the proposer.

472: Symmetric Conics

Mathematics Magazine **35** (1962), 255; proposal January 1962

Source PDF p. 128.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Let (C) be a noncircular ellipse, or a hyperbola with semiaxes $a, b > 0$ such that $a^2 \neq 3b^2$ and $b^2 \neq 3a^2$. Let M be a variable point off the axes; take axis values by continuity.^[E049] Let T be the point symmetric to M with respect to the main axis, and t the tangent line at T . Denote the intersection of the perpendicular from M to t with the line joining T to the center of the conic by I . If M' is symmetric to M with respect to I , prove that:

1. The locus of M' is another conic (C') of the same kind as (C) .
2. The conics (C) and (C') are confocal.

Solution by R. D. H. Jones, College of William and Mary, Virginia (corrected). Let the conic be $x^2/a^2 + y^2/b^2 = 1$, with $a > b > 0$. Let $M = (a \cos \Delta, b \sin \Delta)$, so $T = (a \cos \Delta, -b \sin \Delta)$, and the tangent at T is $(x/a) \cos \Delta - (y/b) \sin \Delta = 1$. The line MI is perpendicular to this tangent, and therefore

$$y - b \sin \Delta = -\frac{a \sin \Delta}{b \cos \Delta}(x - a \cos \Delta). \quad (1)$$

The line joining T to the center is

$$\frac{x}{a \cos \Delta} + \frac{y}{b \sin \Delta} = 0. \quad (2)$$

Away from the axes, their intersection has coordinates

$$I = \left(\frac{a(a^2 + b^2)}{a^2 - b^2} \cos \Delta, -\frac{b(a^2 + b^2)}{a^2 - b^2} \sin \Delta \right).$$

By hypothesis $M' = 2I - M$, and hence

$$x_{M'} = \frac{a(a^2 + 3b^2)}{a^2 - b^2} \cos \Delta, \quad y_{M'} = -\frac{b(3a^2 + b^2)}{a^2 - b^2} \sin \Delta.$$

Let

$$A = \frac{a(a^2 + 3b^2)}{a^2 - b^2}, \quad B = \frac{b(3a^2 + b^2)}{a^2 - b^2}.$$

These are the corrected semiaxes of the new ellipse. They satisfy

$$A^2 - B^2 = \frac{(a^2 - b^2)^3}{(a^2 - b^2)^2} = a^2 - b^2,$$

so the ellipses are confocal. The values at axis points are obtained by continuity.^[E049]

Editorial completion for a hyperbola. For $x^2/a^2 - y^2/b^2 = 1$, the corresponding parametrization gives

$$x_{M'} = \frac{a(a^2 - 3b^2)}{a^2 + b^2} \cosh t, \quad y_{M'} = \frac{b(b^2 - 3a^2)}{a^2 + b^2} \sinh t$$

on one branch, with the other branch obtained by central reflection. The squared new semiaxes sum to $a^2 + b^2$, so the foci are unchanged. If $a^2 = 3b^2$ or $b^2 = 3a^2$, a semiaxis vanishes and the locus degenerates; these cases must be excluded from the assertion of a nondegenerate hyperbola. The source's appeal to an imaginary b omitted these exceptions.^[E049]

487: The Square Root of a Matrix

Mathematics Magazine **36** (1963), 76

Source PDF p. 129–130.

Proposed by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey

Find the square root of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Solution by Maurice Brisebois, Université de Sherbrooke, Canada (notation corrected). The source treats the more general equation $X^m = A$ for a nonsingular matrix. Over $\mathbb{C}$, let $A = UJU^{-1}$, where

$$J = \text{diag}(J_1, \dots, J_s), \quad J_i = \lambda_i I_{p_i} + H_{p_i}, \quad \sum_{i=1}^s p_i = n,$$

and H_{p_i} has 1's on the superdiagonal and 0's elsewhere. Choose an m th root μ_i of each nonzero λ_i , and put

$$R_i = \mu_i \sum_{k=0}^{p_i-1} \binom{1/m}{k} \lambda_i^{-k} H_{p_i}^k.$$

All roots are obtained as

$$X = UZ \text{diag}(R_1, \dots, R_s) Z^{-1} U^{-1}, \quad ZJ = JZ, \quad \det Z \neq 0,$$

allowing all choices of the branches μ_i . This writes out the block-root notation in the original solution and distinguishes the matrix size, the number of blocks, and the root order.^[E050]

For a diagonalizable 2×2 matrix this specializes to

$$X = UZ \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} Z^{-1} U^{-1},$$

with independent choices of the scalar square roots and Z commuting with the diagonal form. The source presents this specialization without the necessary diagonalizability qualification. A nonsingular Jordan block instead has roots

$$\pm \begin{pmatrix} \sqrt{\lambda} & 1/(2\sqrt{\lambda}) \\ 0 & \sqrt{\lambda} \end{pmatrix}.$$

For singular matrices the source discusses a system of “admissible elementary divisors”: the nilpotent blocks of a proposed root must, on taking the m th power, produce the nilpotent blocks of A . The printed block description mixes the blocks of A with those of X ; the explicit 2×2 treatment below avoids that ambiguity.

Remarks (corrected). Roots need not all be polynomials in A . Distinct eigenvalues suffice for all roots to be polynomial, but are not necessary. The invertible matrices Z form an open subset of the commutant of A ; if the degrees of the nonconstant invariant factors are $n_1 \geq \dots \geq n_t$, its dimension is

$$N = \sum_{i=1}^t (2i - 1)n_i.$$

This is the dimension of the available commuting matrices, not in general the number of independent parameters in the resulting roots, since different Z can give the same root. For example, for $A = I_2$ the commutant has dimension 4, whereas the nonscalar involutions form a two-dimensional family.

For results in this line, the source cites Lusternik–Sobolev, *Elements of Functional Analysis*, p. 283; Dunford–Schwartz, *Linear Operators*, Part I, problem 31 on p. 583; and Bellman, *Introduction to Matrix Analysis*, problems 1–3 on p. 93.^[E050]

Editorial explicit answer to the stated problem. Write $\tau = \text{tr } A$ and $\delta = \det A$. If A is nonscalar, all its complex square roots are

$$X = \frac{A + sI_2}{t}, \quad s^2 = \delta, \quad t^2 = \tau + 2s, \quad t \neq 0.$$

Indeed, $A^2 - \tau A + \delta I_2 = 0$ verifies the formula. Conversely, $X^2 = A$ implies $XA = AX$; the commutant of a nonscalar 2×2 matrix consists of $\alpha A + \beta I_2$. Comparing coefficients gives $\alpha^2 \tau + 2\alpha\beta = 1$ and $\beta^2 = \alpha^2 \delta$, hence the displayed expression. It also covers singular nonscalar A with nonzero trace. A nonzero nilpotent 2×2 matrix has no square root, since here $\tau = \delta = 0$ forces $t = 0$.

If $A = \lambda I_2$, $\lambda \neq 0$, the roots are $\pm\sqrt{\lambda}I_2$ and all conjugates of $\text{diag}(\sqrt{\lambda}, -\sqrt{\lambda})$. If $A = 0$, they are precisely the matrices X with $X^2 = 0$, namely zero and the conjugates of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. These descriptions are over $\mathbb{C}$; a real root is obtained precisely when the resulting matrix has real entries.^[E050]

Also solved by Brother U. Alfred, St. Mary’s College, California; J. A. H. Hunter, Toronto, Canada; Francis D. Parker, University of Alaska; Gilbert Labelle, University of Montreal, Canada; C. F. Pinzka, University of Cincinnati; J. L. Stearn, Washington, D. C.; and the proposer.

498: A Property of Multiplicative Functions

Mathematics Magazine **36** (1963), 201; proposal November 1962

Source PDF p. 130–131.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

If m and n are positive integers and δ, D are their g.c.d. and l.c.m., respectively, and $d(n)$ denotes the number of divisors of n , $\varphi(n)$ being the Euler function, prove that

$$d(m)d(n) = d(\delta)d(D), \tag{1}$$

$$\varphi(m)\varphi(n) = \varphi(\delta)\varphi(D). \tag{2}$$

Solution by L. Carlitz, Duke University. The result is a special case of the following theorem. Let $f(n)$ be an arbitrary factorable function, that is, $f(mn) = f(m)f(n)$ for all positive m, n such that $(m, n) = 1$. Then

$$f(m)f(n) = f(\delta)f(D), \quad (*)$$

where $\delta = (m, n)$ and $D = [m, n]$, the greatest common divisor and the least common multiple, respectively.

The proof is immediate. If $m = \prod_p p^{r_p}$, $n = \prod_p p^{s_p}$, then

$$\delta = \prod_p p^{r'_p}, \quad D = \prod_p p^{s'_p}, \quad r'_p = \min(r_p, s_p), \quad s'_p = \max(r_p, s_p).$$

Since the unordered pairs $\{r'_p, s'_p\}$ and $\{r_p, s_p\}$ agree, we have

$$f(\delta)f(D) = \prod_p f(p^{r'_p})f(p^{s'_p}) = \prod_p f(p^{r_p})f(p^{s_p}) = f(m)f(n).^{[E051]}$$

Also solved by Stephen R. Cavior, Duke University; Daniel I. A. Cohen, Brooklyn, New York; George Diderick, University of Wisconsin; Murray S. Klamkin, State University of New York at Buffalo; David A. Klarner, Humboldt College, California; Gilbert Labelle, Université de Montréal; Jerry L. Pietenpol, Columbia University; Robert Prielipp, University of Wisconsin; Sam Sesskin, Hempstead, New York; David L. Silverman, Beverly Hills, California; Irene Williams, Converse College, South Carolina; and the proposer.

509: An American Alphametic

Mathematics Magazine **36** (1963), 321; proposal March 1963

Source PDF p. 132–133.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Solve the cryptarithm

$$\begin{array}{rcccccc} & U & N & I & T & E & D \\ & S & T & A & T & E & S \\ \hline A & M & E & R & I & C & A \end{array}$$

in the base 11, introducing the digit α .

Solution by Anton Glaser, Ogontz Campus, Pennsylvania State University. My solution was obtained as follows:

- (1) $A = 1$. In any numeration system, adding two digits can result in “carrying” at most unity.
- (2) $D \neq 0$. Suppose $D = 0$; then we get the contradiction $S = A = 1$ versus $S \neq A$.
- (3) $S \neq 0$, similarly to (2).
- (4) $E \neq 0$. Suppose $E = 0$; then $C = A = 1$, contradicting $C \neq A$.
- (5) $T \neq 0$. Suppose $T = 0$; then either $I = A = 1$ versus $I \neq A$, or $T = I = 0$ versus $T \neq I$.
- (6) $U \neq 0$ and $S \neq 0$ by the usual rules of cryptarithms.
- (7) $E \neq \alpha$. Suppose $E = \alpha$; then $C = E = \alpha$ versus $C \neq E$.
- (8) $U + S > 9$.

(9) $U + S > \alpha$ if nothing was “carried” from column N .

(10) $D + S = 11_{(\text{eleven})} = 12_{\text{ten}} = \text{twelve}$.

Only the following combinations of D, S are possible:

$$\begin{array}{c|cccccccc} D & 2 & 3 & 4 & 5 & 7 & 8 & 9 & \alpha \\ S & \alpha & 9 & 8 & 7 & 5 & 4 & 3 & 2 \end{array}$$

Since $A = 1$, neither D nor S can be 1. Neither can be 6, since this would imply $D = S = 6$ versus $D \neq S$.

(11) $T \neq 2, 3, 4, 5$. For each of these values and the remaining possible values of E , a contradiction results.^[E052]

(12) For $T = 6$ and $E = 5$, the remaining letters could be assigned a one-to-one correspondence with the remaining digits that would satisfy the cryptarithm:

$$\begin{array}{ccccccc} 8 & \alpha & 2 & 6 & 5 & 3 & \\ 9 & 6 & 1 & 6 & 5 & 9 & \\ \hline 1 & 7 & 5 & 4 & 2 & 0 & 1 \end{array}$$

Editorial verification. Exhaustive enumeration of the base-11 column equations with distinct digits and nonzero leading letters gives exactly this one solution.^[E052]

Also solved by Josef Andersson, Vaxholm, Sweden; Merrill Barneby, University of North Dakota; Maxey Brooke, Sweeny, Texas; Harry M. Gehman, State University of New York at Buffalo; Wahin Ng, San Francisco, California; Norman Harelik, Mather High School, Chicago, Illinois; J. A. H. Hunter, Toronto, Ontario, Canada; Robert Sandling, Columbia University; Anita Skelton, Watervliet Arsenal, New York; David L. Silverman, Beverly Hills, California; Orvan Sommers, West Bend High School, Wisconsin; C. W. Trigg, Los Angeles City College; Hazel S. Wilson, Jacksonville University, Florida; Brother Louis F. Zirkel, Archbishop Molloy High School, Jamaica, New York; and the proposer.

517: Parabolic Areas

Mathematics Magazine **37** (1964), 56; proposal May 1963

Source PDF p. 133–135.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Here the regions share the arc MN of the parabola, rather than the straight chord.^[E030] Let F and d be the focus and directrix of a parabola. If M and N are any two points on the parabola and M', N' are their respective projections on d , show that

$$\frac{\text{Area } FMN}{\text{Area } N'M'MN} = \text{Constant}.$$

I. Solution by Francis D. Parker, University of Alaska. Using a focal length f and orienting the directrix on the x -axis and the focus on the y -axis, we may use $y = x^2/(4f) + f$ as the equation of the parabola. If the abscissas of M, N are a, b , respectively, with $a < b$, straightforward calculations yield

$$\text{Area } MM'N'N = \int_a^b y \, dx = \frac{b-a}{12f} (12f^2 + a^2 + ab + b^2)$$

and

$$\text{Area } FMN = \frac{b-a}{24f}(12f^2 + a^2 + ab + b^2).$$

Hence the ratio of the areas is independent of f, a, b and equals $1/2$. Here f denotes the focal length called F in the source, to distinguish it from the focus point. Both areas use the parabolic arc.^[E030]

Editorial detail. The focus is $(0, 2f)$. The area swept by its radius vector is

$$\frac{1}{2} \int_a^b (xy' - (y - 2f)) \, dx = \frac{1}{2} \int_a^b y \, dx,$$

which supplies the omitted calculation directly.

II. Historical argument by Joel Kugelmass, Stanford University and the National Bureau of Standards (not a valid proof). It is clear that any parabola $f_1(x)$ can be transformed into another parabola $f_2(x)$ by applying a projective transformation P , an orthogonal transformation O and suitable rotations and translations. All of these transformations preserve the ratio of the area of the triangular region to that of the trapezoidal region. Hence we may transform any parabola to $y = x^2$. If we transform again so that $\lim M - N = 0$, the areas clearly approach the length of their altitudes which in turn approaches p , the distance from the focus to the center. Now the function $z = (p + \epsilon_1)/(p + \epsilon_2)$, where the divisor and dividend are the areas of the regions, is monotone after a sufficient number of transformations ($\epsilon_1 + \epsilon_2 < \delta$) and hence approaches the limit. Now as all of the ratios are the same under the given transformations, the original ratio equals a constant and the theorem is proved.

Editorial assessment. General projective transformations do not preserve area ratios or Euclidean focus–directrix data. The limiting argument also does not justify replacing areas by lengths or the asserted monotonicity. Solution I proves the corrected curvilinear statement without those claims.^[E053]

Also solved by Josef Andersson, Vaxholm, Sweden; Michael J. Pascual, Watervliet Arsenal, New York; Hazel S. Wilson, Jacksonville University, Florida; and the proposer.

Editorial note (original journal). Dermott A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; P. R. Nolan, Department of Education, Dublin, Ireland; and Brother Louis F. Zirkel, Archbishop Molloy High School, Jamaica, New York; each pointed out that the proposal is incorrect if the figures FMN and $N'M'MN$ are considered to be the rectilinear areas instead of areas bounded by the arc of the parabola, MN . One incorrect solution was received.

Comment on Problem 517

Mathematics Magazine **37** (1964), 517 [compilation citation; journal page not independently verified].

Source PDF p. 134–135.

Comment by Josef Andersson, Vaxholm, Sweden. This property of the areas characterizes in a way the parabola. In fact, if $r = f(\phi)$ is the equation of a curve K in polar coordinates, associated to orthonormal coordinates $X'OX, Y'OY$, and for each point P of K we construct PP' equivalent to $OQ(r, 0)$, the point P' traces the curve K' . Let A_1, A_2 represent the areas between $X'OX, OP, K$ and $K, PP', K', X'OX$, respectively. The condition

$$\frac{1}{2}r^2 = \frac{dA_1}{d\phi} = \frac{1}{2} \frac{dA_2}{d\phi} = \frac{1}{2} \frac{dA_2}{d(r \sin \phi)} \frac{d(r \sin \phi)}{d\phi} = \frac{1}{2}r \frac{d(r \sin \phi)}{d\phi} \quad (1)$$

gives

$$\frac{d(r \sin \phi)}{r \sin \phi} + \frac{d(\cot(\phi/2))}{\cot(\phi/2)} = 0, \quad 2r \cos^2 \frac{\phi}{2} = \text{constant}.$$

Therefore K is a parabola with O as focus, $X'OX$ as axis, and K' is the directrix. The hypothesis and the ratio $1/2$ are deduced immediately from (1) if we take at first one of the points at the vertex.

529: Center of Curvature

Mathematics Magazine **37** (1964), 124; proposal September 1963

Source PDF p. 135–136.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

A cycloid (cardioid) rolls on a straight line without sliding. Prove that the locus of the center of curvature at the point of tangency lies on a circle (ellipse), and identify the portion traced by one arch (one circuit).^[E054]

Solution by P. R. Nolan, Department of Education, Dublin, Ireland. Cycloid. Taking the regular case and putting $\omega t = \alpha$, we have

$$x = a(\alpha - \sin \alpha), \quad y = a(1 - \cos \alpha).$$

By the usual methods, the arc length from the origin and the radius of curvature are

$$S_\alpha = 4a(1 - \cos(\alpha/2)), \quad P_\alpha = 4a \sin(\alpha/2).$$

If one arch rolls once along the y -axis, the coordinates of the center of curvature at the point of tangency will be (P_α, S_α) . Its locus is therefore

$$x^2 + (y - 4a)^2 = (4a)^2,$$

which is a *semicircle*, negative values of $x(P)$ not being admissible, unless we consider the next arch to roll back along the y -axis to complete the locus-circle.

Cardioid. In polar coordinates, $r = a(1 - \cos \theta)$. As before, this gives

$$S_\theta = 4a(1 - \cos(\theta/2)), \quad P_\theta = \frac{4a}{3} \sin(\theta/2).$$

If the cardioid rolls once along the upper edge of the x -axis, the coordinates of the center of curvature at the point of tangency will be (S_θ, P_θ) . Its locus is therefore

$$(x - 4a)^2 + 9y^2 = (4a)^2,$$

which is the *upper half of an ellipse*, negative values of $y(P)$ not being admissible, unless the same cardioid is also rolled along the lower edge of the axis.^[E054]

Also solved by the proposer.

537: Extreme Overlap

Mathematics Magazine **37** (1964), 277; proposal January 1964

Source PDF p. 136–137.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Determine the relative positions of an equilateral triangle and a square inscribed in the same circle so that their common area shall be an extremum.

Solution by Michael Goldberg, Washington, D. C. The extrema are the symmetric relative positions of the square and the triangle.

The minimum overlap occurs when a side of the square is parallel to a side of the triangle, as in the left-hand diagram. The protruding portions of the triangle are marked A, B . For an infinitesimal rotation from this position, an increase in one of the B areas is compensated by the corresponding decrease in the other B area, while the area A is reduced.

The maximum overlap occurs when a vertex of the square coincides with a vertex of the triangle, as in the right-hand diagram. The equal protruding portions of the triangle are marked C . For an infinitesimal rotation, an increase in one C is compensated by the corresponding decrease in the other C , while a portion of the triangle at the third vertex will now protrude.

If the radius of the circle is unity, the areas, with the first two printed expressions corrected, are

$$A = \frac{(\sqrt{2} - 1)^2}{2\sqrt{3}}, \quad 2B = \frac{\sqrt{3}(\sqrt{3} - \sqrt{2})^2}{4}, \quad 2C = \frac{9 - 5\sqrt{3}}{4}.$$

Hence $A + 2B \approx 0.0933$ is the maximum protrusion (minimum overlap), and $2C \approx 0.0849$ is the minimum protrusion (maximum overlap). The overlap areas are respectively $3\sqrt{3}/4 - A - 2B$ and $3\sqrt{3}/4 - 2C$.^[E055]

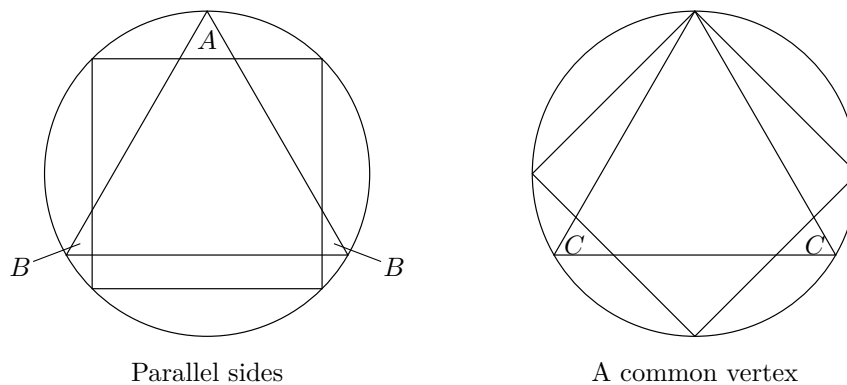


Figure 7.4: The two symmetric positions in Proposal 537: minimum overlap on the left and maximum overlap on the right. Letters mark the parts of the triangle outside the square.

544: A Conditional Alphametic

Mathematics Magazine **37** (1964), 354; proposal March 1964

Source PDF p. 137–138.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Solve the cryptarithm (alphametic)

$$ONE + TWO + SIX = NINE$$

in the base 10, with the following conditions:

(a) $ONE < TWO < SIX$;

(b) $2 \mid TWO$, $6 \mid SIX$, $9 \mid NINE$, where $a \mid b$ means “ a divides b ”.

Solution by Sister Mary Joy, Notre Dame College, St. Louis, Missouri. Since each letter represents a different digit, it can readily be seen from condition (a) that $O < T < S$, $S \geq O + 2$, $T \geq O + 1$, and from condition (b) that TWO and SIX are both even.

Observe that E occupies the units place in the sum. Thus $O + X = 10$. Both X and O are single digits, neither can be zero, nor can the sum be greater than $1E$. From the fact that N occupies the tens place in the sum, it follows that $W + I = 9$. Also $W + I \neq 10$, as there is 1 ten carried from the units column.

Thus there are four possible ordered pairs for X, O : $(8, 2)$, $(2, 8)$, $(6, 4)$, $(4, 6)$. Pairs of addends for W, I are chosen such that neither duplicates a digit already taken. Possibilities for S are chosen such that $6 \mid SIX$, where $S \geq O + 2$. If no duplication has occurred thus far, T is chosen so that $O + T + S + 1 = 10N + I$.^[E056] With still no duplication of digits, E is determined such that $9 \mid NINE$. Consequently the solution is

<i>ONE</i>	217
<i>TWO</i>	392
<i>SIX</i>	408
<i>NINE</i>	1017

Also solved by Josef Andersson, Vaxholm, Sweden; Merrill Barneby, University of North Dakota; Maxey Brooke, Sweeny, Texas; J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; David M. Cohen, East Midwood Day School, Brooklyn, New York; Martin J. Cohen, Beverly Hills, California; Michael P. Commonoff, Lew Wallace High School, Gary, Indiana; John A. Dossy, Illinois State University, Normal, Illinois; Joseph M. Fice, Massachusetts Institute of Technology; C. E. Franti, Berkeley, California; Philip Fung, Fenn College, Ohio; Harry M. Gehman, SUNY at Buffalo, New York; Murray Geller, Jet Propulsion Laboratory, Pasadena, California; Anton Glasser, Pennsylvania State University, Abington, Pennsylvania; Garold F. Gregory, Forest Disease Research Laboratory, Delaware, Ohio; C. T. Haskell, California State Polytechnic College, San Luis Obispo, California; Burton S. Holland, Harpur College, New York; William R. Holt, Delaware, Ohio; J. A. H. Hunter, Toronto, Ontario, Canada; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Janice Langan, Lew Wallace High School, Gary, Indiana; John W. Malsom, Texas A and I, Kingsville, Texas; Wa Hin Ng, San Francisco, California; C. C. Rice, IBM, Endicott, New York; Perry A. Scheinok, Hahnemann Medical College, Philadelphia, Pennsylvania; C. W. Trigg, San Diego, California; A. M. Vaidya, Pennsylvania State University; J. S. Vider, Ottawa, Canada; Thomas Wotain, Lew Wallace High School, Gary, Indiana; Dale Woods, Northeast Missouri State Teachers College; Charles Ziegenfus, Madison College, Virginia; and the proposer.

563: Angles in a Hexagon

Mathematics Magazine **38** (1965), 122; proposal September 1964

Source PDF p. 139.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let A, B', A', B be four consecutive vertices of a regular hexagon. If M is in the open minor arc $A'B'$ of the circumcircle^[E057] and MA, MB intersect BB' and AA' in the points E and F , respectively, then prove that:

$$\angle MEF = 3\angle MAF, \quad (a)$$

$$\angle MFE = 3\angle MBE. \quad (b)$$

Solution by Richard A. Jacobson, South Dakota State University. Let $AB' = x$ and $\angle MAF = a$. Noting that $\angle AB'B = \angle AA'B = \angle AMB = 90^\circ$, we have from triangles $AMB, AB'E, AA'B$ that

$$AM = 2x \cos(30^\circ + a), \quad BM = 2x \sin(30^\circ + a), \quad AE = \frac{x}{\cos(30^\circ - a)}, \quad BF = \frac{x}{\cos a}.$$

Thus in triangle EMF we find that

$$\begin{aligned} \tan \angle MEF &= \frac{MF}{ME} = \frac{BM - BF}{AM - AE} \\ &= \frac{2 \sin(30^\circ + a) - 1/\cos a}{2 \cos(30^\circ + a) - 1/\cos(30^\circ - a)} \\ &= \frac{2 \sin(30^\circ + a) \cos a - 1}{\cos a} \frac{\cos(30^\circ - a)}{2 \cos(30^\circ + a) \cos(30^\circ - a) - 1} \\ &= \frac{2 \sin(30^\circ + 2a) - 1}{\cos a} \frac{\cos(30^\circ - a)}{2 \cos(2a) - 1} \\ &= \frac{2 \sin(30^\circ + 2a) \cos(30^\circ - a) - \cos(30^\circ - a)}{2 \cos(2a) \cos a - \cos a} \\ &= \frac{\sin(60^\circ + a) + \sin(3a) - \cos(30^\circ - a)}{\cos(3a) + \cos a - \cos a} = \frac{\sin(3a)}{\cos(3a)} = \tan(3a). \end{aligned}$$

Since $0 < a < 30^\circ$ in the stated open-arc configuration, $\angle MEF = 3\angle MAF$. Part (b) is done similarly.^[E057]

Also solved by Leon Bankoff, Los Angeles, California; J. D. E. Konhauser, HRB-Singer, State College, Pennsylvania; Stanley Rabinowitz, Far Rockaway, New York; Sidney Spital, California State Polytechnic College; and the proposer.

572: A Memorial Cryptarithm

Mathematics Magazine **38** (1965), 242; proposal January 1965

Source PDF p. 140.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

To the memory of President Kennedy. Mr. J. F. Kennedy was killed on November 22, 1963. That is, on the day 11-22-1963. Solve the cryptarithm

$$JF \cdot (KEN + NEDY) = (11 + 22) \cdot 1963$$

in the decimal system.

Solution by Harry M. Gehman, SUNY at Buffalo, New York. Since $(11 + 22) \cdot 1963$ is the product of the four primes 3, 11, 13, 151, the only possible values of JF are 13 and 39. The latter case leads to a contradiction, and hence $J = 1$, $F = 3$. From this it follows that $KEN + NEDY = 4983$, which leads to $N = 4$, $Y = 9$, and either $K = 2, E = 7, D = 0$ or $K = 7, E = 2, D = 5$. Thus

$$(11 + 22) \cdot 1963 = 13 \cdot (274 + 4709) = 13 \cdot (724 + 4259).$$

The fact that this problem has two solutions means (to a Republican) that *JFK* was *not* unique.

Also solved by Robert H. Anglin, Danville, Virginia; Merrill Barneby, University of North Dakota; Murray Berg, Standard Oil Company, San Francisco, California; Charles R. Berndtson, Institute of Naval Studies, Cambridge, Massachusetts; Dermott A. Breault, Sylvania A.R.L., Waltham, Massachusetts; Robert Brodeur, Lachine, Canada; Maxey Brooke, Sweeny, Texas; Allan Chuck, San Francisco, California; R. J. Cormier, Northern Illinois University; Monte Derham, San Francisco, California; Herta T. Freitag, Roanoke, Virginia; Philip Fung, Fenn College, Ohio; Anton Glaser, Pennsylvania State University, Ogontz Campus; Elmer E. Hunt, Jr., Boise Junior College, Boise, Idaho; J. A. H. Hunter, Toronto, Ontario, Canada; Joel V. Kamer, Cambridge, Massachusetts; John Koelzer, University of Iowa; Wahin Ng, San Francisco, California; C. C. Oursler, Southern Illinois University (Edwardsville); Harry Panish, Pomona, California; Laurence A. Ringenberg, Eastern Illinois University; Sidney Spital, California State Polytechnic College; P. K. Subramanian, Miami University, Ohio; Charles W. Trigg, San Diego, California; William K. Viertel, State University Agricultural and Technical College, Canton, New York; Dale Woods, Northeast Missouri State Teachers College; Charles Ziegenfus, Madison College, Virginia; and the proposer.

587: A Trigonometric Inequality

Mathematics Magazine **39** (1966), 127; proposal May 1965

Source PDF p. 141–142.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Prove the following inequality:^[E029]

$$\left(\frac{\theta + \sin \theta}{\pi} \right)^2 + \cos^4 \frac{\theta}{2} < 1, \quad 0 < |\theta| < \pi.$$

Solution by Samuel Wolf, Linthicum Heights, Maryland. Put

$$F(\theta) = \left(\frac{\theta + \sin \theta}{\pi} \right)^2 + \left(\frac{1 + \cos \theta}{2} \right)^2.$$

Differentiating and setting the result equal to zero gives

$$\frac{2}{\pi^2}(\theta + \sin \theta)(1 + \cos \theta) = \frac{1}{2}(1 + \cos \theta) \sin \theta.$$

Since $\cos \theta \neq -1$ in the open interval, this becomes

$$\frac{4}{\pi^2}\theta + \left(\frac{4}{\pi^2} - 1\right)\sin \theta = 0.$$

One solution is $\theta = 0$; the nonzero solutions satisfy

$$\frac{\sin \theta}{\theta} = \frac{4}{\pi^2 - 4} \approx 0.6815, \quad \theta \approx \pm 1.46$$

(Jahnke and Emde, appendix, p. 33). The corresponding values are $F(0) = 1$ and $F(\pm 1.46) \approx 0.92$. The second derivative is

$$\begin{aligned} F''(\theta) &= \frac{2}{\pi^2}[(1 + \cos \theta)^2 - (\theta + \sin \theta) \sin \theta] \\ &\quad - \frac{1}{2}[-\sin^2 \theta + (1 + \cos \theta) \cos \theta]. \end{aligned}$$

It is negative at zero and positive at the two nonzero stationary points. Thus $F \leq 1$. The equality sign is necessary if zero is retained in the domain.^[E029]

Also solved by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; C. B. A. Peck, State College, Pennsylvania; Simeon Reich, Haifa, Israel; Sidney Spital, California State Polytechnic College; K. L. Yocom, South Dakota State University; and the proposer.

Editorial note (original journal). Raymond E. Whitney, Lock Haven State College, Pennsylvania, pointed out the necessity of including the equals sign along with the inequality.

Editorial completion. To replace the numerical stationary-point argument by an exact one, observe that $\sin \theta / \theta$ decreases strictly from 1 to 0 on $(0, \pi)$. Indeed, the derivative numerator $\theta \cos \theta - \sin \theta$ starts at zero and has derivative $-\theta \sin \theta < 0$. Moreover,

$$F'(\theta) = \frac{1 + \cos \theta}{2} \left[\frac{4\theta}{\pi^2} - \left(1 - \frac{4}{\pi^2}\right) \sin \theta \right].$$

There is exactly one stationary point in $(0, \pi)$, a strict minimum. Since $F(0) = F(\pi) = 1$, we have $F(\theta) < 1$ for $0 < \theta < \pi$; evenness proves the negative half of the assertion.^[E058]

Comment on Problem 587

Mathematics Magazine **39** (1966), 188.

Comment by the proposer (ellipse denominator corrected). The given inequality is equivalent to

$$\left(\frac{\theta + \sin \theta}{\pi}\right)^2 + \left(\frac{1 + \cos \theta}{2}\right)^2 < 1.$$

Now consider the cycloid $x = \theta + \sin \theta$, $y = 1 + \cos \theta$, and the ellipse

$$\frac{x^2}{\pi^2} + \frac{y^2}{4} = 1. \quad [E058]$$

They have common origin and equal diameters. The two curves have points in common at the three vertices. We can prove that at the neighborhoods of these points the cycloid is inside the

ellipse. Since their concavity is in the same direction, the cycloid is wholly inside the ellipse except at the three points. The above inequality is the analytical interpretation for the property just mentioned.

Editorial qualification. The last inference does not follow from common concavity alone. The exact derivative argument above supplies the required global containment proof; the three common points correspond to $\theta = -\pi, 0, \pi$.^[E058]

599: Linearly Dependent Vectors

Mathematics Magazine **39** (1966), 134; proposal September 1965

Source PDF p. 143.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

If a, b, c are any three vectors in 3-space, then show that the vectors

$$a \times (b \times c), \quad b \times (c \times a), \quad c \times (a \times b)$$

are linearly dependent.

Solution by Carl G. Wagner, Duke University. By a well-known theorem of the vector calculus (see p. 90 of Nickerson, Steenrod, and Spencer's *Advanced Calculus* for a proof based on axioms for a vector product),

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C.$$

Writing the other vector products,

$$\begin{aligned} B \times (C \times A) &= (B \cdot A)C - (B \cdot C)A \\ &= (A \cdot B)C - (B \cdot C)A, \\ C \times (A \times B) &= (C \cdot B)A - (C \cdot A)B \\ &= (B \cdot C)A - (A \cdot C)B. \end{aligned}$$

Hence

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0.$$

This is known as the Jacobi identity.

Also solved by Joseph B. Bohac, St. Louis, Missouri; Dermott A. Breault, Sylvania Applied Research Laboratory, Waltham, Massachusetts; Dewey C. Duncan, Los Angeles, California; Philip Fung, Cleveland State University, Ohio; Mrs. A. C. Garstang, Boulder, Colorado; Carl Harris, Western Electric Company, Princeton, New Jersey; Stephen Hoffman, Trinity College, Connecticut; John E. Homer, Jr., St. Procopius College, Illinois; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; John Kiefer, University of Missouri at Rolla; E. S. Langford, U.S. Naval Postgraduate School; Lieselotte Müller, Georgia Institute of Technology; Stanley Rabinowitz, Far Rockaway, New York; Kenneth A. Ribet, Brown University; Richard Riggs, Jersey City State College; Howard L. Walton, Yorktown High School, Arlington, Virginia; K. L. Yocom, South Dakota State University; and the proposer.

600: Related Triangles

Mathematics Magazine **39** (1966), 189; proposal November 1965

Source PDF p. 144.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

If the area of a triangle ABC is S and the areas of the in- and ex-contact triangles are T, T_a, T_b, T_c , then show that

$$T_a + T_b + T_c - T = 2S, \quad (1)$$

$$T_a^{-1} + T_b^{-1} + T_c^{-1} - T^{-1} = 0. \quad (2)$$

Solution by the proposer (missing factor restored). Let I be the incenter and DEF the in-contact triangle of ABC , and let R, r be the circumradius and inradius, respectively. Write $[XYZ]$ for the area of triangle XYZ , and $s = (a + b + c)/2$. Then

$$\frac{[IEF]}{S} = \frac{\frac{1}{2}r^2 \sin(\pi - A)}{\frac{1}{2}bc \sin A} = \frac{r^2}{bc} = \frac{ar^2}{abc} = \frac{ar^2}{4RS},$$

or $[IEF] = ar^2/(4R)$, and similarly $[IFD] = br^2/(4R)$ and $[IDE] = cr^2/(4R)$. Thus

$$T = [IEF] + [IFD] + [IDE] = \frac{(a + b + c)r^2}{4R} = \frac{2sr^2}{4R} = \frac{Sr}{2R}, \quad [\text{E059}]$$

and similarly

$$T_a = \frac{Sr_a}{2R}, \quad T_b = \frac{Sr_b}{2R}, \quad T_c = \frac{Sr_c}{2R}.$$

Using $r_a + r_b + r_c = 4R + r$ and $r_a^{-1} + r_b^{-1} + r_c^{-1} = r^{-1}$, we obtain

$$T_a + T_b + T_c - T = \frac{S(r_a + r_b + r_c - r)}{2R} = 2S,$$

$$T_a^{-1} + T_b^{-1} + T_c^{-1} - T^{-1} = \frac{2R}{S} \left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} - \frac{1}{r} \right) = 0.$$

Also solved by P. N. Bajaj, Western Reserve University; Stanley Rabinowitz, Far Rockaway, New York; G. L. N. Rao, J. C. College, Jamshedpur, India.

609: Crypta-Equivalence

Mathematics Magazine **39** (1966), 248; proposal January 1966

Source PDF p. 145.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Solve the following cryptarithm in the decimal system:

$$4 \cdot \text{NINE} = 9 \cdot \text{FOUR}.$$

Solution by J. D. E. Konhauser, University of Minnesota. The products $4E$ and $9R$ must have the same units digit. Therefore the only possible (E, R) combinations are $(1, 6)$, $(3, 8)$, $(5, 0)$, $(7, 2)$, and $(9, 4)$. Since 4 and 9 are relatively prime, 9 must divide $2N + I + E$.

Case (1, 6). If $E = 1$, 9 must divide $2N + I + 1$, leading to the following (N, I) combinations:

$$(4, 0), (3, 2), (7, 3), (2, 4), (5, 7), (9, 8), (4, 9).$$

The corresponding values for *FOUR* are 1796, 1436, 3276, 1076, 2556, 4396, and 2196. The first two, the fourth, and the last must be rejected since $E = 1$. The third is out since $N = 7$. The fifth is out since 5 is repeated. The sixth is out since $N = 9$.

Similar analysis applied to the remaining cases leads to the solutions

(E, R)	$4 \cdot NINE = 9 \cdot FOUR$
$(3, 8)$	$4 \cdot 4743 = 9 \cdot 2108,$
$(5, 0)$	$4 \cdot 6165 = 9 \cdot 2740,$
$(7, 2)$	$4 \cdot 6867 = 9 \cdot 3052,$
$(9, 4)$	$4 \cdot 5859 = 9 \cdot 2604.$

Also solved by Monte Dernham, San Francisco, California; Samuel P. Hoyle, Jr., University of Virginia; Sidney Kravitz, Dover, New Jersey; C. C. Oursler, Southern Illinois University (Edwardsville); Richard Riggs, Jersey City State College, New Jersey; Jerome J. Schneider, Chicago, Illinois; and Charles W. Trigg, San Diego, California.

Partial solutions were submitted by Merrill Barney, Wisconsin State University (La Crosse); Charles R. Berndtson, Massachusetts Institute of Technology; Lindley J. Burton, Lake Forest College, Illinois; Anton Glaser, Pennsylvania State University (Ogontz); J. A. H. Hunter, Toronto, Canada; Beatriz Margolis, University of Maryland; John W. Milsom, Slippery Rock State College, Pennsylvania; William L. Mrozek, Wyandotte, Michigan; Sam Newman, Atlantic City, New Jersey; C. R. J. Singleton, Petersham, Surrey, England; Lowell Van Tassel, San Diego City College; Gary B. Weiss, New York University, School of Medicine; Donald R. Wilder, Rochester, New York; Dale Woods, Missouri State Teachers College; and the proposer.

628: Pythagorean Alphametic

Mathematics Magazine **40** (1967), 102; proposal September 1966

Source PDF p. 146.

Proposed by B. Süer and Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Solve the alphametic

$$\text{COS}^2 + \text{SIN}^2 = \text{UNO}^2$$

in the decimal system.

Solution by J. A. H. Hunter, Toronto, Ontario, Canada. We have $S^2 + N^2 \equiv O^2 \pmod{10}$, and obviously $S \neq 0$. For each $N = 0, 1, \dots, 9$, we tabulate possible S and corresponding O values, bearing in mind digital “square-endings.”

Since $U > S$, we then test each possible *UNO* value to find its representations (if any) as a sum of two squares, bearing in mind the conditions required for this to be possible. Where representation as a sum of squares is possible, we can then consider corresponding *SIN* and *COS* from the well-known solution

$$(x^2 + y^2)^2 k^2 = (x^2 - y^2)^2 k^2 + (2xy)^2 k^2.$$

The working is somewhat tedious, but not unduly so. It is found that uniquely we have

$$391^2 + 120^2 = 409^2.$$

Also solved by R. H. Anglin, Danville, Virginia; Merrill Barney, Wisconsin State University at La Crosse; Sarah Brooks, Utica Free Academy, New York; Jack Dix, Rutgers University; Charles R. Fleenor, Ball State University, Indiana; Michael Goldberg, Washington, D.C.; Jerome J. Schneider, Chicago, Illinois; Charles W. Trigg, San Diego, California; and the proposers.

639: A Convex Quadrilateral Inequality

Mathematics Magazine **40** (1967), 166; proposal November 1966

Source PDF p. 147.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let $ABCD$ be a convex quadrangle and P be the intersection of diagonals AC and BD . Let the distance of P from the sides $AB = a$, $BC = b$, $CD = c$, $DA = d$ be x, y, z, t , respectively. Prove that

$$x + y + z + t < \frac{3}{4}(a + b + c + d).$$

Solution by Leon Bankoff, Los Angeles, California. Let the bisectors of the angles between the diagonals AC and BD meet AB, BC, CD, DA in R, S, T, U . By a corollary of the Erdős–Mordell theorem,

$$\begin{aligned} 2(PS + PT) &< PB + PC + PD, \\ 2(PT + PU) &< PC + PD + PA, \\ 2(PU + PR) &< PD + PA + PB, \\ 2(PR + PS) &< PA + PB + PC. \end{aligned}$$

Thus

$$4(PR + PS + PT + PU) < 3(PA + PB + PC + PD).$$

This inequality is stronger than the one proposed because

$$x + y + z + t \leq PR + PS + PT + PU$$

and $PA + PB + PC + PD < a + b + c + d$.

Also solved by Leon Bankoff, Los Angeles, California (second solution); Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania; Stanley Rabinowitz, Far Rockaway, New York; and the proposer.

649: An Alphametic

Mathematics Magazine **40** (1967), 279; proposal March 1967

Source PDF p. 147–148.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Solve the cryptarithm

$$\begin{array}{rcccccc} & T & H & R & E & E \\ + & & F & O & U & R \\ \hline S & E & V & E & N \end{array}$$

in the decimal system such that:

- 3 does not divide *THREE*, in which the digit 3 is missing;
- 4 does not divide *FOUR*, in which the digit 4 is missing;
- 7 does not divide *SEVEN*, in which the digit 7 is missing.

Solution by Harry M. Gehman, SUNY at Buffalo, New York. Let us first solve the cryptarithm, given only that (a) the digit 3 is missing from *THREE*; (b) the digit 4 is missing from *FOUR*; and (c) the digit 7 is missing from *SEVEN*. The problem has seven solutions:

- (1) $16544 + 7805 = 24349$
- (2) $47266 + 9102 = 56368$
- (3) $75244 + 9102 = 84346$
- (4) $79244 + 5102 = 84346$
- (5) $17544 + 6805 = 24349$
- (6) $49266 + 7102 = 56368$
- (7) $24811 + 6708 = 31519$

The condition (d) that 3 does not divide *THREE* eliminates solutions (5) and (6). The condition (e) that 4 does not divide *FOUR* eliminates (7). The condition (f) that 7 does not divide *SEVEN* does not eliminate any solution. Therefore the problem as proposed has four solutions: (1)–(4).

If we ignore conditions (d), (e), (f), but retain conditions (a), (b), (c), with the additional condition indicated we have unique solutions as follows:

	Additional condition	Solution
(g)	<i>THREE</i> contains the digit 8	(7)
(h)	<i>SEVEN</i> contains the digit 1	(7)
(i)	<i>FOUR</i> contains both the digits 5 and 6	(5)
(j)	<i>THREE</i> contains neither 6 nor 7	(7)
(k)	<i>THREE</i> contains both 6 and 7	(2)
(l)	<i>THREE</i> contains both 1 and 2	(7)
(m)	<i>THREE</i> contains neither 5, 6, nor 7	(7)
(n)	<i>THREE</i> contains neither 5, 7, nor 9	(7)

And so on. The fact that solution (7) occurs so frequently in this list seems to indicate that it has a pattern of digits essentially different from the other six solutions. From the standpoint of numerology, this has some deep significance, I am sure.

680: A Circular Locus

Mathematics Magazine **41** (1968), 219; proposal January 1968

Source PDF p. 149–150.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let E be an ellipse and t', t'' be two variable parallel tangents to it. Consider a circle C , tangent to t', t'' and to E externally. Show that the locus of the center of C is a circle.

Solution by the proposer (followed by an editorial completion). Let the ellipse be given by

$$x^2/a^2 + y^2/b^2 = 1. \quad (1)$$

Denoting the center and radius of the circle by (α, β) and r , we have

$$r^2 = \frac{a^2\beta^2 + b^2\alpha^2}{\alpha^2 + \beta^2}. \quad (2)$$

The circle is also given by

$$(x - \alpha)^2 + (y - \beta)^2 = r^2, \quad (3)$$

such that the normal at $T = (x, y)$ of the ellipse passes through the center C . The distance CT is an extremal distance from C to the ellipse. To determine it we use the method of Lagrange multipliers. Let

$$F(x, y) = (x - \alpha)^2 + (y - \beta)^2 + \lambda(x^2/a^2 + y^2/b^2 - 1).$$

The point T is determined by (1) and

$$x - \alpha + \lambda x/a^2 = 0, \quad y - \beta + \lambda y/b^2 = 0. \quad (4)$$

Eliminating x, y, α, β gives a quartic equation in λ , so we proceed in a different way. Supposing the statement is true, we have

$$OC^2 = \alpha^2 + \beta^2 = (a + b)^2. \quad (5)$$

Solving for α, β from (4), setting them in (5), and comparing the result with (1) gives $\lambda = ab$. If we choose $\lambda = ab$, equations (1), (2), and (3) are consistent, and this consistency gives $\alpha^2 + \beta^2 = (a + b)^2$.

Editorial completion of necessity and converse. The consistency argument alone does not exclude other centers. Write

$$T = (a \cos t, b \sin t), \quad q = \frac{\cos^2 t}{a^2} + \frac{\sin^2 t}{b^2} > 0.$$

External tangency implies $\lambda > 0$ in (4), and gives

$$\alpha = (a + \lambda/a) \cos t, \quad \beta = (b + \lambda/b) \sin t, \quad r^2 = \lambda^2 q.$$

Substitution in (2) yields the factorization

$$r^2(\alpha^2 + \beta^2) - a^2\beta^2 - b^2\alpha^2 = (\lambda^2 - a^2b^2)(1 + \lambda q)^2 = 0.$$

Consequently $\lambda = ab$ and

$$C = ((a + b) \cos t, (a + b) \sin t).$$

Conversely, every such center, with $r = ab\sqrt{q}$, satisfies (2) and lies on the outward normal at T . The circle lies on the opposite side of the supporting tangent at T from the ellipse, so the tangency is external. Equation (2) gives tangency to the two supporting lines parallel to OC . Thus every point of the circle of radius $a + b$ occurs.^[E060]

Also solved by Michael James Smithson, Bellevue, Washington.

Comment on Problem 680

Mathematics Magazine **42** (1969), 162.

Comment by A. W. Walker, Toronto, Canada. Many interesting properties of an ellipse are associated with its so-called Chasles circles of radius $a \pm b$ concentric with the ellipse. If the center of the variable circle C lies on the inward normal, its locus is the inner Chasles circle. The result in Problem 680 was established by Mannheim, *Nouvelles Annales de Mathématiques*, 4th series, **3** (1903), 483, and is equivalent to the following old Japanese theorem (Iwata, 1862):

If an ellipse touches externally two equal nonoverlapping fixed circles and their parallel common tangents, the sum of its major and minor axes is equal to the distance between the centers of the circles.

See *Tôhoku Mathematical Journal*, 1st series, **11** (1917), 65, where with rather obscure justification, it is asserted that the theorem is untrue!

724: Triangle Probability

Mathematics Magazine **42** (1969), 274; proposal March 1969

Source PDF p. 150–151.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Find the probability that for a point P taken at random in the interior of a triangle ABC ($a \geq b \geq c$), the distances of P from the sides of ABC form the lengths of sides of a triangle.

Solution by L. Carlitz, Duke University. Let the internal angle bisectors meet the sides BC, CA, AB in L, M, N , respectively. Let x, y, z denote the distances from P to BC, CA, AB . The equation $x = y + z$ represents the straight line MN . Similarly $y = z + x$ represents NL , and $z = x + y$ represents LM . The incenter I is in the interior of LMN ; hence P must be restricted to the interior of LMN . With the uniform area distribution, the desired probability is

$$p = \frac{\text{area } LMN}{\text{area } ABC} = \frac{\Delta_0}{\Delta}.$$

Now $CL = ab/(b+c)$ and $CM = ab/(a+c)$, so

$$\text{area } CLM = \frac{1}{2} \frac{a^2 b^2 \sin \gamma}{(a+c)(b+c)} = \frac{ab\Delta}{(a+c)(b+c)}.$$

By symmetry,

$$\text{area } AMN = \frac{bc\Delta}{(b+a)(c+a)}, \quad \text{area } BLN = \frac{ac\Delta}{(a+b)(c+b)}.$$

Thus, with cyclic sums,

$$\begin{aligned} \Delta_0 &= \Delta - \sum_{\text{cyc}} \frac{bc\Delta}{(b+a)(c+a)} \\ &= \Delta \left(1 - \frac{\sum_{\text{cyc}} bc(b+c)}{(a+b)(b+c)(c+a)} \right) = \frac{2abc\Delta}{(a+b)(b+c)(c+a)}. \end{aligned}$$

It follows that

$$p = \frac{2abc}{(a+b)(b+c)(c+a)} \leq \frac{1}{4},$$

with equality only when $a = b = c$, since $a + b \geq 2\sqrt{ab}$ and cyclically.

Also solved by Michael Goldberg, Washington, D.C.; C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania; F. G. Schmitt, Jr., Berkeley, California (partially); Paul J. Zwier, Palo Alto, California; and the proposer.

738: A Ferry and a River Current

Mathematics Magazine **43** (1970), 109

Source PDF p. 151–152.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Historical data (inconsistent for a positive-width river).^[E061] *There is a river with parallel and straight shores. A is located on one shore and B on the other, with $AB = 72$ miles. A ferry boat travels the straight path AB from A to B in four hours and from B to A in nine hours. If the boat's speed on still water is $v = 13$ mph, what is the velocity of the flow?*

Editorial answer. Under the usual constant-current interpretation, these data admit no river of positive width. Let e be the unit vector from A to B and c the current velocity. The ground velocities are $18e$ and $-8e$ miles per hour. Hence

$$|18e - c| = |-8e - c| = 13.$$

Subtracting the squared equations gives $e \cdot c = 5$, and substitution gives $|c|^2 = 25$. Equality in Cauchy–Schwarz forces $c = 5e$: the route must be parallel to the current, hence to the banks. It cannot join opposite shores of a positive-width river. The historical answer of 5 mph is possible only in the zero-width limiting configuration.^[E061]

Historical solution fragment (opening and byline absent from the supplied reproduction).

13 mph + $s \geq 18$ mph, and thus $s \geq 5$ mph.

Returning from B to A, the maximum possible magnitude for the vector sum is 13 mph – s . Thus 13 mph – $s \geq 8$ mph and $s \leq 5$ mph. Since $s \geq 5$ mph and $s \leq 5$ mph, $s = 5$ mph.

An alternate method would be to allow the width of the river to be m miles and find s for any m . It is possible to find m using this method. It involves solving the following equation:

$$13\sqrt{5184 - m^2} = 4\sqrt{13689 - m^2} + 9\sqrt{2704 - m^2}.$$

The only solution is $m = 0$ miles. With this value of m it follows that $s = 5$ mph.

Editorial qualification. The claimed return-speed bound $13 - s$ assumes motion directly against the current and is not valid for an oblique crossing. The concluding zero width exposes the inconsistency rather than supplying a ferry crossing. The source's unit “mph” for m is corrected to “miles”; the missing opening has not been invented.^[E061]

Also solved by Richard L. Breisch, University of Colorado; Bruce A. Broemser, El Sobrante, California; Raphael T. Coffman, Richland, Washington; George F. Corliss, Michigan State University; Mickey Dargitz, Ferris State College, Michigan; Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania; Frank Eccles, Phillips Academy, Massachusetts; W. W. Funkenbusch, Michigan Technological University; Michael Goldberg, Washington, D.C.; John M. Howell, Little Rock, California; Alfred Kohler, Long Island University, New York; Lew Kowarski,

Morgan State College, Maryland; J. R. Kuttler, Johns Hopkins University; Joseph O'Rourke, St. Joseph's College, Pennsylvania; C. D. O'Shaughnessy, University of Saskatchewan; John E. Prussing, University of Illinois; John R. Ray, Clemson University; Simeon Reich, Israel Institute of Technology, Haifa, Israel; Ray B. Robinson, Butler, Tennessee; Steve Rohde, Lehigh University; E. F. Schmeichel, College of Wooster, Ohio; W. A. Schmidt, Texas A&M University; E. P. Starke, Plainfield, New Jersey; Charles W. Trigg, San Diego, California; A. W. Walker, Toronto, Canada; Sam Zaslavsky, City University of New York; and the proposer.

743: Tetrahedral Inequality

Mathematics Magazine **43** (1970), 169; proposal November 1969

Source PDF p. 153–154.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Historical proposal (false as printed).^[E062] Let P be an interior point of a regular tetrahedron, $T \equiv A_1A_2A_3A_4$, with $p_i = PA_i$, and let x_{ij} denote the distance of P from the edge A_iA_j . Then prove

$$\sum_{i=1}^4 p_i \geq \frac{2\sqrt{3}}{3} \sum_{i < j} x_{ij},$$

equality holding if and only if P is at the center O of T .

Historical argument by Michael Goldberg, Washington, D.C. (invalid as a proof of the assertion). Given a base of fixed length and the sum of two other lengths, the triangle of greatest height is obtained when the triangle is isosceles. Similarly, given the same base and the sum of three other lengths to form three triangles by using the three pairs of these three sides, the sum of the heights is maximized when the triangles are congruent isosceles triangles. This can be generalized to n triangles. Hence the sum of the distances of P from the edges of a regular tetrahedron divided by the sum of the distances of P from the vertices is maximized when P is at the center of the tetrahedron.

If e is the edge length, the distance h between opposite edges is given by

$$h^2 + (e/2)^2 + (e/2)^2 = e^2, \quad h = e/\sqrt{2}.$$

The distance R from the center to a vertex is given by

$$R^2 = (e/2)^2 + (h/2)^2 = e^2/4 + e^2/8, \quad R = \frac{\sqrt{3}e}{2\sqrt{2}}.$$

Hence, when P is at the center, the ratio of the sums is

$$\frac{4R}{3h} = \frac{2e\sqrt{3}/\sqrt{2}}{3e/\sqrt{2}} = \frac{2\sqrt{3}}{3}.$$

A similar extremal is obtained for each of the regular polyhedra, and for each of the regular polygons. Of course, the ratio of the two sums depends upon the figure considered.

Editorial note (original journal). Also solved by the proposer. One incorrect solution was received.

Editorial qualification. The passage from separate triangle maximizations to the tetrahedral ratio is invalid, as the following published counterexample demonstrates. The center calculation itself is correct.^[E062]

Comment on Problem 743

Mathematics Magazine **44** (1971), 44.

Comment by E. F. Schmeichel, College of Wooster, Ohio. The inequality should read

$$\sum_{i=1}^4 p_i \geq \frac{2\sqrt{2}}{3} \sum_{i<j} x_{ij}.$$

Apparently a $\sqrt{3}$ was misprinted in place of the $\sqrt{2}$ above. To show that the inequality as printed is false, consider a regular tetrahedron of edge length 1. Let P be the midpoint of edge A_1A_2 . Then

$$\begin{aligned} p_1 = p_2 = \frac{1}{2}, \quad p_3 = p_4 = \frac{\sqrt{3}}{2}, \quad x_{12} = 0, \\ x_{13} = x_{14} = x_{23} = x_{24} = \frac{\sqrt{3}}{4}, \quad x_{34} = \frac{\sqrt{2}}{2}. \end{aligned}$$

Thus

$$\sum_i p_i = 1 + \sqrt{3} < 2.8, \quad \sum_{i<j} x_{ij} = \sqrt{3} + \frac{\sqrt{2}}{2}.$$

Since $\sqrt{6} > 2.4$, we have

$$\frac{2\sqrt{3}}{3} \sum_{i<j} x_{ij} = 2 + \frac{\sqrt{6}}{3} > 2.8,$$

so the printed inequality is reversed at this point. By continuity of the distances involved, this strict reverse inequality remains true for interior points sufficiently near P .

Editorial assessment of the correction. This disproves the historical proposal. Simply replacing its coefficient by $2\sqrt{2}/3$ does not repair the entire statement: at the center the ratio is $2\sqrt{3}/3$, so equality no longer holds there. The comment supplies no proof of the revised bound and does not determine its equality cases. The original assertion is therefore marked as false, and no unverified sharp replacement is asserted in this edition.^[E062]

756: Centrally Symmetric Curves

Mathematics Magazine **43** (1970), 283; proposal March 1970

Source PDF p. 155.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Determine closed and centrally symmetric curves C , other than circles, such that the product of two perpendicular radius vectors (issued from the center) be a constant.

Solution by Harry W. Hickey, Arlington, Virginia. Let us consider a generalized form of the problem: determine closed and centrally symmetric curves, other than circles, such that the product of two radius vectors issued from the center is a constant when their angle is π/N , where N is any positive even integer. Call the center O and the constant product R^2 . Construct a circle K with center O and radius R . For every point of C inside this circle there is a point outside it such that the product of their distances from O is R^2 . Hence C crosses K at some point A , and crosses again at B , where $\angle AOB = \pi/N$.

Draw an arc from A to B which does not pass through O , nor intersect any line through O more than once. Aside from these restrictions, the form of the arc is arbitrary. Let its polar equation be $\rho = f(\theta)$. The restrictions ensure that $1/f$ is finite and f is single-valued;

a multivalued f leads to ambiguities about the radius-vector length. So far f is defined on an interval of length π/N . Extend it by

$$f(\theta + \pi/N) = \frac{R^2}{f(\theta)} \quad \text{for all } \theta.$$

Then the curve C is defined. The function f is periodic with period $2\pi/N$ and, since N is even, $f(\theta + \pi) = f(\theta)$, making C centrally symmetric. Drop the symmetry requirement and N can be odd.

Editorial clarification. Choose a positive continuous, nonconstant initial arc with endpoint radius R to obtain a closed noncircular curve. For instance $f(\theta) = R \exp(\varepsilon \sin N\theta)$, with $\varepsilon \neq 0$ and even N , satisfies the construction. The original perpendicular case is $N = 2$.

Also solved by Michael Goldberg, Washington, D.C.; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Lew Kowarski, Morgan State College, Maryland; John Oman, Wisconsin State University, Oshkosh; E. F. Schmeichel, College of Wooster, Ohio; and the proposer.

763: Quasi Zeta Functions

Mathematics Magazine **44** (1971), 108; proposal May 1970

Source PDF p. 156.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Prove:

$$\left(1 + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \cdots\right) = \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) \left(1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \cdots\right).$$

Solution by M. G. Greening, University of New South Wales, Australia (convergence condition and factor corrected). For real $s > 1$ (or complex s with $\operatorname{Re} s > 1$),

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.^{[E063]}$$

Separating the even terms gives

$$\begin{aligned} 1 + 3^{-4} + 5^{-4} + \cdots &= (1 - 2^{-4})\zeta(4) = \frac{15}{16}\zeta(4), \\ 1 - 2^{-6} + 3^{-6} - 4^{-6} + \cdots &= (1 - 2^{-5})\zeta(6), \\ 1 + 3^{-10} + 5^{-10} + \cdots &= (1 - 2^{-10})\zeta(10) \\ &= (1 - 2^{-5})\frac{33}{32}\zeta(10).^{[E063]} \end{aligned}$$

In the source's unsigned Bernoulli-number convention,

$$\zeta(2n) = \frac{2^{2n-1} B_n \pi^{2n}}{(2n)!}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_5 = \frac{5}{66}.$$

These B_n are the absolute values of the modern B_{2n} . Thus

$$\zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(10) = \frac{\pi^{10}}{93555},$$

and the result follows after simplification.

Also solved by Bernard August, Glassboro State College, New Jersey; Miguel Bamberger, Monterey, California; Walter Blumberg, New Hyde Park, New York; Wray G. Brady, Slippery Rock State College, Pennsylvania; Richard L. Breisch, Pennsylvania State University; Donald R. Childs, Naval Underwater Weapons Research and Engineering Station, Rhode Island; Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania; D. Dummit, San Mateo High School, California; Louis Grinstein, New York, New York; Jeffrey Hoffstein, Bronx High School of Science, New York; Robert F. Jackson, University of Toledo, Ohio; Shiv Kumar, Panjabi University, and Miss Nirmal, Government Girls' High School, Panipat, India (jointly); J. R. Kuttler, Johns Hopkins Applied Physics Laboratory, Maryland; Herbert R. Leifer, Pittsburgh, Pennsylvania; Michael J. Martino, IBM, Poughkeepsie, New York; Kenneth Rosen, University of Michigan; L. E. Schaefer, General Motors Institute, Flint, Michigan; E. F. Schmeichel, College of Wooster, Ohio; E. P. Starke, Plainfield, New Jersey; Paul D. Thomas, Naval Research Laboratory, Washington, D.C.; Graham C. Thompson, Binghamton, New York; Michael R. Wise, University of Colorado; Gregory Walczun, Bucknell University; K. L. Yocom, University of Wyoming; and the proposer.

775: Inverse Functions

Mathematics Magazine **44** (1971), 230; proposal November 1970

Source PDF p. 157–158.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Prove

$$\int_0^1 \sqrt[p]{1-x^p} dx = \int_0^1 \sqrt[q]{1-x^q} dx, \quad p, q > 0.$$

I. Solution by J. C. Binz, Bern, Switzerland (hypotheses clarified). More generally, let f be strictly decreasing and continuous on $[a, b]$. Then its inverse g exists on $[f(b), f(a)]$ and is also strictly decreasing and continuous. When the following change of variable and integration by parts are justified, compute

$$\begin{aligned} \int_{f(b)}^{f(a)} g(y) dy &= \int_b^a g(f(t)) f'(t) dt = \int_b^a t f'(t) dt \\ &= af(a) - bf(b) + \int_a^b f(t) dt. \end{aligned}$$

If additionally $f(a) = b$ and $f(b) = a$, then

$$\int_a^b g(t) dt = \int_a^b f(t) dt.$$

The functions $f(x) = (1-x^q)^{1/p}$ and $g(x) = (1-x^p)^{1/q}$ on $[0, 1]$ form a special case, proving the proposition.

Editorial clarification. A merely nonincreasing function need not have an inverse, and continuity alone does not justify differentiation. The displayed area identity holds for every continuous strictly decreasing function, by the area decomposition between inverse graphs (or integration by parts for Riemann–Stieltjes integrals). For the functions in this problem the derivative calculation is valid, allowing improper endpoint integrals.^[E064]

II. *Solution by Václav Konečný, Jarvis Christian College, Hawkins, Texas (beta factor corrected).* With $z = x^p$ and $p, q > 0$,

$$\begin{aligned}\int_0^1 (1 - x^p)^{1/q} dx &= \frac{1}{p} \int_0^1 z^{-1+1/p} (1 - z)^{1/q} dz \\ &= \frac{1}{p} B\left(\frac{1}{p}, 1 + \frac{1}{q}\right) = \frac{1}{p+q} B\left(\frac{1}{p}, \frac{1}{q}\right). \text{[E064]}\end{aligned}$$

Here B is the beta function. Since $B(x, y) = B(y, x)$, the value of the integral is unchanged if we interchange p and q .

Also solved by Joseph Beer and Bernard August (jointly), Glassboro State College, New Jersey; Walter Blumberg, Flushing High School, New York; Dermott A. Breault, Cyber, Inc., Cambridge, Massachusetts; Robert B. Brennan, Dover, New Jersey; Robert J. Bridgman, Mansfield State College, Pennsylvania; David C. Brooks, Seattle Pacific College, Washington; G. R. Desai, St. Louis University; Robert Desko, Davenport, Iowa; Ellis Detwiler, Adams, New York; Santo M. Diano, Havertown, Pennsylvania; Fred Dodd, University of South Alabama; M. G. Greening, University of New South Wales, Australia; Robert G. Griswold, University of Hawaii, Hilo College; Philip Haverstick, Fort Belvoir, Virginia; Harry W. Hickey, Arlington, Virginia; John E. Homer, Lisle, Illinois; N. J. Kuenzi, Oshkosh, Wisconsin; David E. Mannes, SUNY, Oneonta, New York; Stephen B. Maurer, Phillips Exeter Academy; Edward Moylan, Ford Motor Company, Dearborn, Michigan; Albert J. Patsche, Rock Island Arsenal, Illinois; V. V. Ramana Rao, Andhra University, South India; B. E. Rhoades, Indiana University; Steve M. Rohde, General Motors Research Laboratories, Warren, Michigan; E. F. Schmeichel, College of Wooster, Ohio; Harry Siller, Hofstra University; A. Swanavayanamuti¹, Andhra University, Waltair, South India; R. A. Struble, North Carolina State University; Philip Tracy, APO San Francisco; C. S. Venkataraman, Sree Kerala Varma College, Trichur, South India; John R. Ventura, Jr., Naval Underwater Systems Center, Newport, Rhode Island; R. L. Woodruff, Menlo College, Menlo Park, California; Thomas Wray, Department of Energy, Mines and Resources, Ottawa, Canada; Robert L. Young, Cape Cod Community College, Massachusetts; Paul Zwier, Calvin College, Michigan; and the proposer.

Comment on Problem 775

Mathematics Magazine **45** (1972), 293.

Comment by Ralph Leung, Berkeley, California. The problem would become immediate if we rewrite the equality as

$$\int_0^1 \sqrt[q]{1 - x^p} dx = \int_0^1 \sqrt[p]{1 - y^q} dy.$$

Both sides give the area of the region bounded by the x -axis, the y -axis, and the graph of $x^p + y^q = 1$ in the first quadrant: the left side integrates with respect to x , the right side with respect to y .

¹The reading of this name in the supplied reproduction (source PDF p. 157) is uncertain.

806: Symmetry About a Line

Mathematics Magazine **45** (1972), 171; proposal September 1971

Source PDF p. 158–159.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let H be the orthocenter of a non-right isosceles triangle ABC , and let AH, BH, CH intersect the opposite sides in D, E, F , respectively. Prove that the incenters of the right triangles $HBD, HDC, HCE, HEA, HAF, HFB$ lie on a conic, possibly degenerate.^[E065]

Solution by Vladimir F. Ivanoff, San Carlos, California (symmetry hypothesis made precise). The problem is a special case of the following theorem: six points arranged in three reflected pairs about a line lie on a conic, possibly degenerate. This can be proved by the converse of Pascal's theorem, or analytically.^[E065]

Incidentally, the theorem also holds for three pairs symmetric about a point. Taking that point as origin, the six points have coordinates

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \quad (-x_1, -y_1), (-x_2, -y_2), (-x_3, -y_3).$$

When the three fixed rows are independent, the conic is given by

$$\begin{vmatrix} x^2 & y^2 & xy & 1 \\ x_1^2 & y_1^2 & x_1y_1 & 1 \\ x_2^2 & y_2^2 & x_2y_2 & 1 \\ x_3^2 & y_3^2 & x_3y_3 & 1 \end{vmatrix} = 0.$$

Editorial completion. For reflection in the y -axis, write the pairs as $(\pm x_i, y_i)$. The three equations

$$Ax_i^2 + By_i^2 + Cy_i + D = 0 \quad (i = 1, 2, 3)$$

have a nonzero solution for the four coefficients. This gives a conic through all six points. The isosceles-triangle incenters occur in precisely such pairs. A right isosceles triangle must be excluded because some of the six small triangles collapse and their incenters are undefined.

For central symmetry use $Ax^2 + By^2 + Cxy + D = 0$ instead. If the determinant displayed in the historical solution vanishes identically because its fixed rows are dependent, choose any nonzero coefficient vector in the nullspace of those rows; the existence of a conic still follows. Degenerate conics are allowed.^[E065]

Also solved by Leon Bankoff, Los Angeles, California (three solutions); Ragnar Dybvik, Tingvoll, Norway; Michael Goldberg, Washington, D.C.; M. G. Greening, University of New South Wales, Australia; and the proposer.

839: Probability of No Change

Mathematics Magazine **46** (1973), 169; proposal September 1972

Source PDF p. 159–160.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Given three boxes each containing w white balls and r red balls identical in shape. Take a ball from the first box and put it in the second box, then take a ball from the second box and put it in the third, and finally take a ball from the third box and put it in the first. Find the probability that the boxes have their original contents as to color.

Solution by Thomas Spencer, Trenton State College, New Jersey. A moment's reflection shows that the only events leaving the color composition of all three boxes unchanged are the choices white, white, white or red, red, red. Their probabilities by the multiplication rule are

$$\frac{w}{w+r} \frac{w+1}{w+r+1} \frac{w+1}{w+r+1} = \frac{w(w+1)^2}{(w+r)(w+r+1)^2}$$

and

$$\frac{r}{w+r} \frac{r+1}{w+r+1} \frac{r+1}{w+r+1} = \frac{r(r+1)^2}{(w+r)(w+r+1)^2},$$

respectively. These events are disjoint, so the required probability is their sum:

$$\frac{w(w+1)^2 + r(r+1)^2}{(w+r)(w+r+1)^2}.$$

In the case of N boxes, the solution would be

$$\frac{w(w+1)^{N-1} + r(r+1)^{N-1}}{(w+r)(w+r+1)^{N-1}}.$$

Here the draws are uniform; w, r are nonnegative integers with $w+r > 0$.

Also solved by Gladwin Bartel, La Junta, Colorado; Melvin Billick, Midland High School, Michigan; J. L. Brown, Jr., Pennsylvania State University; Joseph B. Browne, Oklahoma State University; Daniel L. Calloway, Ashville, North Carolina; Abraham L. Epstein, Hanscom Field, Massachusetts; George Fabian, Park Forest, Illinois; Michael Goldberg, Washington, D.C.; Kathleen Harris, New Hampton, Iowa; Karl Heuer, Moorhead, Minnesota; John M. Howell, Little Rock, California; Václav Konečný, Jarvis Christian College, Texas; Lew Kowarski, Morgan State College, Maryland; Michael W. O'Donnell, Carnegie-Mellon University; George Pfeiffer, Old Dominion University, Virginia; Louisa Russo, Michigan Technological University; R. Shantaram, University of Michigan-Flint; and the proposer.

859: A Non-Unique Cryptarithm

Mathematics Magazine **47** (1974), 49; proposal March 1973

Source PDF p. 161–162.

Proposed by B. Sürer and Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Solve the cryptarithm

$$THREE + NINE = EIGHT + FOUR.$$

I. Solution by Harry L. Nelson, Livermore, California (leading-zero convention clarified). There are twelve solutions in decimal base if a leading zero is allowed in *FOUR*. Six representatives are

$$\begin{array}{rclcl} THREE + NINE & = & EIGHT + FOUR \\ 30122 + 4542 & = & 25703 + 8961 \\ 29433 + 7073 & = & 30692 + 5814 \\ 40233 + 5653 & = & 36104 + 9782 \\ 59766 + 4346 & = & 63295 + 0817 \\ 70566 + 2926 & = & 69107 + 4385 \\ 69877 + 5457 & = & 74096 + 1238 \end{array}$$

In each one, interchange G and O to obtain another solution, yielding twelve in all. Under the usual convention that leading digits are nonzero, omit the fourth row and its mate, leaving ten solutions.^[E066]

If we add the condition that $THREE$ is prime, only the pair

$$69877 + 5457 = 74096 + 1238 = 74296 + 1038$$

qualifies. If in addition $FOUR$ is not divisible by 3, the solution is unique: $74096 + 1238$ on the right.

II. Solution by John Tabor and John Beidler (jointly), University of Scranton, Pennsylvania (table corrected). Solutions to additive cryptarithms are now trivia with the TABOR-AUTOMATIC CRYPTARITHM SOLVER. This program will accept any cryptarithm involving several additions and one equal sign and solve it in any base. The program was written as a term project in a course on DATA STRUCTURES.

The cryptarithm $THREE + NINE = FOUR + EIGHT$ proved interesting in that it has ten solutions. The replacements to obtain these solutions are:

E	T	R	N	H	U	I	G	O	F
2	3	1	4	0	6	5	7	9	8
2	3	1	4	0	6	5	9	7	8
3	2	4	7	9	1	0	6	8	5
3	2	4	7	9	1	0	8	6	5
3	4	2	5	0	8	6	1	7	9
3	4	2	5	0	8	6	7	1	9
6	7	5	2	0	8	9	1	3	4
6	7	5	2	0	8	9	3	1	4
7	6	8	5	9	3	4	0	2	1
7	6	8	5	9	3	4	2	0	1

^[E066]

Total CPU time was 20 seconds on an XDS Sigma 5. The program is in FORTRAN.

Also solved by Merrill Barney, University of Wisconsin at La Crosse; Harold Biller, Brooklyn, New York; Dorothy Brunet, Sherman Oaks, California; Robert Copus, Rose Hulman Institute of Technology; H. Marlon Hewit, Reedley High School, California; J. A. H. Hunter, Toronto, Canada; Janice A. McGoldrick, Cranston High School, Rhode Island; Sam Newman, Atlantic City, New Jersey; Erwin Schmidt, Washington, D.C.; S. O. Shachter, Philadelphia, Pennsylvania; Mary F. Turner, Glen Allen, Virginia; C. S. Venkataraman, Trichur, India; and the proposers.

916: Trilinear Coordinates

Mathematics Magazine **48** (1975), 296; proposal November 1974

Source PDF p. 162.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let XYZ *be the pedal triangle of a point* P *with regard to the triangle* ABC . *Then find the trilinear coordinates* x, y, z *of* P *such that*

$$YA + AZ = ZB + BX = XC + CY.$$

Solution by M. S. Klamkin, University of Waterloo. By drawing segments from P parallel to AB and AC , respectively, and terminating on BC , it follows that

$$BX = x \cot B + z \csc B, \quad CX = x \cot C + y \csc C.$$

Here x, y, z are normalized signed distances to the side-lines; the segment expressions are initially directed. The other distances CY, AY, AZ, BZ follow by cyclic interchange. Seeking a point whose feet lie on the side segments, the common sum must be $2s/3$, where s is the semiperimeter. Thus

$$\begin{aligned}(y+z)(\cot A + \csc A) &= (z+x)(\cot B + \csc B) \\ &= (x+y)(\cot C + \csc C) = \frac{2s}{3}.\end{aligned}$$

Solving,

$$\begin{aligned}x &= \frac{s}{3} \left(\tan \frac{B}{2} + \tan \frac{C}{2} - \tan \frac{A}{2} \right), \\ y &= \frac{s}{3} \left(\tan \frac{A}{2} + \tan \frac{C}{2} - \tan \frac{B}{2} \right), \\ z &= \frac{s}{3} \left(\tan \frac{A}{2} + \tan \frac{B}{2} - \tan \frac{C}{2} \right).\end{aligned}$$

Editorial verification of the segment convention. Put $u = s - a$, $v = s - b$, $w = s - c$, all positive. With inradius r , $\tan(A/2) = r/u$ and cyclically, and $r^2 = uvw/s$. Substituting in the projection formulas gives

$$\begin{aligned}AZ &= (u + 2v)/3, & ZB &= (2u + v)/3, \\ BX &= (v + 2w)/3, & XC &= (2v + w)/3, \\ CY &= (w + 2u)/3, & YA &= (2w + u)/3.\end{aligned}$$

All six lengths are positive; they partition their respective sides and have the required equal adjacent sums. Also $ax + by + cz = 2sr$, so the displayed signed distances have the proper normalization. This verifies the proposed point for ordinary segment lengths. The solution does not classify possible additional points whose pedal feet lie outside the side segments; no uniqueness assertion is added.

Also solved by D. M. Bailey, Gordon Bennett, Alfred Brousseau, Michael Goldberg, J. M. Stark, and the proposer.

963: Convex Quadrilaterals

Mathematics Magazine **50** (1977), 53; proposal January 1976

Source PDF p. 163.

Proposed by Hüseyin Demir, Ankara, Turkey

Characterize convex quadrilaterals with sides a, b, c, d such that

$$\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} = 0.$$

Solution by Clayton W. Dodge, University of Maine at Orono. By adding and subtracting rows and columns, the determinant equation is equivalent to

$$(a + c + b + d)(a + c - b - d)[(a - c)^2 + (b - d)^2] = 0.$$

Since a, b, c, d are all positive, either $a + c = b + d$, or $a = c$ and $b = d$. In the first case the convex quadrilateral can be circumscribed about a circle; in the second it is a parallelogram. The argument reverses to show that if a quadrilateral is either a parallelogram or possesses an inscribed circle, then its determinant is zero.

Also solved by Gerald Bergum, M. G. Greening (Australia), Daniel Mark Rosenblum, J. M. Stark, and the proposer.

998: A 120° Triangle

Mathematics Magazine **51** (1978), 199; proposal November 1976

Source PDF p. 163–164.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Characterize all triangles in which the triangle whose vertices are the feet of the internal angle bisectors is a right triangle.

Solution by John Oman, University of Wisconsin–Oshkosh. Let A', B', C' be the feet of the angle bisectors of A, B, C , respectively. Then $\angle A'C'B'$ is a right angle if and only if $\angle ACB = 120^\circ$.

Let a, b, c (respectively a', b', c') be the lengths of the sides opposite A, B, C (respectively A', B', C'). The cosine law and the angle-bisector theorem give

$$\begin{aligned}(c')^2 &= \left(\frac{ab}{a+c}\right)^2 + \left(\frac{ab}{b+c}\right)^2 - 2\frac{ab}{a+c}\frac{ab}{b+c}\frac{a^2+b^2-c^2}{2ab}, \\(b')^2 &= \left(\frac{ac}{b+c}\right)^2 + \left(\frac{ac}{a+b}\right)^2 - 2\frac{ac}{b+c}\frac{ac}{a+b}\frac{a^2+c^2-b^2}{2ac}, \\(a')^2 &= \left(\frac{bc}{a+c}\right)^2 + \left(\frac{bc}{a+b}\right)^2 - 2\frac{bc}{a+c}\frac{bc}{a+b}\frac{b^2+c^2-a^2}{2bc}.\end{aligned}$$

The angle $A'C'B'$ is right if and only if $(a')^2 + (b')^2 - (c')^2 = 0$. After simplification this becomes

$$\frac{2abc^2(a^2 + b^2 - c^2 + ab)}{(a+b)^2(a+c)(b+c)} = 0.$$

Thus $\angle A'C'B'$ is right if and only if $a^2 + b^2 - c^2 + ab = 0$, which by the cosine law is equivalent to $\angle ACB = 120^\circ$. Cyclic interchange completes the characterization.

Also solved by Gordon Bennett, Howard Eves, Michael Goldberg, Leonard D. Goldstone, M. G. Greening (Australia), Hubert J. Ludwig, J. M. Stark, Pambuccian Victor (Romania), Robert L. Young, and the proposer.

1197: Collinear Mid-Altitudes

Mathematics Magazine **58** (1985), 240; proposal September 1984

Source PDF p. 164–165.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Characterize the triangles of which the midpoints of the altitudes are collinear.

I. Solution by J. H. Webb, University of Cape Town, South Africa. The midpoints of the altitudes of a triangle are collinear if and only if the triangle is right-angled.

The altitudes all lie inside an acute triangle, while two lie outside an obtuse triangle. Let P, Q, R be the midpoints of the altitudes from A, B, C , respectively. If D, E, F are the midpoints of BC, CA, AB , then P, Q, R lie, respectively, on EF, FD, DE , produced if necessary. By Pasch's axiom applied to triangle DEF , the points P, Q, R are collinear if and only if two of them coincide with two of D, E, F , in other words lie on sides of triangle ABC . This occurs if and only if triangle ABC is right-angled.

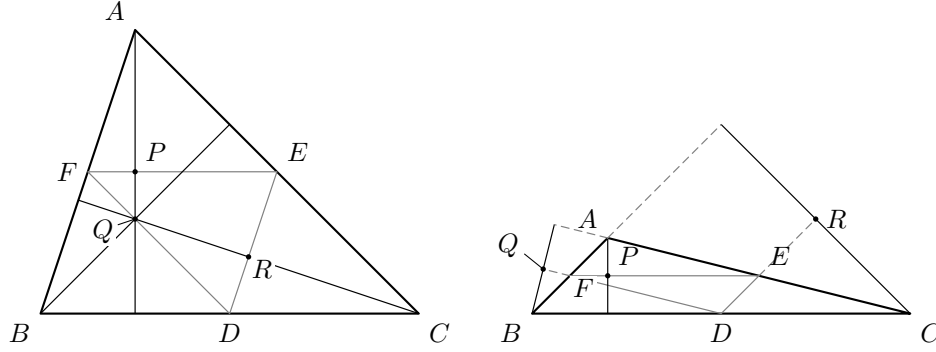


Figure 7.5: Midpoints of altitudes in acute and obtuse triangles; reconstruction of the two source diagrams.

II. Solution by J. F. Rigby, University College, Cardiff, Wales. The altitudes of a triangle are concurrent at the orthocenter. This is the only property of the altitudes that we need use; the answer is a special case of the following more general result.

Let D, E, F be points on the side-lines BC, CA, AB , respectively, of triangle ABC , such that AD, BE, CF are concurrent at P . Their midpoints are collinear if and only if P coincides with a vertex or lies on a side-line of ABC .

If P coincides with a vertex, suppose $P = A$ without loss of generality. Then $E = F = A$, and D lies anywhere on BC ; the midpoints are collinear in a line parallel to BC .

If P is not a vertex, choose oblique coordinate axes AB, AC , with suitable units so that $A = (0, 0)$, $B = (2, 0)$, $C = (0, 2)$. Let $P = (p, q)$. Then

$$D = \left(\frac{2p}{p+q}, \frac{2q}{p+q} \right), \quad E = \left(0, \frac{2q}{2-p} \right), \quad F = \left(\frac{2p}{2-q}, 0 \right).$$

We require $p+q \neq 0$, $2-p \neq 0$, and $2-q \neq 0$, since otherwise at least one of D, E, F is undefined (for instance AP is parallel to BC if $p+q=0$). The midpoints are

$$\left(\frac{p}{p+q}, \frac{q}{p+q} \right), \quad \left(1, \frac{q}{2-p} \right), \quad \left(\frac{p}{2-q}, 1 \right).$$

They are collinear if and only if

$$\begin{vmatrix} p/(p+q) & q/(p+q) & 1 \\ 1 & q/(2-p) & 1 \\ p/(2-q) & 1 & 1 \end{vmatrix} = \frac{2pq(2-p-q)}{(p+q)(2-p)(2-q)} = 0.$$

This occurs if and only if $p = 0$, $q = 0$, or $p+q = 2$, that is, if and only if P lies on a side-line of the triangle, in which case two midpoints coincide.

The orthocenter cannot lie on a side-line unless it coincides with a vertex, that is, unless the triangle is right-angled. Hence the midpoints of the altitudes are collinear if and only if the triangle is right-angled.

Also solved as in Solution I by Jordi Dou (Spain), Howard Eves, Syrous Mairivani, Nikie Molloy (student, Canada), Richard Parris, Cem Tezer (Turkey), and Michael Woltermann; as in the generalized Solution II (but using the theorems of Ceva and Menelaus) by Cem Tezer (Turkey, second solution); using analytic geometry by S. F. Barger, Kenneth Bernstein, Ragnar Dybvik (Norway), Cornelius Groenewoud, Boulkhodra Hacene, L. Kuipers (Switzerland), Hubert J. Ludwig, Bill Oik (student), John Oman, Harry Seidanger, Robert S. Stacy (West Germany), John S. Sumner, Michael Vowe (Switzerland), Jihad Yamout (student), and Robert L. Young; using barycentric or similar coordinate systems by O. Bottema (The Netherlands), J. T. Groenman (The Netherlands, two solutions), J. C. Linders (The Netherlands), and the proposer; using conjugate complex coordinates by Howard Eves (second solution) and Stephanie Sloyan; and using vector analysis by Leonard D. Goldstone and Harry D. Ruderman.

1206: Sum of Inradii of a Dissected Triangle

Mathematics Magazine **59** (1986), 46; proposal January 1985

Source PDF p. 166–167.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let ABC be a triangle with sides a, b, c and semiperimeter s . Let the side BC be subdivided using the points $B = P_0, P_1, \dots, P_{n-1}, P_n = C$ in order. If r_i is the inradius of triangle $AP_{i-1}P_i$ for $i = 1, \dots, n$, prove that

$$r_1 + \dots + r_n < \frac{1}{2} h_a \ln \frac{s}{s-a},$$

where h_a is the length of the altitude from vertex A .

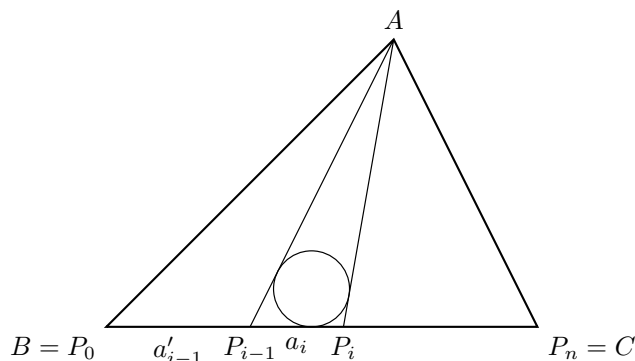


Figure 7.6: A subdivision of the base and the incircle of one component triangle.

Solution by Vania D. Mascioni, student, ETH Zürich, Switzerland. For $i = 1, \dots, n$, let $a_i = P_{i-1}P_i$ and let s_i be the semiperimeter of $AP_{i-1}P_i$. Let a'_i and s'_i be the corresponding quantities for ABP_i . We show that

$$\frac{s'_{i-1} - a'_{i-1}}{s'_{i-1}} \frac{s_i - a_i}{s_i} = \frac{s'_i - a'_i}{s'_i} \quad (2 \leq i \leq n). \quad (1)$$

Induction then yields

$$\frac{s-a}{s} = \prod_{i=1}^n \frac{s_i - a_i}{s_i}.$$

By the arithmetic–geometric mean inequality and $r_i s_i = \frac{1}{2} a_i h_a$,

$$\begin{aligned} \left(\frac{s-a}{s} \right)^{1/n} &\leq \frac{1}{n} \sum_{i=1}^n \frac{s_i - a_i}{s_i} = \frac{1}{n} \sum_{i=1}^n \left(1 - \frac{a_i}{s_i} \right) \\ &= 1 - \frac{2}{n h_a} \sum_{i=1}^n r_i. \end{aligned}$$

The non-strict comparison is needed here.^[E067] Consequently

$$\sum_{i=1}^n r_i \leq \frac{n h_a}{2} \left[1 - \left(\frac{s-a}{s} \right)^{1/n} \right],$$

which is stronger than the proposed inequality. The latter follows from $1 - 1/x < \ln x$ for $x > 1$, with $x = (s/(s-a))^{1/n}$.

Proof of (1). Relabel the sides of ABP_{i-1} and $AP_{i-1}P_i$ as in the figure. Then (1) becomes

$$\frac{u+v-p}{u+v+p} \frac{v+w-q}{v+w+q} = \frac{u+w-p-q}{u+w+p+q}.$$

An easy (though boring) algebraic manipulation shows this is equivalent to

$$(v^2 + p^2 - u^2)q + (v^2 + q^2 - w^2)p = 0.$$

By the cosine law this is equivalent to

$$2pqv(\cos \angle AP_{i-1}B + \cos \angle AP_{i-1}P_i) = 0,$$

which is immediate because the angles sum to π . Compare also Stewart's theorem and Greitzer, *Geometry Revisited*, p. 6.

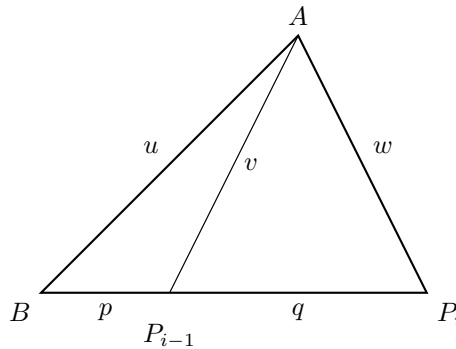


Figure 7.7: The two adjacent triangles in the proof of the product identity (Stewart's theorem).

Also solved by Jordi Dou (Spain), Václav Konečný and Ronald Shepler, L. Kuipers (Switzerland), Syrous Mairivani, William A. Newcomb, Bjorn Poonen (student), J. M. Stark, Paul J. Zwier, and the proposer.

Editorial note (original journal). Most solvers used an estimate of the following kind. Take $A = (0, h_a)$, $B = (t, 0)$, and $C = (t + a, 0)$. Let $P'_j = (x'_j, 0)$ for $j = 0, \dots, m$ be a strict

refinement of $P_0, \dots, P_n$, with $m > n$. If r'_j is the inradius of $AP'_{j-1}P'_j$, then

$$\begin{aligned} \sum_{i=1}^n r_i &\leq \sum_{j=1}^m r'_j \\ &= \sum_{j=1}^m \frac{h_a(x'_j - x'_{j-1})}{x'_j - x'_{j-1} + \sqrt{(x'_{j-1})^2 + h_a^2} + \sqrt{(x'_j)^2 + h_a^2}} \\ &\approx \int_t^{t+a} \frac{h_a dx}{2\sqrt{x^2 + h_a^2}}. \end{aligned}$$

Editorial completion. The final approximation becomes a limit as the refinement mesh tends to zero. The integral is $\frac{1}{2}h_a \ln(s/(s-a))$. The main proof above establishes the required strict inequality without relying on an unspecified approximation.

1211: Isoptic of an Ellipse

Mathematics Magazine **59** (1986), 113; proposal March 1985

Source PDF p. 168–169.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Find the locus of points under which an ellipse is seen under a constant angle.

Solution by Volkhard Schindler, Berlin, East Germany. Consider $x^2/a^2 + y^2/b^2 = 1$. The tangent at (x_1, y_1) has equation $xx_1/a^2 + yy_1/b^2 = 1$. Its intercepts are a^2/x_1 and b^2/y_1 , so the slope of a tangent through an exterior point (x, y) is

$$m = \frac{y}{x - a^2/x_1} = \frac{y - b^2/y_1}{x}.$$

Thus, where these divisions are defined,

$$\frac{x_1}{a} = \frac{ma}{mx - y}, \quad \frac{y_1}{b} = \frac{b}{y - mx}. \quad (1)$$

Since (x_1, y_1) lies on the ellipse, substitution gives

$$(x^2 - a^2)m^2 - 2xym + (y^2 - b^2) = 0. \quad (2)$$

If α is the constant angle subtended by the ellipse, number the roots m_1, m_2 so that $\tan \alpha = (m_1 - m_2)/(1 + m_1m_2)$. Hence

$$\tan^2 \alpha = \frac{(m_1 + m_2)^2 - 4m_1m_2}{(1 + m_1m_2)^2}.$$

This is unchanged on replacing α by $180^\circ - \alpha$. Using the sum and product of the roots gives

$$\tan^2 \alpha = 4 \frac{b^2x^2 + a^2y^2 - a^2b^2}{(x^2 + y^2 - a^2 - b^2)^2}. \quad (3)$$

The 180° limit is the original ellipse. At $\alpha = 90^\circ$, the limiting relation gives the director circle $x^2 + y^2 = a^2 + b^2$ of radius $\sqrt{a^2 + b^2}$; the displayed quotient itself is not defined on that circle.^[E068]

Introduce polar coordinates $x = \rho \cos \theta$, $y = \rho \sin \theta$. For $\alpha \neq 90^\circ$, (3) becomes $\rho^4 - 2A\rho^2 + B = 0$, where

$$A = a^2 + b^2 + 2(b^2 \cos^2 \theta + a^2 \sin^2 \theta) \cot^2 \alpha,$$

$$B = (a^2 + b^2)^2 + 4a^2b^2 \cot^2 \alpha.$$

Therefore

$$\rho^2 = A \pm \sqrt{A^2 - B}. \quad (4)$$

For fixed θ , the distance decreases as α increases. The plus sign gives $0 < \alpha < 90^\circ$ and the minus sign gives $90^\circ < \alpha < 180^\circ$. Both give $\rho^2 = a^2 + b^2$ at 90° by continuity. The loci are near-ellipses for obtuse angles and nearly ovals of Cassini or lemniscates of Booth for acute angles. If $a = b$, all loci are circles.

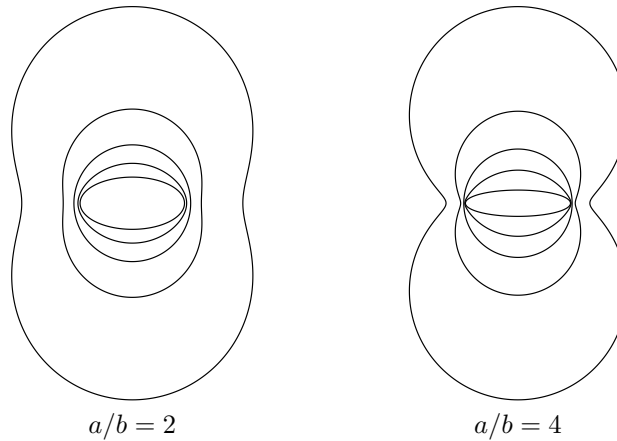


Figure 7.8: Isoptics for $a/b = 2$ and $a/b = 4$. From the outermost curve inward, the angles are 30° , 60° , 90° , 120° , and the limiting 180° ellipse. Each panel is scaled separately.

Editorial completion for exceptional directions and the angle branch. Put $D = b^2x^2 + a^2y^2 - a^2b^2$. For an exterior point $D > 0$, a direction-independent form is

$$\cot \alpha = \frac{x^2 + y^2 - a^2 - b^2}{2\sqrt{D}}, \quad 0 < \alpha < \pi. \quad [\text{E068}]$$

This follows by using direction vectors for the two tangent rays, and extends the slope calculation continuously to vertical tangents and axis points. Its sign distinguishes the acute and obtuse angles lost by squaring (3), and it includes $\alpha = 90^\circ$ directly. Points on the ellipse correspond only to the boundary limit $\alpha \rightarrow \pi$, not to two distinct tangent rays from an exterior point. The figures were generated from (4), with the original ellipse drawn separately for that limit.

Also solved by Michael V. Finn, J. T. Groenman (The Netherlands), L. Kuipers (Switzerland), Vania Mascioni (student, Switzerland), William A. Newcomb, Richard Parris, Stephanie Sloyan, and Robert L. Young. Solved partially by Zachary Franco (student) and the proposer.

Editorial note (original journal). M. S. Klamkin (Canada) found the result in R. C. Yates, *A Handbook on Curves and their Properties*, J. W. Edwards, Ann Arbor, 1947 (reprinted as *Curves and their Properties*, NCTM, 1974), pp. 138–140, where the terms isoptic and orthoptic are defined. None of the solvers considered the exceptional cases arising when, for example, x , x_1 , y , y_1 , or m is zero, or m is infinite in (1).

1298: Circumscribable Quadrangle

Mathematics Magazine **62** (1989), 200; proposal June 1988

Source PDF p. 170–171.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

A quadrilateral $ABCD$ is circumscribed about a circle, and P, Q, R, S are the points of tangency of sides AB, BC, CD, DA , respectively. Let $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$, and $p = |QS|$, $q = |PR|$. Show that

$$\frac{ac}{p^2} = \frac{bd}{q^2}.$$

I. Solution by J. M. Stark, Lamar University, Texas (angle index corrected). Denote by r the radius of the inscribed circle and let $\alpha, \beta, \gamma, \delta$ be the successive central angles subtended by SP, PQ, QR, RS . We have

$$a = |AP| + |PB|, \quad b = |BQ| + |QC|, \quad c = |CR| + |RD|, \quad d = |DS| + |SA|.$$

Right-triangle geometry gives $|AP| = |SA| = r \tan(\alpha/2)$ and, cyclically, the corresponding tangent lengths $r \tan(\beta/2)$, $r \tan(\gamma/2)$, $r \tan(\delta/2)$. Consequently

$$\begin{aligned} ac &= r^2 \left(\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} \right) \left(\tan \frac{\gamma}{2} + \tan \frac{\delta}{2} \right), \\ bd &= r^2 \left(\tan \frac{\beta}{2} + \tan \frac{\gamma}{2} \right) \left(\tan \frac{\delta}{2} + \tan \frac{\alpha}{2} \right). \end{aligned} \quad (1)$$

The identity $\tan x + \tan y = \sin(x+y)/(\cos x \cos y)$ gives

$$\frac{ac}{bd} = \frac{\sin((\alpha+\beta)/2) \sin((\gamma+\delta)/2)}{\sin((\beta+\gamma)/2) \sin((\alpha+\delta)/2)}. \quad (2)$$

Since $\alpha + \beta + \gamma + \delta = 2\pi$, this reduces to

$$\frac{ac}{bd} = \frac{\sin^2((\alpha+\beta)/2)}{\sin^2((\beta+\gamma)/2)}. \quad (3)$$

Finally,

$$p^2 = \left(2r \sin \frac{\alpha+\beta}{2} \right)^2, \quad q^2 = \left(2r \sin \frac{\beta+\gamma}{2} \right)^2, \quad [E069]$$

so (3) gives $ac/bd = p^2/q^2$.

II. Solution by O. Nouhaud, Faculté des Sciences de Limoges, France. Let A', B', C', D' be the inverses of A, B, C, D under inversion in the inscribed circle with center O and radius r . We know that

$$|A'B'| = r^2 \frac{|AB|}{|OA||OB|}$$

(see Howard Eves, *A Survey of Geometry*, Allyn and Bacon, Boston, 1963, Theorem 3.4.20, p. 153). Cyclic permutation gives three similar relations. Moreover, $2|A'B'| = |SQ|$ because A' bisects SP and B' bisects PQ . Similarly $2|C'D'| = |SQ|$ and $2|A'D'| = 2|B'C'| = |RP|$. The desired result follows from these relations.

Also solved by Mangho Ahuja, Wadie A. Bassali (Kuwait), J.-M. Becker (France), Bilkent University Problem Solving Group (Turkey), J. C. Binz (Switzerland), Duane M. Broline, Brown University Fly-Fishing Club, Onn Chan (student), Gang Chang (student), Chico Problem

Group, Timothy Chow, Ragnar Dybvik (Norway), E. C. Friedman, Francis M. Henderson, J. Heuver (Canada), Geoffrey A. Kandall, Hans Kappus (Switzerland), Václav Konečný, L. Kuipers (Switzerland), Helen M. Marston, Richard E. Pfefer, James S. Robertson, Harry D. Ruderman, Raul A. Simon (Chile), László Szűcs, R. S. Tiberio, George Vakanas (student), and the proposer.

1305: Inradii Identity

Mathematics Magazine **62** (1989), 278; proposal October 1988

Source PDF p. 172–173.

Proposed by Hüseyin Demir and Cem Tezer, Middle East Technical University, Ankara, Turkey

Let $P_0 = B, P_1, P_2, \dots, P_n = C$ be points, taken in that order, on the side BC of the triangle ABC . If r, r_i, h denote, respectively, the inradii of the triangles $ABC, AP_{i-1}P_i$ and the common altitude, prove that

$$\prod_{i=1}^n \left(1 - \frac{2r_i}{h}\right) = 1 - \frac{2r}{h}.$$

Solution by Jim Francis, University of Washington, Seattle, Washington. It suffices to prove the case $n = 2$, since the formula then follows by induction. The inradius of a triangle is its area divided by its semiperimeter. Let $x_1 = BP_1, x_2 = P_1C, a_1 = AB, a_2 = AP_1$, and $a_3 = AC$. Then

$$r_1 = \frac{x_1 h}{x_1 + a_1 + a_2}, \quad r_2 = \frac{x_2 h}{x_2 + a_2 + a_3}, \quad r = \frac{(x_1 + x_2)h}{x_1 + x_2 + a_1 + a_3}.$$

Thus

$$\left(1 - \frac{2r_1}{h}\right) \left(1 - \frac{2r_2}{h}\right) = \frac{(a_1 + a_2 - x_1)(a_2 + a_3 - x_2)}{(a_1 + a_2 + x_1)(a_2 + a_3 + x_2)},$$

whereas

$$1 - \frac{2r}{h} = \frac{a_1 + a_3 - x_1 - x_2}{a_1 + a_3 + x_1 + x_2}.$$

It remains to prove

$$\begin{aligned} & (a_1 + a_2 + x_1)(a_2 + a_3 + x_2)(a_1 + a_3 - x_1 - x_2) \\ &= (a_1 + a_2 - x_1)(a_2 + a_3 - x_2)(a_1 + a_3 + x_1 + x_2). \end{aligned}$$

Expanding and removing common terms leaves on the right

$$(-a_1^2 + a_2^2 + x_1^2)x_2 + (-a_3^2 + a_2^2 + x_2^2)x_1,$$

and its additive inverse on the left. This expression is zero by the cosine law: if $\alpha = \angle AP_1B$, then

$$\begin{aligned} -a_1^2 + a_2^2 + x_1^2 &= 2a_2x_1 \cos \alpha, \\ -a_3^2 + a_2^2 + x_2^2 &= 2a_2x_2 \cos(\pi - \alpha) = -2a_2x_2 \cos \alpha. \end{aligned}$$

The proof is complete.

Also solved by S. Belbas, Francisco Bellot-Rosado (Spain), Anne Boettcher and Václav Konečný, Duane M. Broline, Michael V. Finn, John F. Goehl, Jr., Francis M. Henderson, J. Heuver (Canada), Hans Kappus (Switzerland), L. Kuipers (Switzerland), Lamar University Problem Solving Group, J. C. Linders (The Netherlands), Vania Mascioni (Switzerland), The Oxford Running Club, Werner Ratzke (West Germany), John P. Robertson, Hyman Rosen, Volkhard

Schindler (East Germany), Michael Vowe (Switzerland), A. Zulauf (New Zealand), and the proposer.

Editorial cross-reference. This is the product identity used in Proposal 1206, since $1 - 2r_i/h = (s_i - a_i)/s_i$.

1327: Diagonals of Exscribed Quadrangles

Mathematics Magazine **63** (1990), 275; proposal October 1989

Source PDF p. 174–175.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let the sides PQ, QR, RS, SP of a convex quadrangle $PQRS$ touch an inscribed circle at A, B, C, D and let the midpoints of the sides AB, BC, CD, DA be E, F, G, H . Show that the angle between the diagonals PR, QS is equal to the angle between the bimedians EG, FH .

I. Solution by Jordi Dou, Barcelona, Spain. Let the inscribed circle have radius r and center O . Let J, K, L, M be its intersection points with the rays OHP, OEQ, OFR, OGS , respectively. Put $N = JL \cap KM$, $T = PR \cap QS$, and $I = EG \cap FH$.

The lines JL and KM are perpendicular because the arcs JAK and LCM together comprise half the circumference. Since $\triangle JOL$ is isosceles, KM is parallel to the angle bisector of $\angle JOL$. The lines PR, HF are antiparallel with respect to the sides of $\angle JOL$: $\angle OHF = \angle ORP$ and $\angle OFH = \angle OPR$. Indeed $OH \cdot OP = OF \cdot OR = r^2$, so $\triangle OHF$ and $\triangle ORP$ are similar. It follows that PR and HF make equal angles, say α , with JL . Similarly QS and EG make equal angles, say β , with KM . We then have $\angle GIF = 90^\circ - (\alpha + \beta)$ and $\angle RTS = 90^\circ + (\alpha + \beta)$ in the configuration drawn. These are supplementary, so the smaller angles between the respective pairs of lines agree.

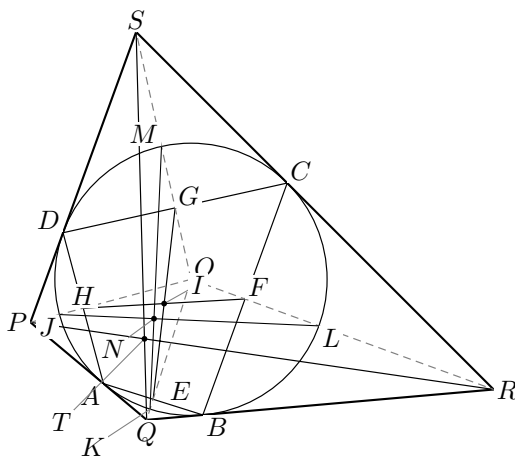


Figure 7.9: Tangency points, the chords JL, KM , and the bimedians EG, FH .

II. Solution by Jiro Fukuta, Motosu-gun, Gifu-ken, Japan. Let O be the center of the inscribed circle, of radius r . Represent P, Q, R, S by complex numbers $\alpha, \beta, \gamma, \delta$ with origin O . Then E, F, G, H correspond to $r^2/\bar{\beta}, r^2/\bar{\gamma}, r^2/\bar{\delta}, r^2/\bar{\alpha}$, respectively. To obtain the conclusion it suffices to show that

$$\mathcal{F} = \frac{\alpha - \gamma}{\beta - \delta} \div \frac{r^2/\bar{\beta} - r^2/\bar{\delta}}{r^2/\bar{\alpha} - r^2/\bar{\gamma}}$$

is real. We have

$$\begin{aligned}\mathcal{F} &= \frac{\alpha - \gamma}{\beta - \delta} \frac{\bar{\alpha} - \bar{\gamma}}{\bar{\beta} - \bar{\delta}} \frac{\beta\delta}{\alpha\gamma} \\ &= \frac{|\alpha - \gamma|^2}{|\beta - \delta|^2} \left(\frac{\delta}{\alpha} \frac{\beta}{\gamma} \right).\end{aligned}$$

But $(\delta/\alpha)(\beta/\gamma)$ is real because the corresponding central angles $\angle POS$ and $\angle ROQ$ sum to π . This completes the proof.

Also solved by Duane M. Broline, Timothy V. Craine, John F. Goehl, Jr., Francis M. Henderson, Paul Martin, and the proposer.

1356: Collinearity and Symmetry

Mathematics Magazine **64** (1991), 278; proposal October 1990

Source PDF p. 175.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let P, Q be points taken on the side BC of a triangle ABC , in the order B, P, Q, C . Let the circumcircles of PAB, QAC intersect at M ($\neq A$) and those of PAC, QAB at $N \neq A$, assuming both distinct second intersections exist.^[E070] Show that A, M, N are collinear if and only if P and Q are symmetric in the midpoint A' of BC .

Solution by Christos Athanasiadis, student, Massachusetts Institute of Technology, Cambridge, Massachusetts. Let K, L be the intersections of BC with AM, AN , respectively. Take BC as the x -axis with origin B , and let a, p, q, k, l denote the coordinates of C, P, Q, K, L . Since K is on the radical axis of the circles PAB, QAC , its powers are equal:

$$k(k - p) = (k - q)(k - a).$$

It follows that $k = aq/(a + q - p)$. Similarly, interchanging p, q , we obtain $l = ap/(a + p - q)$. Since $0 < p < q < a$, the denominators are positive, and $l = k$ if and only if $p + q = a$. This proves the result.

Editorial qualification. The stated second intersections must exist and be distinct from A . This excludes the tangent-at- A degeneracy; the existence of $M \neq A$ is already explicit in the source, and the same convention is required for N .^[E070]

Also solved by Raúl Marín Carrera (student, Mexico), Jordi Dou (Spain), Jiro Fukuta (Japan), Václav Konečný, Alvaro Avila Márquez (student, Mexico), Richard E. Pfefer, Ioan Sadoveanu, Jyotirmoy Sarkar (student), Seshadri Sivakumar (Canada), John S. Sumner, and the proposer.

Editorial note (original journal). Pfefer obtained the solution by inverting the figure through a circle with center A .

1371: A Triangle Invariant

Mathematics Magazine **65** (1992), 133; proposal April 1991

Source PDF p. 176–177.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let A, B, C be vertices of a triangle and let D, E, F be points on the sides BC, AC, AB , respectively. Let U, X, V, Y, W, Z be the midpoints of, respectively, BD, DC, CE, EA, AF, FB . Prove that

$$\text{Area}(\triangle UVW) + \text{Area}(\triangle XYZ) - \frac{1}{2} \text{Area}(\triangle DEF)$$

is a constant independent of D, E, F .

I. Solution by Jordi Dou, Barcelona, Spain. Submitted on the occasion of his 80th birthday. First let D' be any point between C and D , and take U', X' as the midpoints of $BD', D'C$. Then $XX' = UU' = \frac{1}{2}DD'$. Let $h_F, h_A, h_W, \dots$ be the distances from $F, A, W, \dots$ to BC . Since

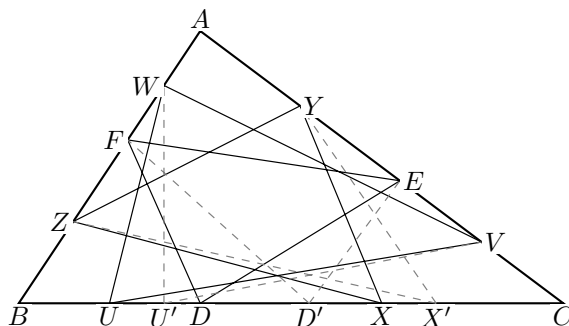


Figure 7.10: Moving D to D' while retaining E, F ; dashed lines indicate the new triangles.

$$h_Z = h_F/2 \text{ and } h_W = (h_F + h_A)/2,$$

$$h_Z + h_W - h_F = \frac{1}{2}h_A = h_Y + h_V - h_E.$$

Write $[PQR]$ for the area of triangle PQR , and put $S = [ABC]$, $\sigma = [DEF]$, $\sigma_1 = [XYZ]$, $\sigma_2 = [UVW]$, $\sigma'_1 = [D'EF]$, $\sigma'_1 = [X'YZ]$, $\sigma'_2 = [U'VW]$. Then

$$\begin{aligned} \sigma' - \sigma &= \frac{1}{2}DD'(h_E - h_F), \\ \sigma'_1 - \sigma_1 &= \frac{1}{2}XX'(h_Y - h_Z) = \frac{1}{4}DD'(h_Y - h_Z), \\ \sigma'_2 - \sigma_2 &= \frac{1}{4}DD'(h_V - h_W). \end{aligned}$$

It follows that

$$\begin{aligned} &(\sigma'_1 + \sigma'_2 - \frac{1}{2}\sigma') - (\sigma_1 + \sigma_2 - \frac{1}{2}\sigma) \\ &= \frac{1}{4}DD'(h_Y - h_Z + h_V - h_W - h_E + h_F) \\ &= \frac{1}{4}DD'((h_Y + h_V - h_E) - (h_Z + h_W - h_F)) = 0. \end{aligned}$$

By symmetry the same holds when D' is between B and D . Moving F along AB and E along AC in the same way proves invariance with respect to DEF .

To find the value, put $E = A$, $F = B$, $D = C$. Then $X = C$, $Y = A$, $Z = B$, while U, V, W are the midpoints of BC, CA, AB . Thus $\sigma = S$, $\sigma_1 = S$, $\sigma_2 = S/4$, and the invariant is $3S/4$.

II. *Solution by László Szűcs, Fort Lewis College, Durango, Colorado.* Write $[ABC] = \text{Area}(\triangle ABC)$. The given expression is

$$\begin{aligned} & [ABC] - ([AWV] + [BUW] + [CVU]) \\ & + [ABC] - ([AZY] + [BXZ] + [CYX]) \\ & - \frac{1}{2}([ABC] - ([AFE] + [BDF] + [CED])). \end{aligned}$$

Since $[AFE] = 4[AWY]$, $[BDF] = 4[BUZ]$, and $[CED] = 4[CVX]$, this becomes

$$\begin{aligned} & \frac{3}{2}[ABC] - ([AWV] - [AWY] + [AZY] - [AWY] \\ & + [BUW] - [BUZ] + [BXZ] - [BUZ] \\ & + [CVU] - [CVX] + [CYX] - [CVX]). \end{aligned}$$

Using $[AWV] - [AWY] = [VYW] = \frac{1}{4}[CAF]$ and five analogous relations gives

$$\frac{3}{2}[ABC] - \frac{1}{4}([CAF] + [EAB] + [ABD] + [FBC] + [BCE] + [DCA]) = \frac{3}{4}[ABC].$$

Also solved by Larry E. Askins, Eynshteyn Averbukh, Seung-Jin Bang (Korea), Karen Benbury, Francisco Bellot Rosado (Spain), Scott D. Cohen (student), C. Patrick Collier, Miguel Amengual Covas (Spain), Jordi Dou (Spain), Ragnar Dybvik (Norway), Kao H. and Irene C. Sze, Milton P. Eisner, Jiro Fukuta (Japan), Thomas E. Gartner, John F. Goehl, Jr., Cornelius Groenewoud, H. Guggenheimer, Francis M. Henderson, Ralph P. Grimaldi, Russell Jay Hendel, Paul Irwin, Geoffrey A. Kandall, Václav Konečný, Philip Lau, Eugene Lee, Peter W. Lindstrom, James Pfagentdner, Richard E. Pfefer, Rolf Rosenkranz (Romania), Ioan Sadoveanu, Jyotirmoy Sarkar, Volkhard Schindler (Germany), Mohammad Parvez Shaikh (student), Ching-Kuang Shene, John S. Sumner, Jordan Tabov (Bulgaria), Michael Vowe, and the proposer.

Editorial note (original journal). Tabov proved the more general result. Consider a triangle $A_1A_2A_3$, a real number $\alpha \neq 0, 1$, and real numbers λ, μ . For arbitrary points X_1, X_2, X_3 on the lines A_2A_3, A_3A_1, A_1A_2 , respectively, define C_{ij} for $i \neq j$ by

$$\overrightarrow{OC_{ij}} = \alpha \overrightarrow{OA_i} + \beta \overrightarrow{OX_j}, \quad \alpha + \beta = 1,$$

where O is any point outside the plane of $A_1A_2A_3$. Let

$$F(X_1, X_2, X_3) = \lambda[C_{23}C_{31}C_{12}] + \mu[C_{13}C_{21}C_{32}] - [X_1X_2X_3],$$

where brackets denote signed area and each X_i varies independently on its line. Then F is constant if and only if $\lambda = \mu = \frac{1}{2}(1 - \alpha)^{-2}$. The given problem corresponds to $\alpha = 1/2$.

1377: A Triangle Invariant

Mathematics Magazine **65** (1992), 199; proposal June 1991

Source PDF p. 178.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Let D, E, F lie on the side segments BC, CA, AB , respectively, of triangle ABC , and let U, X, V, Y, W, Z be the midpoints of the line segments BD, DC, CE, EA, AF, FB , respectively. Show that the expression

$$|UVW| + |XYZ| - \frac{1}{2}|DEF|$$

for areas is constant.^[E071]

Solution by Hans Kappus, Mathematisches Institut der Universität, Basel, Switzerland. Denote the expression by S . We show that $S = \frac{3}{4} \text{Area } ABC$. Since the ratio to $\text{Area } ABC$ is unchanged by affine transformations, choose coordinates $A = (0, 0)$, $B = (1, 0)$, $C = (0, 1)$. Put

$$D = (1 - r, r), \quad E = (0, s), \quad F = (t, 0), \quad 0 \leq r, s, t \leq 1.$$

Then

$$\begin{aligned} U &= (1 - r/2, r/2), & V &= (0, (1 + s)/2), & W &= (t/2, 0), \\ X &= ((1 - r)/2, (1 + r)/2), & Y &= (0, s/2), & Z &= ((1 + t)/2, 0). \end{aligned}$$

Direct area calculations give

$$\begin{aligned} [UVW] &= \frac{1}{8}(2 - r + 2s - t - rs + rt - st), \\ [XYZ] &= \frac{1}{8}(1 + r + t - rs + rt - st), \\ [DEF] &= \frac{1}{2}(s - rs + rt - st). \end{aligned}$$

Hence $S = 3/8 = \frac{3}{4}[ABC]$.

Also solved by Benno Arbel (Israel), H. Guggenheimer, Francis M. Henderson, John G. Heuver (Canada), Thomas Jager, Václav Konečný, Helen M. Marston, Ralph Merrill, José Heberto Nieto (Venezuela), Chrysostom G. Petalas (Greece), F. C. Rembis, Robert L. Young, an unsigned solution, and the proposer.

Editorial note (original journal). Several people mentioned that the problem is incorrect as stated. The intention was that D, E, F should be on line segments BC, CA, AB , respectively. A corrected version appears as Problem 1371 in April 1991, with several solutions in April 1992. Somehow the uncorrected version did not get lifted from the file of accepted proposals, so it inadvertently reappeared. Apologies.^[E071]

1405: Isogonally Related Circles

Mathematics Magazine **66** (1993), 269; proposal October 1992

Source PDF p. 179–180.

Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

Two circles inscribed in distinct angles of a triangle are isogonally related if the tangents from the third vertex not coinciding with the sides are symmetric with respect to the bisector of the third angle. Given three circles inscribed in distinct angles of a triangle, prove that if any two of the three pairs of circles are isogonally related then so is the third pair.

Solution by the proposer (division-free final form). Let $\Gamma_1, \Gamma_2, \Gamma_3$ be circles inscribed in angles BAC, CBA, ACB , respectively. Let I_i, r_i be the center and radius of Γ_i . Let E, F be the points on AB , distinct from A, B , such that CE, CF are tangent to Γ_1, Γ_2 , respectively. Put $\gamma = \angle ACE$ and $\mu = \angle FCB$, and let a, b, c be the lengths of BC, CA, AB .

Considering triangles AI_1C and I_2BC , respectively, gives

$$b = \left(\cot \frac{A}{2} + \cot \frac{\gamma}{2} \right) r_1, \quad a = \left(\cot \frac{B}{2} + \cot \frac{\mu}{2} \right) r_2.$$

The circles Γ_1, Γ_2 are isogonally related if and only if $\gamma = \mu$. The equations show that this is equivalent to

$$\frac{b}{r_1} - \cot \frac{A}{2} = \frac{a}{r_2} - \cot \frac{B}{2},$$

or, using the semiperimeter s and inradius r of ABC ,

$$\frac{b}{r_1} - \frac{s-a}{r} = \frac{a}{r_2} - \frac{s-b}{r}.$$

Equivalently,

$$\frac{1/r_1 - 1/r}{a} = \frac{1/r_2 - 1/r}{b}. \quad (1)$$

Similarly Γ_2, Γ_3 are isogonally related if and only if

$$\frac{1/r_2 - 1/r}{b} = \frac{1/r_3 - 1/r}{c}. \quad (2)$$

Combining (1) and (2) yields

$$\frac{1/r_1 - 1/r}{a} = \frac{1/r_3 - 1/r}{c},$$

which is equivalent to Γ_1, Γ_3 being isogonally related. The source expresses these equalities using the reciprocals of both sides; the present form avoids division by $1/r_i - 1/r$.^[E072]

Also solved by Richard Darley, Francis M. Henderson, Kee-Wai Lau, Francisco Bellot Rosado and María Ascensión López Chamorro (Spain).

Quickies for Mathematics Magazine

The fourteen Quickies composed by Hüseyin Demir are listed, with their journal references, in Chapter 13. The original “Q” prefix distinguishes them from the regular proposals with the same numbers.

Q117

Mathematics Magazine **28** (1954–1955), 37

Source PDF p. 182.

Submitted by Hüseyin Demir

How many squares are there on a chessboard?

Q138

Mathematics Magazine **28** (1954–1955), 241

Source PDF p. 182.

Submitted by Hüseyin Demir

Find the sum $S_n = 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$.

Q141

Mathematics Magazine **28** (1954–1955), 292

Source PDF p. 183.

Submitted by Hüseyin Demir

If the twelve months of the year are written in the order offered by $5n + 2 \pmod{12}$, $n = 1, 2, 3, \dots, 12$, what can be said about the characteristics of the first seven, the next four, and of the last month?

Q166

Mathematics Magazine **29** (1955–1956), 229

Source PDF p. 183.

Submitted by Hüseyin Demir

Can the sum of the cubes of the first m consecutive integers equal the sum of the cubes of the next n consecutive integers?

Q188

Mathematics Magazine **30** (1956–1957), 172

Source PDF p. 183.

Submitted by Hüseyin Demir

At what times must the hands of a clock be interchanged in order to obtain new correct time?

Q227

Mathematics Magazine **32** (1958–1959), 32

Source PDF p. 183.

Submitted by Hüseyin Demir

Find a function $f(n)$ such that $f(1), f(2), f(3), \dots, f(13), f(14)$ be all prime numbers not all equal nor all distinct.^[E019]

Q234

Mathematics Magazine **32** (1958–1959), 113

Source PDF p. 183.

Submitted by Hüseyin Demir

If the sum of the coefficients of $A(x)B(x)$ is zero, the sum of the coefficients of one of the polynomials is necessarily zero.

Q242

Mathematics Magazine **32** (1958–1959), 229

Source PDF p. 183.

Submitted by Hüseyin Demir

Find an $f(n)$ such that $f(\text{even}) = \frac{1}{2}$ and $f(\text{odd}) = 1$.

Q266

Mathematics Magazine **33** (1959–1960), 302

Source PDF p. 184.

Submitted by Hüseyin Demir

If p is a prime number greater than 3, then $p^2 + 2$ is composite.

Q281

Mathematics Magazine **34** (1961), 303

Source PDF p. 184.

Submitted by Hüseyin Demir

Evaluate the radius of the inner tritangent circle to excircles of a triangle.

Q284

Mathematics Magazine **34** (1961), 303

Source PDF p. 184.

Submitted by Hüseyin Demir

What is the locus of points whose projections on the sides of a triangle are collinear?

Q341

Mathematics Magazine **37** (1964), 251

Source PDF p. 184.

Submitted by Hüseyin Demir

Find the limit of the fraction as n approaches infinity:

$$\frac{\varphi(1) + \varphi(2) + \varphi(3) + \cdots + \varphi(n)}{1 + 2 + 3 + \cdots + n},$$

where $\varphi(n)$ is Euler's totient.

Q343

Mathematics Magazine **37** (1964), 251

Source PDF p. 184.

Submitted by Hüseyin Demir

Identify the angle θ satisfying

$$\frac{\sin(\theta + 15^\circ)}{\sqrt{1}} = \frac{\sin(2\theta + 15^\circ)}{\sqrt{2}} = \frac{\sin(3\theta + 15^\circ)}{\sqrt{3}}.$$

Q710

Mathematics Magazine **59** (1986), 112

Source PDF p. 184.

Submitted by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

If n is any positive integer, show that the number $T = \frac{1}{8}n(n+1)(n+2)(n+3)$ is a triangular number.

Solutions of Quickies

Q117

Mathematics Magazine **28** (1954–1955), 37

Source PDF p. 186.

Not 64, but^[E020]

$$1^2 + 2^2 + \cdots + 8^2 = \frac{8 \cdot 9 \cdot 17}{6} = 204.$$

Q138

Mathematics Magazine **28** (1954–1955), 241

Source PDF p. 186.

$$\begin{aligned} S &= \sum_{p=1}^n p \cdot p! = \sum_{p=1}^n (p+1-1)p! \\ &= \sum_{p=1}^n (p+1)! - \sum_{p=1}^n p! \\ &= (n+1)! - 1. \end{aligned}$$

Q141

Mathematics Magazine **28** (1954–1955), 292

Source PDF p. 186.

The first seven months listed will have 31 days, the next four months will have 30 days and the last one has 29 or 28 days.

Q166

Mathematics Magazine **29** (1955–1956), 229^[E021]

Source PDF p. 186.

If

$$1^3 + 2^3 + \cdots + m^3 = (m+1)^3 + \cdots + (m+n)^3,$$

then

$$2(1^3 + 2^3 + \cdots + m^3) = 1^3 + 2^3 + \cdots + (m+n)^3,$$

or

$$2 \left[\frac{1}{2} m(m+1) \right]^2 = \left[\frac{1}{2} (m+n)(m+n+1) \right]^2,$$

so

$$\frac{1}{2}m(m+1)\sqrt{2} = \frac{1}{2}(m+n)(m+n+1).$$

But this last equation is impossible. Therefore the answer is no.

Q188

Mathematics Magazine **30** (1956–1957), 172

Source PDF p. 187.

Let p denote the number of hours and α the fraction of an hour at the time t . When the hands are interchanged we obtain new time t' , the corresponding numbers being p', α' (we may suppose $p' \geq p$). At t the angle in hours $p' + \alpha'$ of the minute hand is 12 times α :^[E022]

$$\begin{aligned} p' + \alpha' &= 12\alpha, & 0 \leq p' < 12, \quad 0 \leq \alpha' < 1, \\ p + \alpha &= 12\alpha', & 0 \leq p < 12, \quad 0 \leq \alpha < 1. \end{aligned}$$

We have

$$\alpha = \frac{12p' + p}{143}, \quad \alpha' = \frac{p' + 12p}{143}.$$

Hence, given any two integers p, p' with $0 \leq p, p' < 12$, except the pair $(11, 11)$, we get α and α' , and therefore the required times. If “new” is required to mean a different time, omit the pairs with $p = p'$.

Q227

Mathematics Magazine **32** (1958–1959), 32

Source PDF p. 187.

$$f(n) = 1 + \varphi(n).^{[E019]}$$

Editorial verification. With φ denoting Euler’s totient, the fourteen values are

$$2, 2, 3, 3, 5, 3, 7, 5, 7, 5, 11, 5, 13, 7.$$

They are all prime, with repetitions and more than one distinct value.

Q234

Mathematics Magazine **32** (1958–1959), 113

Source PDF p. 187.

For $x = 1$ we have $\sum a_i \cdot \sum b_i = \sum c_i$, and the result follows.

Q242

Mathematics Magazine **32** (1958–1959), 229

Source PDF p. 187.

Construct $f(n)$ so that

$$f(n) = \frac{1}{2} \left(n + 1 - 2 \left\lfloor \frac{n}{2} \right\rfloor \right),$$

where $\lfloor n/2 \rfloor$ is the largest integer in $n/2$.

Q266

Mathematics Magazine **33** (1959–1960), 302

Source PDF p. 187.

If p is a prime exceeding 3 then we have

$$p^2 + 2 = (6m \pm 1)^2 + 2 \equiv 0 \pmod{3}.$$

Q281

Mathematics Magazine **34** (1961), 303

Source PDF p. 188.

This circle, being the nine-point circle of the triangle, has radius $\frac{1}{2}R$.

Q284

Mathematics Magazine **34** (1961), 303

Source PDF p. 188.

If the points are restricted to lie on the plane of the triangle, the locus is the circumcircle of the triangle. Since no such restriction is made, the locus is the right cylinder having this circumcircle as section.

Q341

Mathematics Magazine **37** (1964), 251

Source PDF p. 188.

For $k \geq 2$, $\varphi(k)$ denotes the number of positive integers smaller than k and prime to it. Hence,^[E023]

$$\sum_{k=2}^n \varphi(k)$$

is the total number of relatively prime pairs among the first n positive integers, the total number of pairs being $\binom{n}{2}$. The limit of the given fraction is the same as that of

$$\frac{\sum_{k=2}^n \varphi(k)}{\binom{n}{2}}.$$

It will be the limiting probability that two integers chosen from the first n positive integers are relatively prime. This probability is known to have the value $6/\pi^2$. Hence, the limit of the given fraction is $6/\pi^2$.

Q343

Mathematics Magazine **37** (1964), 251

Source PDF p. 188.

The angle θ is evidently 15° .

Q710

Mathematics Magazine **59** (1986), 112

Source PDF p. 188–189.

T is a triangular number if for some positive integer k , one has

$$\frac{1}{8}n(n+1)(n+2)(n+3) = \frac{1}{2}k(k+1).$$

By considering parity and noting that $n(n+3) < (n+1)(n+2)$, one is led to try $k = n(n+3)/2$. Then

$$k+1 = \frac{n(n+3)}{2} + 1 = \frac{(n+1)(n+2)}{2},$$

and the result follows.

Contributed Solutions in Mathematics Magazine

The following list includes all contributions reproduced in this chapter. Seven entries omitted from the compiler's opening list have been restored.^[E095] Journal references identify the reproduced solutions.

Proposal	Volume	Year	Page
192	28	1954–1955	36
203	28	1954–1955	163
204	28	1954–1955	165
221	28	1954–1955	291
226	29	1955–1956	48 ^[E073]
270	30	1956–1957	108
273 ^[E079]	30	1956–1957	166
332	32	1958–1959	52
353	32	1958–1959	226
374	33	1959–1960	113
383	33	1959–1960	228
387	33	1959–1960	233
400	34	1960–1961	53
401	34	1960–1961	55
412	34	1961	175
427	34	1961	303
428	34	1961	303
432	34	1961	365
435	34	1961	368
445	35	1962	317
476	35	1962	312
646	40	1967	226
653	42	1969	283
1199	58	1985	243
1256	61	1988	54

Source PDF p. 190.

192

Mathematics Magazine **28** (1954–1955), 36; proposal January 1954

Source PDF p. 190–191.

Proposed by V. Thebault, Tennie, Sarthe, France

If A', B', C' are the symmetries of the vertices of a triangle ABC with respect to a fixed point, the circumcircles of the three triangles $AB'C'$, $BC'A'$, $CA'B'$ have a point in common which lies on the circumcircle of the triangle ABC .

Editorial qualification. The construction is understood in general position: the indicated triangles must have circumcircles and the intersection used below must be distinct from their common vertex.^[E074]

Solution by Hüseyin Demir, Zonguldak, Turkey. It will suffice to prove that any two of the circumcircles intersect on the circumcircle (O) of ABC . Let I be the intersection other than A' of the circles $BC'A'$ and $CA'B'$. To prove that it belongs to (O) we show that $\angle BIC = \angle BAC = \angle A$. Put $K = CA \cap A'B'$; the following angles are directed modulo π :

$$\begin{aligned}\angle BIC &= \angle BIA' + \angle A'IC \\ &= \angle BC'A' + \angle A'B'C \\ &= \angle B'CA + \angle KB'C \\ &= \angle B'CK + \angle KB'C \\ &= \angle A'KC = \angle BAC = \angle A.\end{aligned}$$

The first cyclic-angle substitutions follow from the fact that B, I, C', A' on one hand and I, B', C, A' on the other lie on the respective circumcircles; the other equalities follow from the parallelisms

$$BC' \parallel CB', \quad A'C' \parallel AC, \quad A'B' \parallel AB.$$

Editorial completion. To make the common-point conclusion explicit, apply the same argument cyclically to the circles $AB'C'$ and $BC'A'$. Their second intersection also belongs to (O). Both this point and I are therefore the second intersection of $BC'A'$ with (O), whose first intersection is B . In general position they coincide, proving the assertion.

Also solved by H. E. Fettis, Dayton, Ohio; O. J. Ramler, Catholic University of America; and the proposer.

203

Mathematics Magazine **28** (1954–1955), 163; proposal May 1954

Source PDF p. 191.

Proposed by Norman Anning, Alhambra, California

Prove that three of the intersections of

$$x^2 - y^2 + ax + by = 0, \quad x^2 + y^2 - a^2 - b^2 = 0$$

trisect the circle through these three points. Here a, b are real and $a^2 + b^2 > 0$.^[E075]

II. Solution by Hüseyin Demir, Zonguldak, Turkey. Set $r^2 = a^2 + b^2$, and, initially assuming $b \neq 0$, substitute the value of y obtained by adding the two equations into the first equation. We get the fourth-degree equation

$$4x^4 + 4ax^3 - 3r^2x^2 - 2ar^2x + a^2r^2 = 0,$$

whose roots are x_1, x_2, x_3, x_4 . If the triangle $A_1A_2A_3$ corresponding to x_1, x_2, x_3 is equilateral, $x_1 + x_2 + x_3$ vanishes, since its center is the center O of $x^2 + y^2 = r^2$. The coefficient of x^3 then gives $x_4 = -a$.

Division gives

$$4x^4 + 4ax^3 - 3r^2x^2 - 2ar^2x + a^2r^2 = (x + a)(4x^3 - 3r^2x + ar^2).$$

The cubic factor has no quadratic term, in agreement with the stated condition. The source concludes that $A_1A_2A_3$ is equilateral.

Editorial completion. The missing converse and the case $b = 0$ follow simultaneously by writing $a = r \cos \phi$, $b = r \sin \phi$, and $(x, y) = r(\cos t, \sin t)$. The first equation becomes

$$\cos 2t + \cos(t - \phi) = 2 \cos \frac{3t - \phi}{2} \cos \frac{t + \phi}{2} = 0.$$

The three values $t = (\phi + \pi)/3 + 2k\pi/3$, $k = 0, 1, 2$, give three distinct, equally spaced points. This remains valid if the remaining intersection coincides with one of them.^[E075]

204

Mathematics Magazine **28** (1954–1955), 165; proposal May 1954

Source PDF p. 192–194.

Proposed by C. W. Trigg, Los Angeles City College

In triangle ABC , let the feet of the median (m_a), the internal angle bisector (t_a), the cevian (p_a) to the contact point of the incircle with a , and the cevian (q_a) to the contact point of the excircle relative to A with a be respectively A_m, A_t, A_p, A_q . Use similar notation for the corresponding lines to b and c .

1. *Determine the relationship between the sides if the following triads are concurrent:*

$$\begin{array}{c|c} p_a, m_b, t_c & S \\ p_a, q_b, m_c & R \\ m_a, p_b, t_c & T \\ q_a, p_b, m_c & V \end{array}$$

2. *Show that A_pB_q and A_qB_p are parallel to AB ; C_tB_p and SV are parallel to BC ; and C_tA_p and RT are parallel to AC .*

Solution by Hüseyin Demir, Zonguldak, Turkey (indices corrected). 1. We determine the positions of the cevians a_u, b_v, c_w , or their feet A_u, B_v, C_w on BC, CA, AB , by the directed ratios

$$k(A_u) = \frac{\overline{A_uB}}{\overline{A_uC}}, \quad k(B_v) = \frac{\overline{B_vC}}{\overline{B_vA}}, \quad k(C_w) = \frac{\overline{C_wA}}{\overline{C_wB}}.$$

Since these are interior points of the sides, all ratios are negative. With $s = (a + b + c)/2$, their values are

	m	t	p	q
$k(A_\bullet)$	-1	$-c/b$	$-(s-b)/(s-c)$	$-(s-c)/(s-b)$
$k(B_\bullet)$	-1	$-a/c$	$-(s-c)/(s-a)$	$-(s-a)/(s-c)$
$k(C_\bullet)$	-1	$-b/a$	$-(s-a)/(s-b)$	$-(s-b)/(s-a)$

Ceva's theorem requires that each product of three ratios be -1 :

Triad	Point	Condition
p_a, m_b, t_c	S	$(s-b)/(s-c) = a/b$
p_a, q_b, m_c	R	$(s-a)(s-b) = (s-c)^2$
m_a, p_b, t_c	T	$(s-c)/(s-a) = a/b$
q_a, p_b, m_c	V	$(s-c)^2 = (s-a)(s-b)$

All four conditions are equivalent to

$$c = \frac{a^2 + b^2}{a + b}.$$

2. (a) To prove $A_p B_q \parallel A_q B_p \parallel AB$, observe that

$$k(A_p) = \frac{1}{k(B_q)}, \quad k(A_q) = \frac{1}{k(B_p)}.$$

(b) To prove $C_t B_p \parallel BC$, use $k(B_p) = 1/k(C_t)$, which is the condition for the third triad. To prove $SV \parallel BC$, apply Menelaus's theorem to triangles BCB_m and BCC_m , cut respectively by ASA_p and AVA_q :

$$\begin{aligned} \frac{\overline{A_p B}}{\overline{A_p C}} \frac{\overline{AC}}{\overline{AB_m}} \frac{\overline{SB_m}}{\overline{SB}} &= 1, & \frac{\overline{SB}}{\overline{SB_m}} &= 2k(A_p), \\ \frac{\overline{A_q B}}{\overline{A_q C}} \frac{\overline{VC}}{\overline{VC_m}} \frac{\overline{AC_m}}{\overline{AB}} &= 1, & \frac{\overline{VC}}{\overline{VC_m}} &= \frac{2}{k(A_q)}. \end{aligned}$$

Hence

$$\frac{\overline{SB}}{\overline{SB_m}} = 2k(A_p) = \frac{2}{k(A_q)} = \frac{\overline{VC}}{\overline{VC_m}}.$$

Thus S, V divide BB_m, CC_m in the same ratio. Since $B_m C_m \parallel BC$, the property follows.

(c) To prove $C_t A_p \parallel AC$, use $k(C_t) = 1/k(A_p)$, the condition for the first triad. To show $RT \parallel AC$, apply Menelaus's theorem to triangles CAC_m and CAA_m , cut respectively by BRB_q and BTB_p :

$$\begin{aligned} k(B_q) \frac{\overline{BA}}{\overline{BC_m}} \frac{\overline{RC_m}}{\overline{RC}} &= 1, & \frac{\overline{RC}}{\overline{RC_m}} &= 2k(B_q), \\ k(B_p) \frac{\overline{TA}}{\overline{TA_m}} \frac{\overline{BA_m}}{\overline{BC}} &= 1, & \frac{\overline{TA}}{\overline{TA_m}} &= \frac{2}{k(B_p)}. \end{aligned}$$

Therefore

$$\frac{\overline{RC}}{\overline{RC_m}} = 2k(B_q) = \frac{2}{k(B_p)} = \frac{\overline{TA}}{\overline{TA_m}}.$$

Hence R, T divide CC_m, AA_m in the same ratio. Since $C_m A_m \parallel CA$, we also have $RT \parallel CA$.^[E076]

Also solved by Sister M. Stephanie, Georgian Court College, N. J.; and the proposer.

221

Mathematics Magazine **28** (1954–1955), 291; proposal November 1954

Source PDF p. 194.

Proposed by E. P. Starke, Rutgers University

On a conical surface there is traced a spiral which crosses each of the linear elements at a fixed angle ψ . Find a simple expression for the length of this spiral between any two of its points.

Solution by Hüseyin Demir, Zonguldak, Turkey (integration notation clarified). The cone is a developable surface. When developed, the ψ -spiral on the cone is transformed into a ψ -logarithmic spiral on the plane. For $0 < \psi < \pi/2$, choose the direction in which the slant distance r from the vertex increases. Its polar equation is

$$r = Ce^{\theta \cot \psi}, \quad C > 0.$$

Then

$$ds = \sqrt{dr^2 + r^2 d\theta^2} = \frac{C}{\sin \psi} e^{\theta \cot \psi} d\theta.$$

Between the points corresponding to $\theta_0 < \theta_1$,

$$\ell = \frac{C}{\sin \psi} \int_{\theta_0}^{\theta_1} e^{\theta \cot \psi} d\theta = \frac{C}{\cos \psi} \left[e^{\theta \cot \psi} \right]_{\theta_0}^{\theta_1} = \frac{b - a}{\cos \psi},$$

where a, b denote the distances of the endpoints from the vertex of the cone, with $b > a$.^[E077]

Editorial endpoint cases. At $\psi = 0$ the curve follows a generator and its length is $|b - a|$. At $\psi = \pi/2$ it follows a parallel circle; both endpoint slant distances are equal and the length depends on the traversed angular interval, so it cannot be recovered from a, b alone.

Also solved by Walter B. Carver, Cornell University; M. S. Klamkin, Polytechnic Institute of Brooklyn; S. H. Sesskin, Hofstra College, New York; A. Sisk, Maryville College, Tennessee; and the proposer.

226

Mathematics Magazine **29** (1955–1956), 48; proposal January 1955

Source PDF p. 194–195.

Proposed by P. D. Thomas, Eglin Air Force Base, Florida

Tangents are drawn from a point P to an ellipse. If R, Q are the points of contact and O is the center of the ellipse, find the locus of P if the area of the quadrilateral $PQOR$ remains constant.

II. Solution by Hüseyin Demir, Zonguldak, Turkey. The ellipse is an orthogonal projection of a circle. Let $P'Q'O'R'$ be the corresponding quadrangle. The locus of P' is a concentric circle, for the two quadrangles are in a constant ratio in area. Hence the locus of P , the projection of P' , is an ellipse homothetic with the original one.

Also solved by M. S. Klamkin, Brooklyn Polytechnic Institute; S. H. Sesskin, Hofstra College; E. P. Starke, Rutgers University; Chih-yi Wang, University of Minnesota; and the proposer.

270: Circles in a Crescent

Mathematics Magazine **30** (1956–1957), 108; proposal March 1956

Source PDF p. 195–196.

Proposed by Leon Bankoff, Los Angeles, California

A maximum circle is inscribed in a crescent formed by a semicircle and a quadrant of a circle. Find a general expression for the radii of consecutively tangent circles touching the sides of the crescent, the first touching the maximum circle, the second touching the first, and so on.

Solution by Hüseyin Demir, Kandilli, Bolgesi, Turkey (intermediate formulas corrected). Let the given circles $(O), (O')$ intersect at A, B , and denote the center and radius of the n th circle by O_n, r_n . Write $AB = 2R$ and index the maximum circle by $n = 0$. Invert with center A and power $k^2 = AB^2 = 4R^2$. The circle (O) becomes its tangent line BH_0 , and (O') becomes BO' , the two lines forming an angle $2\alpha = 45^\circ$. The inverse circles (Ω_n) form a chain inscribed in this angle. If (Ω_n) touches BH_0 at H_n , its radius is

$$\rho_n = \left(\frac{1 - \sin \alpha}{1 + \sin \alpha} \right)^n \rho_0, \quad \rho_0 = \Omega_0 H_0 = BH_0 \tan \alpha = 2R \tan \alpha.$$

Draw the common tangent $AT_n T'_n$ to $(O_n), (\Omega_n)$. Then

$$r_n = \frac{AT_n \rho_n}{AT'_n} = \frac{k^2 \rho_n}{A\Omega_n^2 - \rho_n^2}.$$

If K_n is the projection of Ω_n onto AB , then

$$\begin{aligned} A\Omega_n^2 - \rho_n^2 &= AK_n^2 + K_n\Omega_n^2 - \rho_n^2 \\ &= (2R + \rho_n)^2 + BH_n^2 - \rho_n^2 \\ &= 4R^2 + 4R\rho_n + \rho_n^2 \cot^2 \alpha. \end{aligned}$$

Consequently^[E078]

$$r_n = \frac{4R^2 \rho_n}{4R^2 + 4R\rho_n + \rho_n^2 \cot^2 \alpha}.$$

Substitution gives the desired result, with $\alpha = \pi/8$:

$$r_n = \frac{R}{1 + \frac{1}{2}[(1 + \sin \alpha)^{2n} + (1 - \sin \alpha)^{2n}] \sec^{2n} \alpha \cot \alpha}.$$

Also solved by J. W. Clawson, Collegeville, Pennsylvania; and the proposer.

273: A Harmonic Range

Mathematics Magazine **30** (1956–1957), 166; proposal May 1956

Source PDF p. 197.

Proposed by N. A. Court, University of Oklahoma

The intersection of the tangents to the circumcircle of a triangle at the ends of one side is collinear with the two further intersections of the circle with the median from the opposite vertex and with the parallel through that vertex to the side considered.

Editorial identification. The compilation calls this Proposal 272; the reproduced journal entry is numbered 273. Its journal number is used here.^[E079]

II. Solution by Hüseyin Demir, Kandilli, Bolgesi, Turkey. Let the median and exmedian relative to A intersect the circumcircle at E, F , respectively, and let K_a be the intersection of the tangents at B, C . From the harmonic ratios,

$$A(B, C, E, F) = (AB, AC; AE, AF) = -1.$$

Editorial completion of the supplied fragment. The reproduction stops at this equality. Here is a direct completion. Put $B = (-1, 0)$, $C = (1, 0)$, $A = (u, v)$, $v \neq 0$. The circumcircle has equation $x^2 + y^2 + Dy = 1$, where $D = (1 - u^2 - v^2)/v$. Its tangents at B, C meet at $K_a = (0, 2/D)$ if $D \neq 0$. The other point on the parallel through A is $F = (-u, v)$, while the other point on the median AM , $M = (0, 0)$, is $E = -A/(u^2 + v^2)$. Direct substitution gives $\det(E - K_a, F - K_a) = 0$. Coincident intersections and parallel tangents require the corresponding tangent or projective limiting interpretation.^[E079]

332: A Polynomial Factor

Mathematics Magazine **32** (1958–1959), 52; proposal January 1958

Source PDF p. 197.

Proposed by Norman Anning, Alhambra, California

Prove that no polynomial of degree 22 with rational coefficients is an exact divisor of $x^{45} + 1$.^[E080]

II. Solution by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey (factor argument clarified). The roots are all distinct. The primitive roots among them form one irreducible cyclotomic factor of degree $\varphi(90) = \varphi(45) = 24$. A rational factor of degree 22 cannot contain this factor; it must therefore use only the $45 - 24 = 21$ remaining roots. Hence there is no such factor.

Editorial factorization. Explicitly,

$$x^{45} + 1 = \Phi_2(x)\Phi_6(x)\Phi_{10}(x)\Phi_{18}(x)\Phi_{30}(x)\Phi_{90}(x),$$

with respective degrees 1, 2, 4, 6, 8, 24. The coefficient field matters: over $\mathbb{R}$, a product of eleven conjugate quadratic factors has degree 22.

Also solved by D. A. Breault, Station Hospital, Fort Monmouth, New Jersey; C. F. Pinzka, University of Cincinnati; Norman Anning, Alhambra, California; and the proposer.

Editorial note (original journal). One incorrect solution was received.

353: Tangent Circles

Mathematics Magazine **32** (1958–1959), 226; proposal September 1958

Source PDF p. 198–199.

Proposed by Karl M. Herstein, New York City, New York

Given a line and two points not on the line, construct two equal circles whose centers are on the given line, which pass through the given points respectively and are tangent to each other.

Solution by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey (classification corrected). Let A, B, d be the given points and line. Distinct equal circles cannot touch internally. The source also discusses coincident circles (whose center is the intersection of d with the perpendicular bisector

of AB) and limiting straight lines with contact at infinity; these are not pairs of distinct tangent circles.

For external tangency, take d as the x -axis, reflect either given point across it if necessary, and write $A = (-u, a)$, $B = (u, b)$, with $u > 0$ and $a, b > 0$. Interchanging the points if necessary, suppose $a \geq b$. Let the tangency point be $L = (\lambda, 0)$, and let the other intercepts of the two circles be $A' = (\xi, 0)$ and $B' = (\eta, 0)$. The right triangles $A'AL$ and LBB' give

$$a^2 = (\lambda + u)(-u - \xi), \quad b^2 = (\eta - u)(u - \lambda),$$

so

$$\xi = -\frac{a^2}{\lambda + u} - u, \quad \eta = -\frac{b^2}{\lambda - u} + u.$$

Equating diameters $\lambda - \xi = \eta - \lambda$ yields

$$2\lambda^3 + (a^2 + b^2 - 2u^2)\lambda - u(a^2 - b^2) = 0.$$

Set $a^2 - b^2 = 2c^2$. The equation becomes

$$\lambda^3 + p\lambda + q = 0, \quad p = b^2 + c^2 - u^2, \quad q = -uc^2.$$

For $\mathcal{D} = 4p^3 + 27q^2$, there is one real root if $\mathcal{D} > 0$, a repeated root if $\mathcal{D} = 0$, and three distinct real roots if $\mathcal{D} < 0$. Every real root lies in $(-u, u)$: the original cubic is negative at $-u$, positive at u , and strictly increasing outside that interval. Each root gives centers $((\xi + \lambda)/2, 0)$ and $((\lambda + \eta)/2, 0)$, with common positive radius $(\lambda - \xi)/2$.

As in the source, $\mathcal{D} \leq 0$ implies $p \leq 0$. Writing $b^2 + c^2 = u^2 \cos^2 t$, $0 \leq t < \pi/2$, gives

$$\mathcal{D} = -4u^6 \sin^6 t + 27u^2 c^4 \leq 0, \quad \sqrt{27} c^2 \leq 2u^2 \sin^3 t \leq 2u^2.$$

This last necessary bound is not an equivalent discriminant test; the classification must use $\mathcal{D}$.^[E081]

Editorial construction qualification. If the perpendicular projections of A, B onto d coincide, no pair of distinct externally tangent circles exists. In general the cubic cannot be solved by straightedge and compass alone: $u = 1, a = 2, b = 1$ gives $2\lambda^3 + 3\lambda - 3 = 0$, irreducible over $\mathbb{Q}$. Thus the construction in unrestricted generality requires solving a cubic, not merely Euclidean quadratic constructions.

Also solved by Sam Kravitz, East Cleveland, Ohio.

374: Equivalent Triangles

Mathematics Magazine **33** (1959–1960), 113; proposal March 1959

Source PDF p. 199–200.

Proposed by Victor Thebault, Tennie, Sarthe, France

Let a straight line d cut BC, CA, AB at A_1, B_1, C_1 . For any point P , define A'_1, B'_1, C'_1 by $\overrightarrow{PA'_1} = -3\overrightarrow{PA_1}$ and its cyclic companions. If A_2, B_2, C_2 are the midpoints of AA'_1, BB'_1, CC'_1 , their triangle has the same area as ABC .

II. Solution by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey. We express the relations vectorially and use vector products. First, P need not lie on d . We have

$$\overrightarrow{PA'_1} = -3\overrightarrow{PA_1}, \quad 2\overrightarrow{PA_2} = \overrightarrow{PA} - 3\overrightarrow{PA_1},$$

and hence

$$2\overrightarrow{A_2B_2} = \overrightarrow{AB} - 3\overrightarrow{A_1B_1}, \quad 2\overrightarrow{A_2C_2} = \overrightarrow{AC} - 3\overrightarrow{A_1C_1}.$$

Use $[XYZ]$ for twice the oriented area, so that $[ABC] = \overrightarrow{AB} \times \overrightarrow{AC}$. Then

$$\begin{aligned} 4[A_2B_2C_2] &= [ABC] - 3(\overrightarrow{AB} \times \overrightarrow{A_1C_1} + \overrightarrow{A_1B_1} \times \overrightarrow{AC}) + 9\overrightarrow{A_1B_1} \times \overrightarrow{A_1C_1} \\ &= [ABC] - 3(\overrightarrow{AB} \times \overrightarrow{A_1A} + \overrightarrow{A_1A} \times \overrightarrow{AC}) \\ &= [ABC] - 3(\overrightarrow{AB} - \overrightarrow{AC}) \times \overrightarrow{A_1A} \\ &= [ABC] + 3\overrightarrow{BC} \times \overrightarrow{CA} = 4[ABC]. \end{aligned}$$

The last product in the first line vanishes because A_1, B_1, C_1 are collinear.

The problem generalizes as follows. If $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$, and $\overrightarrow{PA_1'} = -n\overrightarrow{PA_1}$, $\overrightarrow{AA_2} = m\overrightarrow{AA_1'}$ cyclically, then the corrected identity is^[E082]

$$[A_2B_2C_2] = (1-m)(1-m+mn)[ABC] + m^2n^2[A_1B_1C_1].$$

For $m = 1/2, n = 3$ and collinear A_1, B_1, C_1 , it gives the original result.

Also solved by Christopher Henrich (partially); and the proposer.

383: Dissecting a Square

Mathematics Magazine **33** (1959–1960), 228; proposal September 1959

Source PDF p. 201–202.

Proposed by Raphael T. Coffman, Richland, Washington

Cut any square into not more than six pieces which can be reassembled to form a cube having its surface area equal to the area of the square. Bending of the pieces is permissible.

I. Solution by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey. Let a be the side of the square. The edge u of the cube satisfies $6u^2 = a^2$, so $u = a\sqrt{6}/6$. Develop the cube as in panel (1), and assemble the rectangles into the $3u$ by $2u$ rectangle $ABCD$ of panel (2). Take

$$AP = \sqrt{AB \cdot AD} = \sqrt{3u \cdot 2u} = u\sqrt{6}.$$

Let E be the perpendicular foot of B on AP .^[E083] The similar right triangles ABE and APD give

$$\frac{BE}{AB} = \frac{AD}{AP}, \quad BE = \frac{AB \cdot AD}{AP} = \frac{AP^2}{AP} = AP.$$

Thus the square shown in (2) can be drawn. Comparing (2) with (3) shows the equivalence of $ABCD$ and the square, whose side is a . The number of pieces is six, less than seven. The cube is obtained as in (3a), (3b), and (4). For cutting the square directly without first using the rectangle, use (5).

Editorial construction details. The redraw uses $u = 1$, $A = (0, 0)$, $B = (3, 0)$, $C = (3, 2)$, $D = (0, 2)$, $P = (\sqrt{2}, 2)$, $E = (1, \sqrt{2})$. The square has one side on AP , the opposite side through B , and an adjacent side through C . Horizontal cuts at heights 0, 1, 2 and the boundary $x = 3$ give the six numbered pieces. Moving piece 1 by $(-\sqrt{2}, -2)$, pieces 2 and 3 by $(-3, 0)$, and piece 4 by $(-3 + \sqrt{2}, 2)$ fills the rectangle; pieces 5 and 6 remain fixed. These coordinates specify the cuts without relying on proportions in the old sketch.

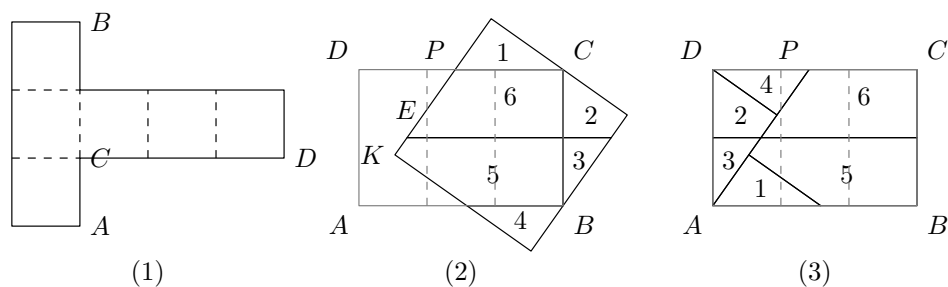


Figure 10.1: Square-to-rectangle dissection: cube net (1), overlaid square (2), and rearranged rectangle (3). Solid internal lines are cuts; dashed grid lines indicate folds.

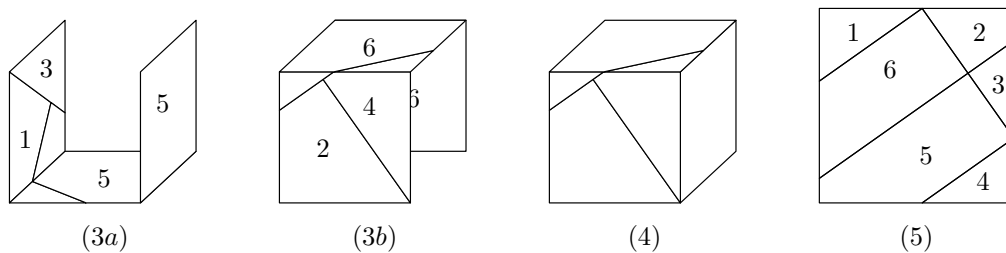


Figure 10.2: Folding views (3a), (3b), assembled cube (4), and the square's cut pattern (5). Folding views are schematic; the planar pattern is generated from the construction.

387: An Induction Proof

Mathematics Magazine **33** (1959–1960), 233; proposal September 1959

Source PDF p. 203–205.

Proposed by D. S. Mitrinovich, University of Belgrade, Yugoslavia

Prove by induction, for every nonnegative integer n , that

$$\left[\frac{\partial^n}{\partial t^n} \left(\frac{1}{1-t} e^{-xt/(1-t)} \right) \right]_{t=0} = e^x \frac{d^n}{dx^n} (x^n e^{-x}).$$

Solution by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey (operator signs and indices corrected). The equality is evident for $n = 0$. Suppose it holds for $n = p$. Write $D = d/dx$. By Leibniz's formula,

$$\begin{aligned} e^x D^p (x^p e^{-x}) &= e^x \sum_{k=0}^p \binom{p}{k} D^k x^p D^{p-k} e^{-x} \\ &= \sum_{k=0}^p (-1)^{p-k} \binom{p}{k} \frac{p!}{(p-k)!} x^{p-k} \\ &= \sum_{\lambda=0}^p (-1)^\lambda \binom{p}{\lambda} \frac{p!}{\lambda!} x^\lambda. \end{aligned} \tag{1}$$

Put $u = e^{-x/(1-t)}$. Then

$$\frac{1}{1-t} e^{-xt/(1-t)} = -e^x \frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial t} = -x \frac{\partial^2 u}{\partial x^2}.$$

Commuting the x, t derivatives and then setting $t = 0$ gives

$$\left[\partial_t^p \left(\frac{1}{1-t} e^{-xt/(1-t)} \right) \right]_{t=0} = (-1)^{p+1} e^x D(xD^2)^p e^{-x}. \quad (2)$$

Thus the induction statement is equivalent to

$$D(xD^2)^p e^{-x} = P_p(x) e^{-x}, \quad P_p(x) = \sum_{\lambda=0}^p a_\lambda x^\lambda, \quad a_\lambda = (-1)^{p+\lambda+1} \binom{p}{\lambda} \frac{p!}{\lambda!}. \quad (3)$$

To pass to $p+1$, observe that

$$\begin{aligned} D(xD^2)^{p+1} e^{-x} &= DxD[D(xD^2)^p e^{-x}] \\ &= DxD(P_p e^{-x}) \\ &= [xP_p'' + (1-2x)P_p' + (x-1)P_p] e^{-x}. \end{aligned}$$

The coefficient of x^λ in the bracket is

$$b_\lambda = (\lambda+1)^2 a_{\lambda+1} - (2\lambda+1)a_\lambda + a_{\lambda-1},$$

where out-of-range coefficients are zero. For $1 \leq \lambda \leq p-1$, this equals

$$\begin{aligned} b_\lambda &= (-1)^{p+\lambda+2} \frac{(p!)^2}{(\lambda!)^2 (p-\lambda+1)!} [(p-\lambda)(p-\lambda+1) \\ &\quad + (2\lambda+1)(p-\lambda+1) + \lambda^2] \\ &= (-1)^{p+\lambda+2} \binom{p+1}{\lambda} \frac{(p+1)!}{\lambda!}. \end{aligned}$$

The boundary coefficients b_0, b_p, b_{p+1} give the same formula directly. Therefore

$$D(xD^2)^{p+1} e^{-x} = \sum_{\lambda=0}^{p+1} (-1)^{p+\lambda+2} \binom{p+1}{\lambda} \frac{(p+1)!}{\lambda!} x^\lambda e^{-x},$$

which proves (3) for $p+1$ and completes the induction.^[E084]

400: A Triangle Construction

Mathematics Magazine **34** (1960–1961), 53; proposal January 1960

Source PDF p. 205.

Proposed by W. B. Carver, Cornell University

Given a point, a circle, and any curve in a plane, construct an equilateral triangle having a vertex on each of them.

II. Solution by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey. Let $A, (b), (c)$ be the given point, circle, and curve. If ABC is a solution, C on (c) is obtained from B on (b) by a rotation about A through 60° or -60° . Hence the unknown vertices on (c) are obtained by intersecting (c) with the two rotated positions of (b) . The triangles are then immediate, discarding any degenerate intersection at A .

Also solved by Harry M. Gehman, University of Buffalo; Rostyslaw J. Lewyckij, University of Toronto; Harvey Walden (partial solution); and the proposer.

401: A Fibonacci Series

Mathematics Magazine **34** (1960–1961), 55; proposal January 1960

Source PDF p. 206–209.

Proposed by John M. Howell, Los Angeles City College

Given $F(0) = c$, $F(1) = d$, and $F(n) = aF(n-1) + bF(n-2)$ for $n \geq 2$, where a, b, c, d are real, find a general form for $F(n)$.

II. Solution by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey. Successively,

$$\begin{aligned} F(0) &= c, & F(1) &= d, \\ F(2) &= ad + bc, \\ F(3) &= (a^2 + b)d + abc, \\ F(4) &= (a^3 + 2ab)d + (a^2b + b^2)c. \end{aligned}$$

In general, for $n \geq 2$,

$$\begin{aligned} F(n) &= \left[\binom{n-1}{0} a^{n-1} + \binom{n-2}{1} a^{n-3} b + \binom{n-3}{2} a^{n-5} b^2 + \cdots \right] d \\ &\quad + \left[\binom{n-2}{0} a^{n-2} + \binom{n-3}{1} a^{n-4} b + \binom{n-4}{2} a^{n-6} b^2 + \cdots \right] bc \\ &= d \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-j}{j} a^{n-1-2j} b^j + bc \sum_{j=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-2-j}{j} a^{n-2-2j} b^j, \end{aligned}$$

which follows by induction. The finite sums contain only nonnegative powers, interpreted as polynomial monomials even when a or b is zero.

III. Alternate solution by Hüseyin Demir (determinants corrected). Write the equations in the unknowns $F(n), F(n-1), \dots, F(2)$:

$$\begin{aligned} F(n) - aF(n-1) - bF(n-2) &= 0, \\ &\vdots \\ F(4) - aF(3) - bF(2) &= 0, \\ F(3) - aF(2) &= bd, \\ F(2) &= ad + bc. \end{aligned}$$

The coefficient matrix is upper triangular with diagonal 1 and first and second superdiagonals $-a, -b$, so its determinant is 1. Cramer's rule replaces its first column by $(0, \dots, 0, bd, ad + bc)^T$. Expanding that determinant gives

$$F(n) = (ad + bc)D_{n-2} + bdD_{n-3}, \quad n \geq 2, \text{ [E085]}$$

where $D_{-1} = 0$, $D_0 = 1$, and

$$D_k = \det \begin{pmatrix} a & b & 0 & \cdots & 0 \\ -1 & a & b & \ddots & \vdots \\ 0 & -1 & a & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b \\ 0 & \cdots & 0 & -1 & a \end{pmatrix}_{k \times k}.$$

Indeed $D_k = aD_{k-1} + bD_{k-2}$, giving the required recurrence.

IV. Alternate solution by Hüseyin Demir. Where the denominators are nonzero, set $u_n = F(n)/F(n-1)$. Then

$$u_1 = \frac{d}{c}, \quad u_n = a + \frac{b}{u_{n-1}} = a + \frac{b}{a + \frac{b}{a + \frac{b}{\ddots + \frac{b}{d/c}}}}.$$

Multiplying $F(j) = u_j F(j-1)$ for $j = 1, \dots, n$ gives

$$F(n) = c u_1 u_2 \cdots u_n.$$

This ratio form applies only when its denominators exist; the polynomial and determinant formulas cover all real initial data.^[E085]

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; R. G. Buschman, University of Oregon; F. D. Parker, University of Alaska; Charles F. Pinzka, University of Cincinnati; Chih-yi Wang, University of Minnesota; and the proposer.

412: Projective Correspondence

Mathematics Magazine **34** (1961), 175; proposal May 1960

Source PDF p. 210.

Proposed by D. Moody Bailey, Princeton, West Virginia

Let P lie on the circumcircle of ABC . The rays BP, CP meet CA, AB at E, F . With directed segments, and when the displayed ratios are defined, prove

$$\frac{b^2}{a^2} \frac{BF}{FA} + \frac{c^2}{a^2} \frac{CE}{EA} = -1,$$

where a, b, c are the opposite side lengths.

Solution by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey. The points E, F correspond projectively. Put $e = CE/EA$, $f = BF/FA$; their bilinear relation is

$$\mathcal{A}ef + \mathcal{B}e + \mathcal{C}f + \mathcal{D} = 0.$$

Let P tend successively to A, B, C . At A , both e, f are infinite, so $\mathcal{A} = 0$. At B , the limiting BE is the exsymmedian, giving $e = -a^2/c^2$, $f = 0$; hence

$$-\mathcal{B} \frac{a^2}{c^2} + \mathcal{D} = 0, \quad \mathcal{B} = \frac{c^2}{a^2} \mathcal{D}.$$

Similarly $\mathcal{C} = (b^2/a^2)\mathcal{D}$. Substitution proves the result. The limiting positions are used to determine the correspondence, not as values of undefined ordinary ratios.^[E086]

Also solved by Josef Andersson, Vaxholm, Sweden; Leon Bankoff, Los Angeles, California; A. F. Hordam, University of New England, Armidale, NSW, Australia; and the proposer.

427: A Cevian Relation

Mathematics Magazine **34** (1961), 303; proposal November 1960

Source PDF p. 211.

Proposed by D. Moody Bailey, Princeton, West Virginia

Through a point P draw cevians from B, C , meeting CA, AB at E, F . Let M be the midpoint of BC . The line MP meets CA at N and AB at O . Extend EF to meet BC at G , and let the parallel to AG through B meet CF at H . Show that $HO \parallel CA$, for nondegenerate defined intersections.

Solution by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey. Fix A, B, C, M, F and vary the geometrically related points P, O, N, G, E, H . Projection from M gives $O \leftrightarrow P$; projection from B gives $P \leftrightarrow E$; projection from F gives $E \leftrightarrow G$; the pencil AG corresponds to the parallel pencil BH , and thence to H on CF . Thus $O \leftrightarrow H$ is projective. Their common point F corresponds to itself, so the correspondence is a perspectivity: OH passes through a fixed point L . When O is at infinity on AB , H is at infinity on CF , so L is at infinity and OH has fixed direction. When $O = B$, the limiting configuration gives $P = E = G = C$, and $OH = BH \parallel CA$. This proves the assertion.^[E087]

Also solved by the proposer.

428: Permuted Digits

Mathematics Magazine **34** (1961), 303; proposal November 1960

Source PDF p. 211–212.

Proposed by Murray S. Klamkin, AVCO, Wilmington, Massachusetts

The number $N = 142857$ has the property that $2N, 3N, 4N, 5N, 6N$ are permutations of its digits. Does there exist a number M such that $2M, 3M, 4M, 5M, 6M, 7M$ are all permutations of M ?

I. Solution by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey (historical argument, not a valid nonexistence proof). The published argument assumes that, because the seven multiples give seven permutations, M must have seven digits. It then writes $M = \overline{abcdefg} = 10G + g$, $G = \overline{abcdef}$, and assumes that one of the multiples is the cyclic shift $\overline{gabcdef} = 10^6g + G$. Under these assumptions,

$$p(10G + g) = 10^6g + G, \quad G = \frac{(10^6 - p)g}{10p - 1} = \frac{N_p}{D_p}g.$$

The source's arithmetic table is

p	N_p	D_p	Reduced quotient information
1	999999	9	$N_p/D_p = 111111$
2	999998	19	irreducible
3	999997	29	irreducible
4	999996	$39 = 3 \cdot 13$	$(N_p/3)/D_p$ irreducible
5	999995	$49 = 7 \cdot 7$	irreducible
6	999994	59	irreducible
7	999993	$69 = 3 \cdot 23$	$(N_p/3)/D_p$ irreducible

It concludes that the only possible G has six repeated digits and hence that no solution exists. Neither the seven-digit assumption nor the assumed occurrence of that cyclic shift follows from the problem. The nonexistence conclusion is false.^[E088]

Editorial counterexample. The ordinary 18-digit integer

$$M = 105263157894736842$$

satisfies the requirement:

$2M$	210526315789473684
$3M$	315789473684210526
$4M$	421052631578947368
$5M$	526315789473684210
$6M$	631578947368421052
$7M$	736842105263157894

All six rows are cyclic permutations of M , with no leading-zero convention needed.

Editorial note (original journal). II. Comment by Dermott A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts. The source asserts the same property through multiplier 16 for $M = 5882352941176470$ and through multiplier 28 for $L = 3448275862068965517241379310$, explaining their connection with the repeating periods of $1/17$ and $1/29$. It further conjectures that, when p is prime and the period of $1/p$ has length $p-1$, multiplying the period by $1, \dots, p-1$ permutes its digits.

Editorial correction to the comment. The two printed integers above do not satisfy the stated ordinary-integer property: their doubles already gain a digit. The relevant fixed-width periods are 0588235294117647 and 0344827586206896551724137931, with the zero at the beginning. For a full-period prime, multiplication gives cyclic permutations of this fixed-width block, counting leading zeros. The counterexample above avoids that ambiguity.^[E088]

432: Cevian Lines

Mathematics Magazine **34** (1961), 365; proposal January 1961

Source PDF p. 212–213.

Proposed by Lee Tih-Ming, Taipei, Taiwan

Join an interior point O of ABC to the vertices. Drop perpendiculars OX, OY, OZ to BC, CA, AB , respectively. Let $D = AO \cap YZ$, $E = BO \cap ZX$, $F = CO \cap XY$. Show that

$$\frac{AZ}{ZB} \frac{BX}{XC} \frac{CY}{YA} = \frac{ZD}{DY} \frac{YF}{FX} \frac{XE}{EZ}.$$

II. Solution by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey. The point O need not be inside the triangle, provided directed ratios and nondegenerate intersections are used.^[E089] Put

$$\alpha = \angle BAO, \beta = \angle CBO, \gamma = \angle ACO, \quad \alpha' = \angle OAC, \beta' = \angle OBA, \gamma' = \angle OCB.$$

The sine law in AZD, ADY gives

$$\frac{ZD}{\sin \alpha} = \frac{AZ}{\sin D}, \quad \frac{DY}{\sin \alpha'} = \frac{YA}{\sin D}, \quad \frac{ZD}{DY} = \frac{AZ \sin \alpha}{YA \sin \alpha'}.$$

Multiply this relation and its two cyclic companions:

$$\frac{ZD}{DY} \frac{YF}{FX} \frac{XE}{EZ} = \frac{AZ}{ZB} \frac{BX}{XC} \frac{CY}{YA} \left(\frac{\sin \alpha}{\sin \alpha'} \frac{\sin \beta}{\sin \beta'} \frac{\sin \gamma}{\sin \gamma'} \right).$$

The parenthesis is 1 by trigonometric Ceva, since AO, BO, CO are concurrent. This proves the identity wherever the ratios are defined. Clearing denominators yields the corresponding product identity at its continuous limiting configurations.

Also solved by Brother Alfred, St. Mary's College, California; Josef Andersson, Vaxholm, Sweden; C. W. Trigg, Los Angeles City College; Dale Woods, Oklahoma State University; and the proposer.

435: Triangular Extrema

Mathematics Magazine **34** (1961), 368; proposal January 1961

Source PDF p. 213–214.

Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts

Determine the largest and smallest equilateral triangles that can be inscribed in an ellipse.

Solution by Hüseyin Demir, Kandilli, Ereğli, Kdz., Turkey. Let $A_1A_2A_3$ be equilateral and inscribed in

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0, \quad (1)$$

and let its circumcircle be

$$(x - u)^2 + (y - v)^2 = r^2. \quad (2)$$

Write $c^2 = a^2 - b^2$. The circle meets the ellipse at a fourth point $A_4 = (x_4, y_4)$, with multiplicities allowed. Eliminating y gives a quartic whose leading terms are $c^4x^4 - 4a^2c^2ux^3 + \dots$; eliminating x gives $c^4y^4 + 4b^2c^2vy^3 + \dots$. Since the equilateral triangle has centroid (u, v) ,

$$x_1 + x_2 + x_3 = 3u, \quad y_1 + y_2 + y_3 = 3v.$$

Thus

$$x_4 = \frac{(a^2 + 3b^2)u}{c^2}, \quad y_4 = -\frac{(b^2 + 3a^2)v}{c^2}. \quad (3)$$

Substituting in (1) gives the center ellipse

$$\frac{u^2}{\alpha^2} + \frac{v^2}{\beta^2} = 1, \quad \alpha = \frac{ac^2}{a^2 + 3b^2}, \quad \beta = \frac{bc^2}{b^2 + 3a^2}, \quad \alpha > \beta. \quad (4)$$

Moreover,

$$\begin{aligned} r^2 &= (x_4 - u)^2 + (y_4 - v)^2 \\ &= (a - \alpha)^2 \frac{u^2}{\alpha^2} + (b + \beta)^2 \frac{v^2}{\beta^2}. \end{aligned}$$

The two coefficients are the squared extreme radii. As $b + \beta > a - \alpha$, the maximum occurs at $(u, v) = (0, \pm\beta)$ and the minimum at $(\pm\alpha, 0)$:

$$r_{\max} = b + \beta = \frac{4a^2b}{3a^2 + b^2}, \quad r_{\min} = a - \alpha = \frac{4ab^2}{a^2 + 3b^2}.$$

The largest triangles have a vertex at an extremity of the minor axis; the smallest have a vertex at an extremity of the major axis. That axis is an axis of symmetry of the triangle. There are two of each when $a > b$. If $a = b$, every inscribed equilateral triangle has the same size and there are infinitely many orientations.^[E090]

Also solved by Josef Andersson, Vaxholm, Sweden; J. W. Clawson, Collegeville, Pennsylvania (two solutions); and J. W. Mellender, University [affiliation cut off in the supplied reproduction].

445: Correction to a Published Proof

Mathematics Magazine **35** (1962), 317; references March and November 1961

Source PDF p. 215.

Comment by Hüseyin Demir, Middle East Technical University, Ankara, Turkey

The proof needs a little modification. Read: let the orthogonal projections of A, B, P on the line OM be A', B', P' . In the remaining part of the proof all letters P are to be replaced by P' , and in conclusion we have

$$\frac{MA^2 - k}{MB^2 - k} = \frac{2OM \cdot A'P'}{2OM \cdot B'P'} = \frac{A'P'}{B'P'} = \frac{PA}{PB}.$$

Editorial source note. Only this correction is reproduced in the supplied collection; the underlying proposal and original proof are absent. Its notation is retained without inventing the missing configuration.^[E091]

476: Perimeter and Area Bisector

Mathematics Magazine **35** (1962), 312; proposal March 1962

Source PDF p. 216–218.

Proposed by Kaidy Tan, Fukien Normal College, China

Draw a straight line bisecting both the perimeter and the area of a given quadrilateral.

Solution by Hüseyin Demir, Middle East Technical University, Ankara, Turkey (feasibility conditions corrected). We consider the convex case. A cutting line meets two adjacent sides or two opposite sides; either case includes an endpoint at a vertex. Write $AB = a$, $BC = b$, $CD = c$, $DA = d$, and let $K = AC \cap BD$, $AK = \alpha$, $KC = \gamma$, $AC = p = \alpha + \gamma$.

I. Adjacent sides. Let $M \in AB$, $N \in AD$. Draw $CC' \parallel BD$, meeting AB extended at C' . The areas satisfy

$$[ABCD] = [ABD] + [BCD] = [ABD] + [BC'D] = [AC'D].$$

Let I be the midpoint of AC' and construct $IN \parallel MD$. Then

$$[AMN] = [AIN] + [IMN] = [AIN] + [IDN] = [AID] = \frac{1}{2}[ABCD].$$

With $AM = m$, $AN = n$, similar triangles give

$$\frac{AC'}{AC} = \frac{AB}{AK} = \frac{a}{\alpha}, \quad AC' = \frac{pa}{\alpha}, \quad mn = AI \cdot AD = \frac{pad}{2\alpha}.$$

Perimeter bisection requires $m + n = (a + b + c + d)/2$. Thus m, n are the roots of

$$2x^2 - (a + b + c + d)x + \frac{pad}{\alpha} = 0.$$

Both assignments of the roots must be tested against $0 \leq m \leq a$, $0 \leq n \leq d$. Necessary consequences are $b + c \leq a + d$, $p \leq 2\alpha$ (equivalently $\gamma \leq \alpha$), and

$$(a + b + c + d)^2 - 8pad/\alpha \geq 0.$$

The root interval tests, not these weaker necessary conditions alone, decide whether the constructed segment lies on the intended sides.^[E092]

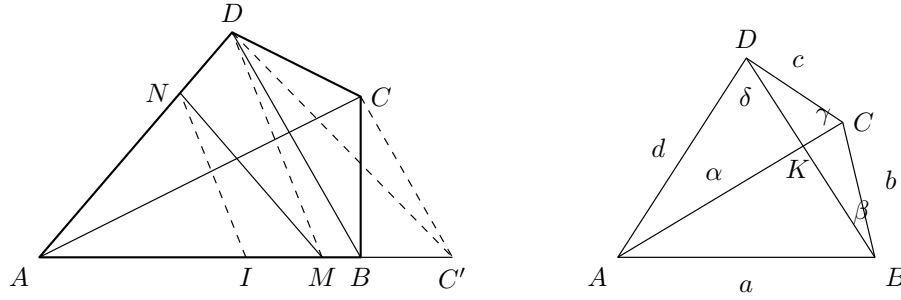


Figure 10.3: Adjacent-side construction and diagonal notation for Proposal 476.

II. Opposite sides. In the configuration shown, extend AB, DC to meet at O . For $M \in AB$, draw $BB' \parallel MC$ and $AA' \parallel MD$, with A', B' on ODC . The area identities give

$$[ABCD] = [AMD] + [MCD] + [BCM] = [A'MD] + [MCD] + [B'CM] = [MB'A'].$$

If N is the midpoint of $A'B'$, MN bisects the area. Put

$$OM = m, ON = n, OA = a', OB = b', OC = c', OD = d', OA' = a'', OB' = b''.$$

Similarity gives $m/c' = b'/b'', m/d' = a'/a''$, and hence

$$2n = a'' + b'' = \frac{a'd' + b'c'}{m}, \quad 2mn = a'd' + b'c'.$$

Perimeter bisection is $MA + AD + DN = MB + BC + CN$, that is,

$$(m - a') + d + (n - d') = (b' - m) + b + (c' - n).$$

Therefore

$$2(m + n) = a' + b' + c' + d' + b - d,$$

and m, n are the roots of

$$2x^2 - (a' + b' + c' + d' + b - d)x + (a'd' + b'c') = 0.$$

A candidate is valid only when the roots are real and an assignment satisfies $a' \leq m \leq b'$, $d' \leq n \leq c'$. These inequalities are not automatically guaranteed by the triangle inequalities quoted in the source.^[E092]

Editorial completion for parallel opposite sides. The point O does not exist as a finite point when $AB \parallel CD$. The following coordinate construction covers that case as well as the other convex configurations. Place A at the origin and put $M = tB$, $N = D + v(C - D)$, $0 \leq t, v \leq 1$. Let $s = (a + b + c + d)/2$ and $T = [ABCD]$ be ordinary area. Perimeter bisection is $at + cv = s - d$. Area bisection is

$$t(B \times D) + tv B \times (C - D) + v(C \times D) = T.$$

Eliminate $v = (s - d - at)/c$, solve the resulting equation of degree at most two, and retain only solutions with $0 \leq t, v \leq 1$. Check the other pair of opposite sides and the four adjacent-side cases likewise. These cases exhaust all line cuts of a convex quadrilateral; the supplied argument does not treat arbitrary nonconvex quadrilaterals.^[E092]

Write this for successive indices, multiply by the appropriate factors, and add. The result is

$$y'_n = \frac{1}{x} \sum_{k=0}^n \left(\prod_{j=n-k-1}^n y_j \right) (\ln x)^k.$$

The $k = n$ term includes $y_{-1} = 1$; the formula also gives $y'_0 = 1$. At $x = 1$ the constant-power term is interpreted in the usual polynomial sense.^[E093]

Also solved by Pierre Bouchard, Université de Montréal, Canada; Nicholas C. Bystrom, St. Paul, Minnesota; Richard W. Feldman, Lycoming College, Pennsylvania; David Fetter, City College of New York; Reinaldo E. Giudici, University of Pittsburgh; Michael Goldberg, Washington, D.C.; Sandra A. Gossum, University of Tennessee; J. M. Howell, Los Angeles City College; Richard A. Jacobson, Houghton College, New York; Lew Kowarski, Morgan State College, Maryland; Fred Lambie, Lexington, Massachusetts; Douglas Lind, University of Virginia; Edwin A. Power, University College, London, England; and the proposer.

Editorial note (original journal). A number of incorrect or undecipherable solutions were received.

1199: Perpendicular Lines in an Isosceles Triangle

Mathematics Magazine **58** (1985), 243; proposal September 1984

Source PDF p. 220.

Proposed by Aristomenis Siskakis, University of Illinois

In an isosceles triangle ABC with AB = AC, let H be the foot of the altitude from A, let E be the perpendicular foot from H to AB, and let M be the midpoint of EH. Show that AM ⊥ EC.

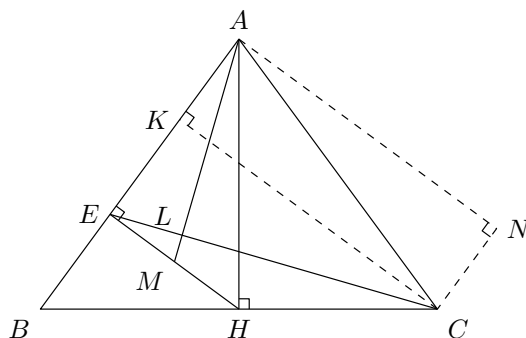


Figure 10.5: The altitude construction, cyclic quadrilateral $HCAL$, and auxiliary rectangle $AKCN$.

I. Solution by Hüseyin Demir, Middle East Technical University, Ankara, Turkey. Let CK be the altitude from C . In the similar right triangles EHA and KBC , the corresponding medians AM and CE make equal angles with the hypotenuses HA and BC . Let the medians meet at L . The quadrilateral $HCAL$ is cyclic. Hence $\angle ALC = \angle AHC = 90^\circ$.

II. Solution by Hüseyin Demir. Use harmonic pencils. Let CK be the altitude from C , and construct the rectangle $AKCN$. Since $HE \parallel AN$ and AM bisects HE ,

$$(AB, AH; AM, AN) = -1.$$

Similarly, $BK \parallel CN$ and CE bisects BK , so

$$(CK, CB; CE, CN) = -1.$$

In these two harmonic pencils three corresponding line pairs are perpendicular. Hence the fourth pair, AM, CE , is perpendicular. Directed or projective interpretations cover the relevant extended lines.

Editorial note (original journal). Also solved by sixty-two others (including the proposer and eight students), who submitted seventy-three solutions. Joseph Konhauser located the problem in the *Monthly*, E1476, with three published solutions in volume 69 (1962), p. 233. Four other solvers forgot to mention that source. P. J. Pedler (Australia) and J. H. Webb (South Africa) found the problem in Loren C. Larson, *Problem Solving Through Problems*, p. 27; Geoffrey A. Kandall found it in M. N. Aref and W. Wernick, *Problems and Solutions in Elementary Geometry*, p. 32, exercise 92. O. Bottema (The Netherlands) and Webb supplied converses: (1) for any triangle, $AM \perp EC$ if and only if $AB = AC$; (2) for any point E on line AB , $AM \perp EC$ if and only if $HE \perp AB$ or A is the midpoint of BE . Other points are as in the statement.

1256: Cyclic Quadrilateral

Mathematics Magazine **61** (1988), 54; proposal December 1986

Source PDF p. 221–222.

Proposed by R. S. Luthar, University of Wisconsin Center, Janesville

Let $ABCD$ be a convex cyclic quadrilateral. Its internal angle bisectors at A, B meet at E . The line through E parallel to CD meets AD at L and BC at M . Prove that $LA + MB = LM$.

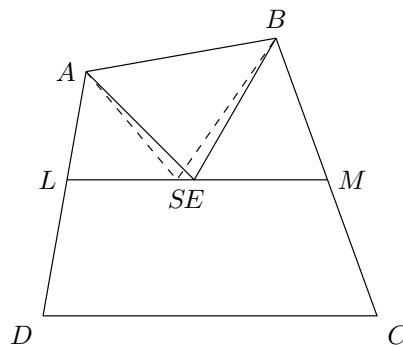


Figure 10.6: The bisectors and parallel segment, with the auxiliary point S in Demir and Tezer's proof.

II. Solution by H. Demir and C. Tezer, Middle East Technical University, Ankara, Turkey. Put $\angle DAB = 2\alpha$, $\angle ABC = 2\beta$, $\angle BCD = 2\gamma$, $\angle CDA = 2\delta$. Then $\angle ELA = 2\delta$, $\angle BME = 2\gamma$, and $\alpha = \pi/2 - \gamma$, $\beta = \pi/2 - \delta$. Suppose first that $\alpha > \beta$; interchange the ends if necessary.

Choose S on LM , on the same side of AD as M , with $LS = LA$. Then $\angle ASL = \angle LAS = \beta$, so $ASEB$ is cyclic. Since $\angle LAS = \beta < \alpha = \angle LAE$, S lies between L, E . Moreover,

$$\begin{aligned} \angle SBM &= \angle SBE + \angle EBM \\ &= \angle SAE + \angle EBM \\ &= \angle LAE - \angle LAS + \beta = \alpha = \angle BSM. \end{aligned}$$

Thus MBS is isosceles and $MS = MB$. Consequently $LM = LS + SM = LA + MB$. The equality case $\alpha = \beta$ follows by continuity (then $S = E$).^[E094]

III. Solution by John P. Hoyt, Lancaster, Pennsylvania. Produce DA, CB to meet at X , in the configuration shown. Draw FH through E parallel to AB , with $F \in XD$, $H \in XC$, and join XE . Since E is the intersection of two exterior angle bisectors of triangle XAB , XE bisects

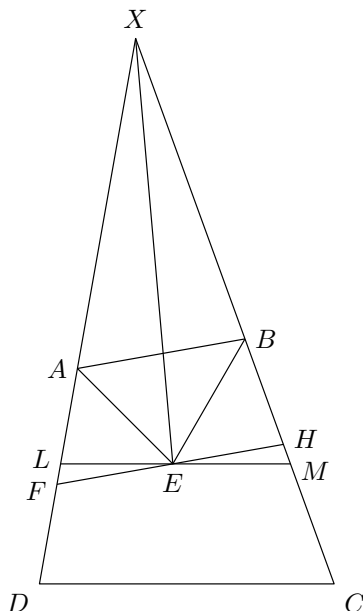


Figure 10.7: Hoyt's construction with the external angle bisector XE and $FH \parallel AB$.

$\angle AXB$. Triangles MLX and HFX have equal corresponding angles and a common angle-bisector segment; hence they are congruent. The angle equalities follow because opposite angles of a cyclic quadrilateral are supplementary. Thus $ME = FE$, $HE = LE$, and $HM = LF$.

Since $FH \parallel AB$ and AE bisects $\angle DAB$, $\angle FAE = \angle AEF$. Hence FAE is isosceles and $AF = FE$. Similarly, $BH = EH$. Therefore

$$\begin{aligned} LA + MB &= (AF - LF) + (BH + HM) \\ &= (AF + BH) + (HM - LF) \\ &= AF + BH = FE + EH = LM. \end{aligned}$$

This finite- X diagram treats the indicated configuration; parallel sides are covered by the first proof or a limit.^[E094]

Also solved by Frank Allen; Farid G. Bassiri (student); Andreas Bender (student), Switzerland; Nirdosh Bhatnagar; David Earnshaw (Canada); Howard Eves; Herta T. Freitag; Richard A. Gibbs; J. T. Groenman (Netherlands); Michael B. Handelsman; P. L. Hon (Hong Kong); King Jamison; Geoffrey A. Kandall; Tsz-Mie Ko (student); Mary S. Krimmel; L. Kuipers (Switzerland); Kee-wai Lau (Hong Kong); J. C. Linders (The Netherlands); David Morin (student, four solutions); Anna Michaelides Penk; Farhood Pouryoussefi (student, Iran); Harry D. Ruderman; Kiran Lall Shrestha (Nepal); J. M. Stark; M. Vowe (Switzerland); Harry Weingarten; and Brent Young (student).

Editorial note (original journal). Most solutions used trigonometric arguments, with an impressive variety of identities. A purely geometric solution due to Gregg Patrino (U.S.A.) appears in Murray Klamkin's *International Mathematical Olympiads, 1978–1985*, New Mathematical Library, no. 31, MAA (solution to Problem 1, 1985).

Problems and Solutions in Mathesis

Question d'examen 10

Mathesis **69** (1960), 249–250

Source PDF p. 224.

Hüseyin Demir, Ereğli, Turquie

Si h_a, h_b, h_c sont les hauteurs et R, r les rayons des cercles circonscrit et inscrit d'un triangle ABC , le rayon R' du cercle circonscrit au triangle des points de contact A', B', C' de BC, CA, AB avec les cercles exinscrits dans les angles A, B, C est donné par

$$R' = \frac{1}{2r} \sqrt{2R(2R - h_a)(2R - h_b)(2R - h_c)}.$$

Solution (R. D.). L'aire S' de $A'B'C'$, triangle podaire du symétrique du centre I du cercle inscrit par rapport au centre O du cercle circonscrit, est liée à celle S de ABC par

$$S' = S \frac{R^2 - OI^2}{4R^2} = S \cdot \frac{r}{2R}. \quad (1)$$

Le triangle $AB'C'$ donne, puisque $AB' = p - c = r \cot(C/2)$, $AC' = p - b = r \cot(B/2)$,

$$\begin{aligned} B'C'^2 = a'^2 &= r^2 \left[\left(\cot \frac{B}{2} + \cot \frac{C}{2} \right)^2 - 4 \cot \frac{B}{2} \cot \frac{C}{2} \cos^2 \frac{A}{2} \right] \\ &= r^2 \frac{\cos^2(A/2)}{\sin^2(B/2) \sin^2(C/2)} (1 - \sin B \sin C) \\ &= a^2 \left(1 - \frac{bc}{4R^2} \right) \end{aligned}$$

ou, à cause de $bc = 2Rh_a$,

$$a'^2 = \frac{a^2}{2R} (2R - h_a).$$

Il suffit de comparer à (1) l'égalité

$$S'^2 = \frac{a'^2 b'^2 c'^2}{16R'^2} = \frac{a^2 b^2 c^2}{16R^2} \cdot \frac{1}{8RR'^2} (2R - h_a)(2R - h_b)(2R - h_c)$$

pour obtenir l'expression annoncée pour R' .

Question d'examen 11

Mathesis **69** (1960), 250–251

Source PDF p. 225.

Hüseyin Demir

Si $\varphi(n)$ désigne l'indicateur d'un entier $n > 4$ et donnant lieu aux nombres premiers jumeaux $n - 1, n + 1$, on a

$$3\varphi(n) \leq n.$$

Solution (R. Deaux). Cette propriété est incluse dans la suivante : tout entier n divisible par 6 est égal ou supérieur au triple de son indicateur, qui est immédiate. Si en effet $2, 3, c, \dots$ sont les facteurs premiers de n , de l'égalité d'Euler

$$\varphi(n) = n \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{c}\right) \dots$$

on déduit $\varphi(n) \leq n/3$, l'égalité n'ayant lieu que si 2 et 3 sont les seuls facteurs premiers de n .

Or, de l'énoncé on déduit d'abord que n est pair. On n'a pas $n = \mathcal{M}3 + 1$, sinon $n - 1$, premier, vaut 3 et $n = 4$, cas exclu, ni $n = \mathcal{M}3 - 1$, sinon $n + 1$, premier, vaut 3, d'où $n - 1 = 1$, non premier. Donc $n = \mathcal{M}3$ et par suite $\mathcal{M}6$.

Note de l'édition. La notation historique $\mathcal{M}k$ désigne un multiple de k .

Question d'examen 12

Mathesis **69** (1960), 251

Source PDF p. 226.

Hüseyin Demir

1° Quatre coniques $\alpha, \beta, \gamma, \delta$ qui ont deux points communs X, Y et sont respectivement circonscrites aux triangles bcd, cda, dab, abc formés par les côtés d'un quadrilatère q ont un troisième point commun Z .

2° Si X, Y sont sur deux côtés e, f du trilatère diagonal t de q , le point Z est sur le troisième g et les triangles XYZ, t sont homologues.

Solution (R. Deaux). Le 1° a été énoncé et démontré (*Mathesis*, 1947, p. 209) par nous-même. Il résulte de ce que les quatre cubiques $(a, \alpha), (b, \beta), (c, \gamma), (d, \delta)$ ayant en commun X, Y et les six sommets de q ont un neuvième point commun Z . Si X est sur e et Y sur f , la cubique (e, f, g) contient les huit premiers points, donc aussi Z , nécessairement sur g ; comme efg est homologue au triangle abc par exemple, inscrit à δ , il résulte d'un théorème de Steiner que efg est aussi homologue au triangle XYZ des seconds points de rencontre de δ avec les côtés de efg .

Question 3954*Mathesis* **69** (1960), 255

Source PDF p. 226.

*Hüseyin Demir**Démontrer l'égalité entre matrices*

$$\begin{bmatrix} \cos^2 \theta & -\sin 2\theta & \sin^2 \theta \\ \cos \theta \sin \theta & \cos 2\theta & -\sin \theta \cos \theta \\ \sin^2 \theta & \sin 2\theta & \cos^2 \theta \end{bmatrix}^n = \begin{bmatrix} \cos^2 n\theta & -\sin 2n\theta & \sin^2 n\theta \\ \cos n\theta \sin n\theta & \cos 2n\theta & -\sin n\theta \cos n\theta \\ \sin^2 n\theta & \sin 2n\theta & \cos^2 n\theta \end{bmatrix},$$

*n étant un entier positif.***Question 3957***Mathesis* **69** (1960), 375

Source PDF p. 226.

Hüseyin Demir

La somme des indicateurs des carrés des diviseurs d'un entier n est égale à la somme algébrique des diviseurs de n^2 , chacun de ceux-ci étant affecté du signe $+$ ou du signe $-$ selon que la somme des exposants de ses facteurs premiers est paire ou impaire.

Other Publications

Source PDF p. 228.

Other Publications of Hüseyin Demir

- [1] H. Demir. “Les cercles podaires dans le polygone inscriptible.” *Mathesis* **68** (1959), 51–54.
- [2] H. Demir. “A theorem analogous to Morley’s theorem.” *Mathematics Magazine* **38** (1965), 228–230.
- [3] H. Demir. “Maximum area of a region bounded by a polygon with given sides.” *Mathematics Magazine* **39** (1966), 228–231.
- [4] H. Demir. “On the geometry of n dimensions and extensions in E^2 of Erdős–Mordel–Oppenheim inequality.” *METU Journal of Pure and Applied Sciences* **1** (1968), 175–197.
- [5] H. Demir. “A trigonometric proof of Morley’s theorem.” *Delta* **1** (1968), 11–14.
- [6] H. Demir. “Incircles within.” *Mathematics Magazine* **59**(2) (1986), 77–83.
- [7] H. Demir and C. Tezer. “More on incircles.” *Mathematics Magazine* **62**(2) (1989), 107–114.
- [8] H. Demir and C. Tezer. “Reflections on a problem of V. Thébault.” *Geometriae Dedicata* **39**(1) (1991), 79–92.
- [9] B. V. Kutuzov. *Geometri. Cilt I*. No. 19, Turkish Mathematical Society Publications. Türk Matematik Derneği, İstanbul, 1963. Translated by Hüseyin Demir.
- [10] B. V. Kutuzov. *Geometri. Cilt II*. No. 20, Turkish Mathematical Society Publications. Türk Matematik Derneği, İstanbul, 1963. Translated by Hüseyin Demir.
- [11] B. V. Kutuzov. *Geometri. Cilt III*. No. 21, Turkish Mathematical Society Publications. Türk Matematik Derneği, İstanbul, 1964. Translated by Hüseyin Demir.

The *Geometri* books can be downloaded from the web site of Türk Matematik Derneği:

<https://tmd.org.tr/tmd-yayinlari-kitaplar/>

Items [1, 4, 5] are not readily available online. However they can be downloaded from <http://sertoz.bilkent.edu.tr/HuseyinDemir.html>.

Complete List of Problems Composed by Hüseyn Demir

Source PDF p. 230–233.

The 117 entries retain the numbering of the compiler’s complete list. Bibliographic punctuation is standardized; corrections to two year ranges and one volume number are recorded in Appendix A.^[E018]

American Mathematical Monthly: Advanced Problems

No.	Problem	Volume	Year	Page(s)
1	4102	50	1943	638
2	4125	51	1944	252
3	4134	51	1944	475
4	4193	53	1946	160
5	4215	53	1946	470
6	4679	63	1956	191
7	4695	63	1956	426
8	4710	63	1956	669
9	4735	64	1957	277
10	4818	65	1958	779

American Mathematical Monthly: Elementary Problems

No.	Problem	Volume	Year	Page(s)
11	E1134	61	1954	568
12	E1160	62	1955	182
13	E1197	63	1956	39
14	E1209	63	1956	186
15	E1217	63	1956	342
16	E1778	72	1965	420
17	E1779	72	1965	420
18	E1877	73	1966	410
19	E1878	73	1966	410
20	E2100	75	1968	670
21	E2101	75	1968	670
22	E2109	75	1968	780
23	E2110	75	1968	780
24	E2124	75	1968	899
25	E2160	76	1969	300

No.	Problem	Volume	Year	Page(s)
26	E2213	77	1970	79
27	E2311	78	1971	793
28	E2312	78	1971	793
29	E2363	79	1972	662
30	E2462	81	1974	281
31	E2625	83	1976	812
32	E3135	93	1986	215
33	E3164	93	1986	566
34	E3422	98	1991	158
35	E3469	98	1991	955

Mathematics Magazine: Proposals

No.	Problem	Volume	Year	Page(s)
36	208	28	1954–1955	27
37	217	28	1954–1955	103
38	227	28	1954–1955	160
39	234	28	1954–1955	234
40	242	28	1954–1955	284
41	248	29	1955–1956	46
42	258	29	1955–1956	163
43	266	29	1955–1956	222
44	298	30	1956–1957	164
45	304	30	1956–1957	223
46	334	31	1957–1958	228
47	349	32	1958–1959	47
48	372	32	1958–1959	220
49	380	32	1958–1959	278
50	384	33	1959–1960	51
51	398	33	1959–1960	165
52	407	33	1959–1960	225
53	415	33	1959–1960	296
54	419	34	1960–1961	49
55	425	34	1960–1961	109
56	437	34	1961	174
57	440	34	1961	237
58	458	34	1961	364
59	472	35	1962	55
60	487	35	1962	186
61	498	35	1962	309
62	509	36	1963	133
63	517	36	1963	197
64	529	36	1963	264
65	537	37	1964	55
66	544	37	1964	119
67	563	37	1964	276
68	572	38	1965	52

No.	Problem	Volume	Year	Page(s)
69	587	38	1965	179
70	599	38	1965	241
71	600	38	1965	317
72	609	39	1966	69
73	628	39	1966	246
74	639	39	1966	306
75	649	40	1967	100
76	680	41	1968	42
77	724	42	1969	96
78	738	42	1969	214
79	743	42	1969	267
80	756	43	1970	103
81	763	43	1970	166
82	775	43	1970	278
83	806	44	1971	228
84	839	45	1972	228
85	859	46	1973	103
86	916	47	1974	286
87	963	49	1976	43
88	998	49	1976	252
89	1197	57	1984	238
90	1206	58	1985	46
91	1211	58	1985	111
92	1298	61	1988	195
93	1305	61	1988	261
94	1327	62	1989	274
95	1356	63	1990	274
96	1371	64	1991	132
97	1377	64	1991	197
98	1405	65	1992	265

Mathematics Magazine: Quickies

No.	Problem	Volume	Year	Page(s)
99	Q117	28	1954–1955	37
100	Q138	28	1954–1955	241
101	Q141	28	1954–1955	292
102	Q166	29	1955–1956	229
103	Q188	30	1956–1957	172
104	Q227	32	1958–1959	32
105	Q234	32	1958–1959	113
106	Q242	32	1958–1959	229
107	Q266	33	1959–1960	302
108	Q281	34	1961	303
109	Q284	34	1961	303
110	Q341	37	1964	251
111	Q343	37	1964	251

No.	Problem	Volume	Year	Page(s)
112	Q710	59	1986	112

***Mathesis:* Questions d'examen**

No.	Problem	Volume	Year	Page(s)
113	Ex. 10	69	1960	249–250
114	Ex. 11	69	1960	250
115	Ex. 12	69	1960	251

***Mathesis:* Questions**

No.	Problem	Volume	Year	Page(s)
116	3954	69	1960	255
117	3957	69	1960	375

Source list's date: “Last updated on 26 June 2026”.

Errors and Corrections in the Previous Edition

This appendix records errors and discrepancies found in Ali Sinan Sertöz's expanded September 2026 compilation. Page references identify physical PDF pages, including the preliminary pages; in the main text, subtract seven to obtain the old printed page number. Each numbered record identifies the affected problem, the original reading or omission, the correction made in this edition, and its justification. Superscripts in the text link to these records.

The present list contains 95 records and covers the material examined for this working proof. It includes historical corrections already reported by the journals, identified as such, and new corrections established during reconstruction. It also records the earlier Q227 omission that is already repaired in the supplied corrected copy. This is a cumulative record; transcription is complete, but the mathematical audit is not exhaustive.

Confirmed errors are corrected in the new text. Unresolved readings, if any, are explicitly marked as unresolved. Purely typographic changes, such as consistent journal styling, restored Turkish diacritics, and standardized display punctuation, are not presented as mathematical errors. Newly supplied material is identified as an editorial addition and is not attributed to the historical journal.

Confirmed errors and documented corrections

- E001** Corrected. Problem 4102, source PDF p. 12: “thus property (2) will be proved” changed to “property (3)”, matching the part under proof.
- E002** Corrected. Problem 4125, source PDF p. 14: [1944, 352] changed to [1944, 252], in agreement with the proposal and the compiler's list.
- E003** Problem 4134: “inscribed triangle” is the original terminology; the proof uses the contact triangle of the incircle. The quotient in part (1) is undefined when $A_i = \pi/3$; it is understood through the equivalent identity $C_i^n - \pi/3 = (-1)^n 2^{-n} (A_i - \pi/3)$, which also covers this case. In the telescoping angle sum on source PDF p. 16, $C_1 C_2$ has been corrected to $C_2 C_3$ in both arguments, matching the side in the limit on the left.
- E004** Corrected. Problem 4679, source PDF p. 20: the first equality sign in the displayed Simson-line equation was a misprint for a plus sign. The perpendicular foot on $A_3 A_4$ has coordinates $\frac{1}{2}(\cos \theta_3 + \cos \theta_4 + \cos \theta_j - \cos(\theta_3 + \theta_4 - \theta_j), \sin \theta_3 + \sin \theta_4 + \sin \theta_j - \sin(\theta_3 + \theta_4 - \theta_j))$, which verifies the corrected left-hand side.
- E005** Corrected compilation error. Source PDF p. 41 is headed “Solution to Problem 1878” but reproduces the E1877 solution from p. 40. The duplicate has been removed. An explicitly identified editorial proof of E1878 is supplied, with the proposal reference retained; the source's alleged solution citation is not represented as verified.
- E006** Corrected degeneracy. E2110, source PDF pp. 27 and 45: the conclusion requires A, B, C not all coincident. Three cyclic orderings of the same equilateral triangle satisfy the premise with $A = B = C$. The determinant argument gives an affine map $z \mapsto \alpha z + \beta$; it

- is a direct similarity only when $\alpha \neq 0$. The needed qualification is added to the statement and proof.
- E007** Corrected. E2160, source PDF p. 48: $\sum_i x_i = 2\Delta$ is dimensionally and mathematically incorrect; the area identity is $\sum_i a_i x_i = 2\Delta$, as also required by the immediately following maximum calculation. The repeated identical intermediate product expression is consolidated. The boundary case is explicitly obtained by continuity because the preceding formula divides by $\sin P_i$.
- E008** Corrected numerical example. E2213, source PDF p. 50 gives the vertices $(0, 0)$, $(19, 0)$, $(6, 24)$, $(0, 24)$ and $N = (7, 12)$. Those side lengths do not satisfy $3b = a + c + d$. Replacing 6 by $38/7$ gives lengths 19, $193/7$, $38/7$, 24, with $3 \cdot 19 = 193/7 + 38/7 + 24$, and the claimed concurrence at $(7, 12)$. This exact repair preserves the other three vertices and the stated Nagel point.
- E009** Corrected. E2363, source PDF p. 55: the proposal page 663 in the solution heading has been changed to 662, matching the proposal reference on p. 29 and the complete list. The displayed cosine law requires $\cos A'$ rather than $\cos A$. The phrase describing the two other equations is clarified as the cyclic permutations, since the printed pair of permutations did not describe them correctly.
- E010** Corrected. Dodge's E2462 article, source PDF p. 60: the first display has $2r_2 + r_3$ on the right. The missing coefficient is restored to give $2r_2 + 2r_3$, as required by the stated inequality and the subsequent cases.
- E011** Corrected. Dodge's Lemma 1, source PDF p. 61: the preliminary value 0.044 is a misplaced decimal; the same page later gives 0.004419. The exact minimum is $1 - \sqrt{5/4 - \sin 15^\circ}$. Derivatives are taken with respect to the radian-valued variable; the displayed degree values identify the corresponding angles.
- E012** Corrected endpoint convention. Dodge's quadrilateral corollary, source PDF p. 63: $R_i > 0$ is replaced by $R_i \geq 0$, since the stated arbitrary point P may coincide with a vertex.
- E013** Corrected. E3422, source PDF p. 67: the second chord equation lacks s and is restored to $(-f)(1 + p's) = p' + s$. The claim that $p + p' \neq 0$ merely because s is nonreal is false when F, F' lie on the circle; the required nonvanishing follows from the problem's finite intersection T . Equation (*) is read after clearing denominators when $s^2 = -1$. The unique-intersection and distinct-point conditions are implicit in the original construction; coincident tangents do not define a unique T .
- E014** Corrected. E2625, source PDF p. 64: the two triangles are polar under the newly defined polarity $\tau = \sigma\pi$, rather than under the original conic polarity π of K . The phrase "with respect to K " is corrected to "with respect to τ ".
- E015** Corrected. E3469, source PDF p. 69: the list of six midpoint labels omitted 3. The complete list 1, 2, 3, 4, 5, 6 is restored, in agreement with both solutions and the six named segments.
- E016** Corrected. E1687, source PDF pp. 76–77: the displayed matrices use row coordinates, despite the word "columns"; that word is corrected. The second occurrence of the coordinate matrix incorrectly changes three plus signs to minus signs. The entries $1 + a'$, $1 + b'$, and $1 + c'$ are restored to agree with (β) and the verified product matrix. Consistent normalization is specified for matrix multiplication. The final reference to point P is changed to T , the concurrence point. Equality of areal coordinates is stated for the affine setting, since general projective transformations do not preserve area ratios.

- E017** Corrected mathematical factor. E1142, source PDF p. 73: the numerator factor 16 is replaced by 4. If $x = \sin \alpha$, $y = \sin \beta$, $z = \sin \gamma$ and K is the area of the triangle with sides x, y, z , the displayed determinant is $-16K^2$. The three unit generator vectors have mutual chord lengths $2x, 2y, 2z$ and circumradius $\sin \phi = xyz/(2K)$; hence $\sin^2 \phi = -4x^2y^2z^2/\det$. As a direct check, three mutually perpendicular generators have $\alpha = \beta = \gamma = 45^\circ$ and $\sin^2 \phi = 2/3$; the old factor would give $8/3$.
- E018** Corrected bibliographic misprints. Source PDF pp. 80, 85 and 231 give the year range 1969–1960 for proposals 384 and 398 in volume 33 of *Mathematics Magazine*. Both ranges are corrected to 1959–1960, matching the same volume’s other entries and the chronological sequence. On pp. 81 and 232 the volume of proposal 756 is printed “43I”; the extraneous letter is removed, giving 43.
- E019** Q227, source PDF pp. 183 and 187. The previously reported cut-off statement is complete in the supplied corrected copy; its ending, “all prime numbers not all equal nor all distinct”, is retained here in native LaTeX. The answer on p. 187 prints $f(n) = 1 - \varphi(n)$, which gives $f(1) = 0$ and nonpositive values thereafter. The sign is corrected to $+$; the resulting fourteen values are explicitly checked in the text.
- E020** Corrected. Q117, source PDF p. 186: the plus sign between $1^2 + \dots + 8^2$ and $8 \cdot 9 \cdot 17/6$ is replaced by an equality sign. Each expression equals 204; their sum does not.
- E021** Corrected bibliographic discrepancy. Q166, source PDF p. 186: the solution heading gives page 29, whereas the proposal and complete list give 229. The missing leading digit is restored, following the paired Quickies citation.
- E022** Corrected. Q188, source PDF p. 187: the first clock equation has 12 instead of 12α on the right, despite the preceding words “12 times α ”. The missing factor is restored. Nonnegative hour indices are allowed, so midnight is not omitted, and $(p, p') = (11, 11)$ is excluded because it yields $\alpha = \alpha' = 1$, contrary to the required half-open ranges. The distinction between a valid interchange and a different resulting time is made explicit.
- E023** Corrected counting convention. Q341, source PDF p. 188: with the usual $\varphi(1) = 1$, the number of distinct unordered coprime pairs from $\{1, \dots, n\}$ is $\sum_{k=2}^n \varphi(k)$, not $\sum_{k=1}^n \varphi(k)$. The lower limit is corrected in the counting argument. The denominator $\binom{n}{2}$ and the problem’s denominator $n(n+1)/2$ give the same limit, not the same finite ratio. The limiting argument is stated accordingly; the answer $6/\pi^2$ is unchanged.
- E024** Corrected missing hypotheses. Proposal 384, source PDF p. 85: matrices whose row sums are 1 need not be invertible (the matrix with every entry $1/n$ is singular for $n > 1$), so they do not all form a group. The historical solution on p. 115 already identifies the need for nonsingularity. “Nonsingular real” is supplied, and $n \geq 2$ is stipulated for the assertion of infinite order. If $A\mathbf{1} = \mathbf{1}$, then $A^{-1}\mathbf{1} = \mathbf{1}$, proving closure under inversion; for $n = 1$ the group consists only of (1).
- E025** Corrected missing exceptional case. Proposal 425, source PDF p. 86: the twin primes 3 and 5 give $n = 4$, for which $3\varphi(4) = 6 > 4$. The condition $n > 4$ is restored; it is explicitly present in the same assertion as Mathesis Question d’examen 11.
- E026** Corrected radius dependence. Proposal 440, source PDF p. 87: the source replaces $\rho = (R/r - 1)/2$ by $n = \lfloor \rho \rfloor$ both in the summation limit and inside the square root. Rounding inside the square root loses circles in partially completed radial shells. The correction agrees with Sutcliffe’s published solution on p. 125. Only the summation limit is rounded in the corrected formula. For example, $R/r = 1 + 2\sqrt{3}$ gives 13 contained

- circles, whereas the printed formula gives 7. In one open 60° sector the lattice centers have squared radial distance $(2r)^2(p^2 + pq + q^2)$, $p, q \geq 1$; solving for the admissible integers q gives the corrected summand.
- E027** Corrected subscript. Proposal 372, source PDF p. 85: the first angle inside the cosine is printed θ_1 although the summand is indexed by i, j . It is corrected to θ_i . For adjacent indices the required term is $\cos(\theta_i + \theta_{i+1})$, and the corrected general expression agrees with expansion of the squared sine of the angle sum. In the solution on pp. 112–113, the same first-index error recurs; in the induction step, θ_{p-1} is changed to θ_{p+1} , the minus sign preceding twice the sum is changed to plus, and $\sin \theta_1$ under the sum is changed to $\sin \theta_i$. The telescoping calculation then agrees with the corrected induction equation.
- E028** Corrected ambiguous notation. Proposal 437, source PDF p. 86: the source writes $(a + b)^{[n]}$ and then describes $[n]$ as the count of binary 1's in n , leaving the exponent of the claimed power of 2 unstated. The standard intended assertion is written explicitly as $2^{s_2(n)}$ odd coefficients in $(a + b)^n$, where $s_2(n)$ counts those 1's. This agrees with the restatement in the journal's comment on p. 123. The original “prove or disprove” instruction is retained.
- E029** Corrected strict-inequality domain. Proposal 587, source PDF p. 90: the stated interval $-\pi < \theta < \pi$ includes $\theta = 0$, where the left side is 1. The strict inequality therefore requires $0 < |\theta| < \pi$. Equivalently, one may retain the full open interval and replace $<$ by $\leq$, with equality only at zero.
- E030** Clarified from the original journal's correction. Proposal 517, source PDF pp. 88 and 133–134: the assertion is false for the straight-sided triangle FMN and trapezoid $N'M'MN$. The two regions must instead share the parabola's arc MN . The journal explicitly reports this distinction after the solutions on p. 134, crediting Breault, Nolan and Zirkel. This qualification is included in the statement.
- E031** Corrected formulas. Proposal 208, source PDF p. 99: the first geometric quotient must begin $x^{2n-2} + x^{2n-4}$, the second polynomial on the left must be $x^{2n+1} + 1$, the first cosine product must use $k\pi/(2n)$, and the second sine-product identity requires a missing equality sign. The corrected factorizations follow directly from the roots of unity and give the same three final values as the first solution.
- E032** Corrected algebraic signs. Proposal 227, source PDF p. 101: the six off-diagonal weight coefficients in the final formulas for R_1, R_2, R_3 have the wrong signs. Each factor $k_i - 1$ in those terms is replaced by $1 - k_i$. This follows by expanding $R_i = W_i - r_i + r_{i+1}$ cyclically, using the preceding, correct formulas for r_i . It also restores the equilibrium check $R_1 + R_2 + R_3 = W_1 + W_2 + W_3$.
- E033** Corrected summation limit. Proposal 234, source PDF p. 102: the displayed upper limit m is replaced by n , since the problem assumes $m \geq n$ and a square cannot have more than n lattice points on a bounding-box side. The subsequent sums and final answer already use n . The historical note that no solution of part (b) was received is retained; an explicitly identified new finite-sum solution of that part is added.
- E034** Corrected vector label. Proposal 242, source PDF p. 102: the first numerator in the formula for B'' is $B' + kA'$ in the source but must be $C' + kA'$, as confirmed both by the cyclic construction and by its immediately following expanded numerator.
- E035** Corrected sign. Proposal 217, source PDF p. 99: the displayed squared bracket has three positive products of forward distances. For the stated skew-symmetric determinant the

third product must have a minus sign, or equivalently use the reversed distance A_4A_2 . The corrected bracket is exactly Ptolemy's relation.

- E036** Corrected missing divisions. Proposal 248, source PDF p. 104: the intermediate derivative line multiplies by CM_1, CM_2 although the preceding and following lines divide by those distances. The intermediate expressions are corrected to $(ds_i/d\alpha)/CM_i$.
- E037** Corrected orthocenter calculation. Proposal 258, source PDF pp. 105–106: substituting the stated coordinates of C into its altitude equation gives $-a \cos(t + 3A/2)$, not the printed positive value. Consequently the sine and cosine factors in the two ellipse denominators and semiaxes are interchanged. The corrected parametrization is $x = a \sin t \sin(3A/2)/\sin(A/2)$, $y = -a \cos t \cos(3A/2)/\cos(A/2)$. It agrees with the second solution's final factors $1 \pm 2 \cos A$. Absolute values are needed for semiaxis lengths; the zero-semiaxis cases $A = 60^\circ, 120^\circ$ are explicitly included. The garbled intermediate semidiameter chain in the second solution is replaced by the resulting length formulas.
- E038** Corrected angle convention and sign. Proposal 266, source PDF pp. 83 and 107–108: the statement requires oriented angles modulo 2π , as Edy's solution and the journal's note about Bankoff explicitly explain. It is not valid for unrestricted ordinary unsigned angles. The final relation in Guy's solution prints an equality between $\angle AMB$ and $\angle AM'B$ where their sum is required; the plus sign is restored.
- E039** Corrected hypothesis. Proposal 298, source PDF p. 109: the repeated statement has $f''(x) = 0$ instead of $f''(x) > 0$ in the proposal on p. 84. The strict inequality is restored, consistently with the positive density formula in the solution.
- E040** Corrected sign. Proposal 380, source PDF p. 114: equation (11) prints $x = (b^2/a^2)z - b(c-b)u/(ca)$. The sign before the last term must be plus. From (8), using $u = abc/(ab+bc+ca)$, one obtains the corrected equation directly. Substitution of $z = au/c$ then gives $x = bu/a$, as claimed. The original minus sign generally fails this substitution.
- E041** Corrected omitted term. Proposal 334, source PDF p. 112: the expanded area computation drops the sector contribution $-a^2\theta/2$ on its second line. It is restored so that the expansion actually simplifies to $a^2(\theta - \sin\theta) = ax$. The final answer in the source is already correct.
- E042** Corrected reindexing display. Proposal 415, source PDF p. 119: the substitution $p = n - p'$ prints a sum with lower index n and upper index 0, and omits x in the sine argument. The corrected finite sum is $\sum_{p'=0}^n \binom{n}{n-p'} \sin((2p' - n)x)$, which equals $-s$ by binomial symmetry and the oddness of sine.
- E043** Corrected mathematical conclusion; historical invalid proof explicitly retained for comparison. Proposal 419, source PDF pp. 120–121: a particle constrained to a fixed frictionless path under gravity has conserved kinetic-plus-potential energy. Thus constant nonzero speed requires constant height, so the path in a vertical plane is horizontal. The published projection argument does not describe this motion. It also evaluates $\int_{y_0}^y \sqrt{y_0/(t-y_0)} dt$ as $\frac{1}{2}\sqrt{y_0}\sqrt{y-y_0}$ rather than $2\sqrt{y_0}\sqrt{y-y_0}$, and describes the vertex tangent of $y = 4x^2/y_0 + y_0$ as vertical although it is horizontal. The corrected answer and explanation precede and follow the labeled historical argument; the invalid formulas are not offered as a valid solution.
- E044** Historical error corrected in the working solution. Proposal 425, source PDF pp. 121–123: Breault applies multiplicativity as $\varphi(6m) = 2\varphi(m)$ without assuming $(6, m) = 1$. Klarner's published comment already identifies the error using $m = 2$. The repaired

- computation is $\varphi(6m) = 2m \prod_{p|m, p \neq 2,3} (1 - 1/p)$, which gives the desired inequality for all $m \geq 1$. The historical comment is reproduced and its erroneous equality is explicitly identified as the value produced by the flawed method.
- E045** Corrected formula and specified factorization. Proposal 425, Carlitz's solution, source PDF p. 122: the final term of (2) is printed as $1/(3n)$; it is corrected to $n/3$, equal to $2^\alpha 3^{\beta-1}m$. In the converse argument $n = 3^\alpha k$ is taken with $(k, 3) = 1$, a needed condition for the totient evaluation that follows. The mathematical equivalence $\varphi(n) = n/3$ if and only if $n = 2^\alpha 3^\beta$ with $\alpha, \beta \geq 1$ is unchanged.
- E046** Clarified domains. E1288, source PDF pp. 124–125: the counting proof includes $h = 0, n$ despite initially defining h only for $0 < h < n$. The conventions $L(0) = 0$ and containment at both endpoints are stated. The quotient rule for $C(a/b)$ is used when the quotient is a positive integer. Moser's base- p generalization needs p prime; this condition is supplied explicitly. It is false for arbitrary composite bases: for $p = 4, n = 6$, where four coefficients are coprime to 4 but the base-4 digit product is six.
- E047** Corrected coefficient. Proposal 440, source PDF p. 125: the second form of Sutcliffe's sum has $3/5$ under the square root. It must be $3/4$, as shown both by expanding $\frac{1}{2}\sqrt{4\rho^2 - 3\rho^2}$ and by the right-triangle computation and final formula on p. 126. The unrounded radius variable is called ρ throughout the new solution to distinguish it from its integer part.
- E048** Corrected historical comment. Proposal 440, source PDF p. 127: the claim that Sutcliffe's count includes partly contained circles is false. His center-radius bound is $R - r$, exactly the condition for a radius- r circle to be entirely contained in the radius- R circle. At $R/r = 1 + 2\sqrt{3}$, there are 13 fully contained circles, including six internally tangent boundary circles. The original formula gives only 7. The published comment is retained as historical text and immediately followed by the verified correction.
- E049** Corrected semiaxes and missing hypotheses. Proposal 472, source PDF p. 128: after obtaining the coordinates of M' , the source defines A, B using the coefficients of I instead. The correct ellipse semiaxes are $a(a^2 + 3b^2)/(a^2 - b^2)$ and $b(3a^2 + b^2)/(a^2 - b^2)$; their squared difference is $a^2 - b^2$. A parabola has no center, a circle gives no unique intersection for generic M , and axis points need a limiting convention. In the hyperbola case the new coefficients are $a(a^2 - 3b^2)/(a^2 + b^2)$ and $b(b^2 - 3a^2)/(a^2 + b^2)$; either can vanish. The exceptional ratios $a^2 = 3b^2$ and $b^2 = 3a^2$ are excluded from the nondegenerate conclusion, and both cases are explicitly evaluated.
- E050** Corrected and completed. Proposal 487, source PDF pp. 129–130: the Jordan-block discussion conflates root order, number of blocks, and matrix dimension. The displayed diagonal 2×2 specialization requires diagonalizability and does not cover a defective matrix. The claim that repeated eigenvalues prevent polynomial roots is too strong: I has the polynomial roots $\pm I$, and every root of a nonscalar 2×2 matrix is polynomial in it. The commutant dimension is not the number of independent parameters in the family of roots. The general nonsingular construction is rewritten with explicit block roots and precise notation; an editorial formula gives every square root of a nonscalar 2×2 matrix, and the scalar and singular cases are included. All statements are made over $\mathbb{C}$, with real roots selected by real entries. The singular-block discussion is clarified rather than reproducing inconsistent block identifications as valid equalities.
- E051** Corrected proof. Proposal 498, source PDF p. 131: the final display replaces $f(\delta)f(D)$ by a product of powers $p^{r'+s'}$, which evaluates δD rather than the function values. Multiplicativity requires products $f(p^{r'})f(p^{s'})$. Equality follows from equality of the unordered

- exponent pairs, not merely from $r' + s' = r + s$. The corrected display retains the prime-wise argument. Positive integer inputs are specified.
- E052** Completed uniqueness check. Proposal 509, source PDF pp. 132–133: the historical search reaches a successful assignment at $T = 6$ after excluding $T = 2, 3, 4, 5$, but does not explicitly exclude every larger value. Exhaustive base-11 column enumeration, with distinct digits and nonzero leading letters, finds exactly one assignment: $(U, N, I, T, E, D, S, A, M, R, C) = (8, 10, 2, 6, 5, 3, 9, 1, 7, 4, 0)$. This confirms the stated answer and its uniqueness; the computation is identified as an editorial verification.
- E053** Invalid historical proof identified. Proposal 517, source PDF p. 134: Kugelmass’s argument asserts that arbitrary projective transformations preserve the relevant area ratio and then replaces areas by altitude lengths in a limit. General projective maps preserve neither area ratios nor Euclidean focus–directrix data, and the stated limiting and monotonicity claims are not justified. The historical argument is retained under an explicit invalid-proof label. Parker’s valid curvilinear-area proof is supplemented by the focus-based area integral. The original journal’s distinction between arc-bounded and rectilinear regions remains recorded in E030.
- E054** Qualified locus. Proposal 529, source PDF pp. 135–136: the original statement says the locus is a circle or ellipse, but the solution itself shows that one arch of the cycloid traces only a semicircle and one circuit of the cardioid along one side of the line traces only a half-ellipse. The statement now asks for the containing conic and the portion traced. The historical explanation of the additional motion required for a full conic is retained; cusp values are understood as limiting values.
- E055** Corrected area formulas. Proposal 537, source PDF pp. 136–137: the source prints $A = (2 - \sqrt{2})/2$ and $2B = \sqrt{3}(\sqrt{3} - \sqrt{2})^2/2$, inconsistent with the diagram and its numerical total. The correct formulas are $A = (\sqrt{2} - 1)^2/(2\sqrt{3})$ and $2B = \sqrt{3}(\sqrt{3} - \sqrt{2})^2/4$. For a unit circumcircle, the top protrusion has height $1 - 1/\sqrt{2}$ and base twice that height divided by $\sqrt{3}$; each lower protrusion has horizontal leg $(\sqrt{3} - \sqrt{2})/2$ and vertical leg $\sqrt{3}$ times that. The corrected total is approximately 0.093272, agreeing with the source’s rounded 0.0933. The expression for $2C$ is unchanged.
- E056** Restored carry. Proposal 544, source PDF p. 137: the hundreds-column condition is printed $O + T + S = NI$. The tens column supplies a carry of 1, so it must be $O + T + S + 1 = 10N + I$. The printed answer $217 + 392 + 408 = 1017$ satisfies the corrected relation ($2 + 3 + 4 + 1 = 10$). Exhaustive enumeration of distinct decimal digits confirms that it is the unique solution subject to the stated order and divisibility conditions.
- E057** Restricted ordinary-angle statement. Proposal 563, source PDF p. 139 and its proposal: “an arbitrary point of the circumcircle” is too broad for the displayed equalities of ordinary angles. The solution uses the open minor arc between B' and A' , which ensures $0 < a < 30^\circ$ and the indicated segment orders. This restriction is made explicit, excluding degenerate endpoints. For example, take $A = (-1, 0)$, $B' = (-1/2, \sqrt{3}/2)$, $A' = (1/2, \sqrt{3}/2)$, $B = (1, 0)$ and $M = (0, -1)$: the ordinary angle MAF is 75° , so its triple cannot equal the ordinary angle MEF . No unsupported extension to the entire circle is claimed.
- E058** Corrected ellipse and completed proof. Proposal 587, source PDF pp. 141–142: the comment prints $x^2/\pi + y^2/4 = 1$; its first denominator must be π^2 . Common direction of concavity does not by itself establish containment. The numerical stationary-point calculation is supplemented by a proof that $\sin \theta/\theta$ is strictly decreasing on $(0, \pi)$, giving

- a unique minimum of F there and the strict bound between its endpoint values. E029 records the separate equality-at-zero correction.
- E059** Corrected missing factor. Proposal 600, source PDF p. 144: the chain for the in-contact area ends with $S/(2R)$ instead of $Sr/(2R)$. Since $S = sr$ and $T = 2sr^2/(4R)$, the missing r is required. The subsequent calculation in the source already uses the corrected expression.
- E060** Completed a necessity-and-converse gap. Proposal 680, source PDF p. 149: assuming the proposed center locus and checking consistency does not exclude other centers. With $q = \cos^2 t/a^2 + \sin^2 t/b^2$ and the outward-normal multiplier $\lambda > 0$, the tangency equations factor as $(\lambda^2 - a^2b^2)(1 + \lambda q)^2 = 0$. Therefore $\lambda = ab$, which forces $C = ((a + b) \cos t, (a + b) \sin t)$. The converse and external tangency are also established.
- E061** Inconsistent data; incomplete reproduction and wrong unit identified. Proposal 738, source PDF pp. 151–152: the reproduced solution starts mid-calculation with no byline. No missing text or attribution is invented. Its return-speed bound $13 - s$ is unjustified for oblique motion. The exact velocity equations force a 5 mph current parallel to AB , so the route cannot connect distinct parallel banks. The historical answer is only a zero-width limiting case. In the alternative computation m denotes width; its unit is corrected from mph to miles.
- E062** False assertion and incomplete published correction. Proposal 743, source PDF pp. 153–154: Schmeichel's edge-midpoint counterexample disproves the coefficient $2\sqrt{3}/3$, and continuity gives interior counterexamples. Goldberg's center calculation is correct but the preceding maximization argument is not. The published replacement coefficient $2\sqrt{2}/3$ is incompatible with the retained claim of equality at the center, where the ratio is $2\sqrt{3}/3$. Both historical contributions are retained and their status is explicit. A sharp corrected inequality and its equality cases remain unresolved in this edition; no unverified replacement is stated as a theorem.
- E063** Corrected convergence domain and denominator. Proposal 763, source PDF p. 156: the Dirichlet series for $\zeta(s)$ is asserted to converge for $|s| > 1$, which is false (take $s = -2$). The condition is $\operatorname{Re} s > 1$, or $s > 1$ for real s . The factor printed $33/22$ in the odd tenth-power sum must be $33/32$, since $1 - 2^{-10} = (1 - 2^{-5})(33/32)$. The older unsigned Bernoulli indexing is explicitly distinguished from modern B_{2n} notation.
- E064** Corrected beta factor and inverse-function hypotheses. Proposal 775, source PDF p. 157: $p^{-1}B(1/p, 1+1/q)$ equals $(p+q)^{-1}B(1/p, 1/q)$, not $(pq)^{-1}B(1/p, 1/q)$. Already $p = q = 1$ disproves the printed value. Strict decrease is specified for existence of a single-valued inverse. Continuity alone does not justify the displayed derivative calculation; the general identity follows by area decomposition or Stieltjes integration, while the specific functions admit the improper derivative calculation.
- E065** Specified nondegeneracy and symmetry hypotheses. Proposal 806, source PDF pp. 158–159: a right isosceles triangle gives collapsed small triangles with undefined incenters and is excluded. The six-point reflection theorem is stated for three reflected pairs, not for an arbitrary reflection-invariant set with fixed points on the axis. A conic may be degenerate. The central-symmetry determinant defines a nonzero quadratic only when its three fixed rows are independent; a nullspace construction covers dependent rows. The added analytic proof supplies the missing qualifications.

- E066** Reconciled conventions and corrected table. Proposal 859, source PDF pp. 161–162: Nelson’s twelve solutions include two with a leading zero in *FOUR*; Tabor and Beidler’s ten exclude them. Both counts are correct under their different conventions, now explicit. In the second solution’s table, the seventh column is printed F although it represents I (the final column is F). In rows five and six the entry for T is printed 3; it must be 4, giving $THREE = 40233$. Exhaustive distinct-digit enumeration verifies all twelve assignments, ten with nonzero leading digits, and the prime-plus-nondivisibility uniqueness claim.
- E067** Corrected strictness in an intermediate inequality. Proposal 1206, source PDF p. 166: the arithmetic–geometric mean comparison is printed with $<$, although equality is possible (already for $n = 1$). It is corrected to $\leq$. The final claimed strict bound remains valid because $1 - 1/x < \ln x$ for $x > 1$. The journal’s concluding integral approximation is identified as a limit as the partition mesh tends to zero.
- E068** Clarified exceptional cases and angle branches. Proposal 1211, source PDF pp. 168–169: the squared tangent formula is undefined at 90° and does not distinguish supplementary angles; the slope derivation also divides by quantities that may vanish. The signed relation $\cot \alpha = (x^2 + y^2 - a^2 - b^2)/(2\sqrt{b^2x^2 + a^2y^2 - a^2b^2})$ covers every exterior point, including vertical-tangent and axis cases, and separates acute from obtuse angles. The director circle is the 90° locus; the original ellipse is only the boundary limit as $\alpha \rightarrow 180^\circ$. This supplies the exceptional-case discussion explicitly noted as missing by the original journal.
- E069** Corrected central-angle index. Proposal 1298, source PDF p. 171: the formula for $q^2 = |PR|^2$ uses $\beta + \delta$ in the source. The successive arcs PQ, QR have angles β, γ , so the correct formula is $q^2 = 4r^2 \sin^2((\beta + \gamma)/2)$. This agrees with equation (3) immediately above it.
- E070** Clarified existence of the second intersections. Proposal 1356, source PDF p. 175: the source specifies $M \neq A$ but does not repeat this requirement for N . Both distinct second intersections must exist for the lines AM, AN to be defined. Tangency at A is possible, for example $B = (0, 0)$, $C = (1, 0)$, $P = (1/4, 0)$, $Q = (3/4, 0)$, $A = (1/2, \sqrt{2}/4)$; these degenerate cases are excluded explicitly.
- E071** Integrated the original journal’s correction. Proposal 1377, source PDF p. 178: saying only that DEF is inscribed in ABC does not specify the required correspondence of vertices to sides. The conditions $D \in BC$, $E \in CA$, $F \in AB$ are now explicit. The journal explains that corrected Problem 1371 had already appeared, but the uncorrected duplicate remained in its accepted-proposals file. Both historical entries and that explanation are retained.
- E072** Removed an unnecessary division in the final algebra. Proposal 1405, source PDF p. 180: the source writes ratios with denominators $1/r_i - 1/r$. The equivalent equalities $(1/r_1 - 1/r)/a = (1/r_2 - 1/r)/b = (1/r_3 - 1/r)/c$ divide only by positive side lengths and exhibit the required transitivity directly. This avoids an unstated nonzero-denominator qualification; it does not assert existence of non-side tangents in a geometrically degenerate configuration.
- E073** Corrected citation. Chapter 10 list, source PDF p. 190: Proposal 226 is listed at page 488. The contribution heading on p. 194 gives page 48, which is adopted.
- E074** Supplied construction details and a degeneracy qualification. Proposal 192, source PDF pp. 190–191: the angle chain uses an undefined point K . The required point is $CA \cap A'B'$. The second intersection I is distinguished from A' , and angles are treated as directed

- modulo π . In the second cyclic quadruple, the printed B is corrected to I , the point actually used in the angle substitution. The cyclic common-point argument is made explicit. The claim requires the indicated circumcircles and a nondegenerate intersection configuration; for example, taking the symmetry center at the midpoint of AB makes $A = B'$ and the first purported triangle has repeated vertices. Exceptional limiting configurations are not classified here.
- E075** Completed a missing inference and clarified the domain. Proposal 203, source PDF p. 191: absence of the quadratic term in the cubic establishes only that the three x -coordinates sum to zero, not by itself that the triangle is equilateral. An explicitly identified trigonometric completion proves the assertion, covers $b = 0$ (excluded by the source's elimination of y), and includes coincident fourth intersections. The degenerate case $a = b = 0$, where the second equation is not a circle, is excluded.
- E076** Corrected indices and made directed ratios explicit. Proposal 204, source PDF pp. 192–193: the second triad's p_z is corrected to p_a , agreeing with the solution table. The reciprocal in part 2(a) is $1/k(B_p)$, not $1/k(b_p)$. The line cutting CAA_m is BTB_p , not the repeated BTB_q , because T lies on the cevian p_b . The negative ratios throughout are directed segment ratios, now marked consistently. The four Ceva conditions reduce to the displayed side relation.
- E077** Corrected variable reuse and integration limits. Proposal 221, source PDF p. 194: the letter a is used both for the spiral's scale and an endpoint distance, while a, b are also placed as angular integration limits. The scale is renamed C , the angular limits are θ_0, θ_1 , and a, b retain their final meaning as slant distances. The formula is stated for an acute crossing angle with increasing slant distance; the generator and parallel-circle endpoint cases are explained separately.
- E078** Corrected intermediate inversion formulas. Proposal 270, source PDF p. 196: the denominator printed with n^2 must contain ρ_n^2 . The next displayed numerator must be $4R^2\rho_n$, since the inversion power is $4R^2$; the source's $R^2\rho_n \tan \alpha$ does not follow from its preceding equation. The final radius formula agrees with the corrected calculation.
- E079** Corrected identification and documented truncation. The compiler's list and heading on source PDF pp. 190 and 197 say 272, whereas the reproduced journal entry clearly says 273. The journal number is adopted. Its supplied solution ends after the harmonic-ratio equality, without a concluding argument. That fragment is retained and a separate coordinate completion is supplied; missing historical text is not reconstructed speculatively.
- E080** Clarified the coefficient field and factor argument. Proposal 332, source PDF p. 197: the assertion needs rational coefficients (or integer coefficients), and is false over $\mathbb{R}$ or $\mathbb{C}$. The source's reference to the “greatest degree of an exact factor” is not literally correct. The obstruction is the irreducible degree-24 factor Φ_{90} : a degree-22 rational divisor cannot include it, while the remaining factors total degree 21.
- E081** Corrected geometric cases and discriminant classification. Proposal 353, source PDF pp. 198–199: “externally” and “internally” are interchanged in the two cases; distinct equal circles can only touch externally. Coincident circles and straight-line limits are not ordinary tangent pairs. The cubic reduction is valid, but the final comparison involving $\sqrt{27}\sqrt{a^2 - b^2}$ is not an equivalent discriminant criterion. For example $a = b = 2, u = 1$ gives one real root, although that comparison predicts three. The full cubic discriminant and admissible-root interval are retained. A rational-coordinate example yields an irreducible cubic, so a general straightedge-and-compass construction is impossible.

- E082** Corrected the generalized area identity. Proposal 374, source PDF p. 200: for $\overrightarrow{PA'_1} = -n\overrightarrow{PA_1}$ and $\overrightarrow{AA'_2} = m\overrightarrow{AA'_1}$, the coefficients must be $(1-m)(1-m+mn)$ and m^2n^2 . These follow by expanding oriented vector areas, and specialize to the proved equality at $m = 1/2, n = 3$ with collinear A_1, B_1, C_1 . Twice-oriented-area notation is made explicit so the vector products do not lose a factor of 2.
- E083** Corrected perpendicular and reconstructed metric cuts. Proposal 383, source PDF p. 201: BE must be perpendicular to AP , not AB . This is required by the similar triangles ABE, APD and the equality $BE = AP$. All seven drawing panels on pp. 201–202 are replaced by native diagrams; the planar cuts are generated from the stated $3u$ by $2u$ rectangle and a square of side $u\sqrt{6}$. Coordinates and translations make the six-piece dissection reproducible independently of the old schematic proportions.
- E084** Corrected derivatives, operator signs, and induction indices. Proposal 387, source PDF pp. 203–205: derivatives on the right of the statement are with respect to x , not t . For $u = e^{-x/(1-t)}$, one has $u_t = -xu_{xx}$ and $(1-t)^{-1}e^{-xt/(1-t)} = -e^xu_x$; both minus signs are missing in the reproduced derivation. Consequently $D(xD^2)^pe^{-x}$ has coefficient signs $(-1)^{p+\lambda+1}$, not the printed $(-1)^\lambda$. The corrected induction keeps the same operator method, including its coefficient recurrence, and corrects the final exponent from $p-1$ to $p+1$.
- E085** Corrected recurrence domain and determinant formula. Proposal 401, source PDF pp. 206–209: the recurrence starts at $n = 2$, and $F(2) = ad+bc$, not $ad-bc$. The tridiagonal continuants need subdiagonal -1 when their superdiagonal is b ; the printed pair $b, -b$ generates b^2 . The second determinant has order $n-3$, not $n-1$. The resulting formula is $(ad+bc)D_{n-2}+bdD_{n-3}$, with $D_0 = 1, D_{-1} = 0$. The continued-fraction proof is explicitly restricted to nonzero denominators.
- E086** Clarified limiting positions and undefined ratios. Proposal 412, source PDF p. 210: at $P = A$ the displayed ordinary ratios have zero denominators, and at $P = B, C$ one secant is a tangent limit. These positions determine the projective relation by limits, rather than by substituting into undefined ratios. Calligraphic letters distinguish the bilinear coefficients from the triangle's vertices.
- E087** Corrected the final parallelism. Proposal 427, source PDF p. 211: the last line says $OH = BH \parallel AB$, which contradicts the requested conclusion. When $O = B$, the limiting point $G = C$, so the construction gives $BH \parallel AG = AC$. The terminal AB is corrected to CA .
- E088** Disproved the nonexistence claim and corrected decimal-period conventions. Proposal 428, source PDF pp. 211–212: having seven permuted multiples does not force seven digits, and the specific cyclic shift used in the proof is not guaranteed to occur. The ordinary integer 105263157894736842 and its six displayed multiples provide a counterexample. Breault's two printed integers also fail as ordinary integers; their doubles gain a digit. The intended periods of $1/17$ and $1/29$ begin with zero and have lengths 16 and 28. Their cyclic-multiple interpretation requires preserving that fixed width. The historical argument and comment are identified as such, with corrections separately explained.
- E089** Qualified the extension to exterior and exceptional points. Proposal 432, source PDF pp. 212–213: the solution concludes “all points in the plane,” although some intersections or denominators can be undefined. Directed ratios extend the proof to nondegenerate exterior configurations; clearing denominators gives the corresponding product identity where a continuous limit exists. No undefined ordinary ratio is assigned a value.

- E090** Clarified the ellipse parameters and circular limit; recorded source truncation. Proposal 435, source PDF pp. 213–214: the quartic uses c without defining it; $c^2 = a^2 - b^2$ is supplied. The four-extrema statement assumes $a > b > 0$; a circle has infinitely many equal-sized equilateral triangles. The final solver affiliation is cut off after “J. W. Mellender, University” and remains explicitly marked as incomplete.
- E091** Recorded missing context, not a reconstructed error. Proposal 445, source PDF p. 215: only Demir’s correction to an earlier proof is supplied, with neither the proposal nor that proof. The correction and its formula are transcribed faithfully; the underlying configuration and the meaning of k cannot be established from this excerpt alone.
- E092** Corrected side bounds, sign, and feasibility claims. Proposal 476, source PDF pp. 216–218: the adjacent-side bound is $m \leq a$, not $m \leq b$, because $m = AM$ and $AB = a$. In the opposite-side perimeter equation the contribution of AD is $+d$, not $-d$. The second interval bound concerns n , not a repeated m . Real roots and the actual side intervals must be checked; the listed necessary inequalities do not alone establish existence for a chosen pair. The finite intersection O excludes parallel opposite sides, so a separate coordinate construction supplies that case. The reconstructed argument covers convex quadrilaterals; arbitrary nonconvex cases are not asserted to follow from it.
- E093** Clarified the tower indexing and real domain. Proposal 653, source PDF pp. 218–219: the tower’s bracket is replaced by the explicit recurrence used in the solution, with $n + 1$ copies of x , $y_0 = x$, and $y_{-1} = 1$. The logarithmic differentiation applies for $x > 0$. The sum and its final product, including y_{-1} , are written with explicit limits.
- E094** Made geometric qualifications explicit. Proposal 1256, source PDF pp. 221–222: the supplied proof assumes convexity and $\alpha > \beta$. Convexity is included in the statement; interchanging the ends and the limit $\alpha = \beta$ complete the first proof. Hoyt’s diagram requires a finite intersection of DA, CB ; parallel sides are covered by the first argument or its limiting case, rather than an undefined finite point X .
- E095** Completed the chapter’s contribution list. The opening list on source PDF p. 190 omits the contributions to 412, 427, 428, 432, 435, 445 and 476 that are actually reproduced on pp. 210–218. These seven entries are restored with the references printed above their reproductions, bringing the chapter total to 25.